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A PUSHING UP THEOREM FOR CHARACTERISTIC 2 TYPE GROUPS

BY
MICHAEL ASCHBACHER[1]

1. Introduction

A finite group G is of *characteristic* 2 *type* if $F^*(M) = O_2(M)$ for each 2-local subgroup M of G. It seems probable that in the near future the problem of classifying the finite simple groups will be reduced to the classification of groups of characteristic 2 type. With the exception of certain sporadic groups, the simple groups of characteristic 2 type are the Chevalley groups over fields of even order. The structure of these groups is determined by the maximal parobolics, that is the maximal 2 locals containing a Sylow 2-subgroup. Hence given a simple group G of characteristic 2 type it appears advisable to study the set $\mathcal{M}$ of maximal 2-local subgroups of G and attempt to force $\mathcal{M}$ to resemble the collection of maximal parobolics in some Chevalley group.

Let $M \in \mathcal{M}$ and $T \in \mathrm{Syl}_2(M)$. If G is indeed a Chevalley group then $N_G(T) \leq M$. Ideally one would like to show this holds in general, modulo a set of known exceptions. In practice $M = N_G(L)$ for some subgroup L of G with the property that M is the unique maximal 2-local containing LT. Hence $N_G(B) \leq M$ for each nontrivial normal subgroup B of LT. In particular neither $J(T)$ nor $Z(T)$ is normal in LT. In many interesting cases $L/O_2(L)$ is simple, so that the Thompson factorization fails. This seems to force $L/O_2(L)$ to be a Chevalley group of even characteristic. Perhaps the most troublesome case occurs when $L/O_2(L)$ is isomorphic to $L_2(2^e)$. The main result of this paper deals with that case.

THEOREM 1. *Let G be a finite group of characteristic 2 type, $H \leq G$, $M = N_G(O^2(H))$ and $T \in \mathrm{Syl}_2(H)$. Assume $H^* = O^2(H/O_2(H)) \cong Z_3$ or $L_2(2^n)$, $O_2(H) \in \mathrm{Syl}_2(C_M(H^*))$, and M is the unique maximal 2-local subgroup of G containing H. Then either*

(1) $N_G(T) \leq M$,

or

(2) *G has sectional 2-rank at most 4.*

The largest Janko group is an example where $N(T)$ is not contained in M. G. Mason called this to the author's attention.

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Theorems 2 and 3 in Sections 3 and 4 are also of independent interest. For example Theorem 2 figures in the proof of [1].

Many of the ideas used here are due to Glauberman and Sims. The author would like to thank Professor Glauberman in particular for generously sharing some of these ideas. A recent result of Harada [4] is also quite useful.

Most of the notation used here is reasonable standard. In addition given a group G, denote by $\mathscr{A}(G)$ the set of elementary abelian 2-subgroups of G of maximal order and $J(G) = \langle \mathscr{A}(G) \rangle$. $\tilde{Z}(G) = \Omega_1(Z(J(G)))$ in case G is a 2-group. $\mathscr{M}$ is the set of maximal 2-local subgroups of G and for $X \leq G$, $\mathscr{M}(X)$ is the set of members of $\mathscr{M}$ containing X.

2. Preliminary lemmas

(2.1) *Let G be a group with $F^*(G) = O_2(G) = Q$ and $G/Q \cong S_3$. Let $T \in \mathrm{Syl}_2(G)$. Then either*

(1) *there is a nontrivial characteristic subgroup of T normal in G,*

or

(2) *there is a unique noncentral chief factor of G contained in Q.*

Proof. See [3].

(2.2) *Let G be a group with $F^*(G) = O_2(G) = Q$ and $G/Q \cong S_3$. Let $T \in \mathrm{Syl}_2(G)$, $S = C_T(\tilde{Z}(T))$ and $H = \langle S^G \rangle$. Then $S \in \mathrm{Syl}_2(H)$, or $\Omega_1(Z(T)) \leq Z(G)$.*

Proof. See 2.11.1.4 in [7].

(2.3) *Let G be a group with $F^*(G) = O_2(G) = Q$ and $G/Q \cong S_5$. Let $V = \Omega_1(Z(O_2(G)))$, $T \in \mathrm{Syl}_2(G)$, $Z = \Omega_1(Z(T))$, $Y = O^2(C_G(Z))$, and X the preimage in G of the centralizer in G/Q of a transposition in T/Q. Assume some element of T induces a transvection on V. Then:*

(1) *$[V, G]$ is the natural module for $O_4^-(2)$.*
(2) *$TY/O_2(YT) \cong S_3$ and $J(O_2(YT)) = J(Q)$.*
(3) *$J(T \cap X) = J(T)$ and $X/O_2(X) \cong S_3$.*

Proof. This is an easy calculation. See [1] for example.

(2.4) *Let G be a group with $F^*(G) = O_2(G) = Q$. Let $T \in \mathrm{Syl}_2(G)$, $L = O^2(G)$, and $V = \Omega_1(Z(Q))$. Assume $L/O_2(L) \cong L_2(2^e)$ or Z_3, $[V, L] \neq 1$, and $\mathscr{A}(G) \nsubseteq Q$. Then either*

(1) *$G/Q \cong S_5$ and some involution induces a transvection on V,*

or

(2) *$V/C_V(L)$ is the natural module for $L_2(2^e)$ and if G is not solvable and $A \in \mathscr{A}(G) - \mathscr{A}(Q)$ then $AQ = T \cap LQ$.*

Proof. This follows easily from some elementary facts about the 2-modular representations of $L_2(2^e)$. See [1] for example for details.

(2.5) *Let $G \cong O_4^-(2)$, V a $GF(2)$ module for G, and U a submodule of V with $|V: U| = 2$ and U the natural module for G. Then $V = C_V(G) \oplus U$.*

Proof. Assume not. Then there exists $v \in V - U$ with $|v^G| = 6$. Hence V is a homomorphic image of the permutation module for G on 6 letters. But then $U = [V, G]$ is the natural module for $L_2(4)$.

3. $L_3(2^e)$

In this section we assume G to be a finite group of characteristic 2 type. L_i, $i = 1, 2$, are distinct subgroups of G such that $V_i = O_2(L_i)$ is the natural module for $L_i/V_i \cong L_2(q)$, $q = 2^e > 2$, with $V_1 V_2 = J$ Sylow in L_1 and L_2. Assume $M_i = N_G(L_i)$ is a maximal 2-local of G, $T \in \mathrm{Syl}_2(M_1)$ and $O_2(\langle T, L_1, L_2\rangle) = 1$.

THEOREM 2. *Under the hypothesis above either*

(1) *G has sectional 2-rank 4,*

or

(2) $F^*(G) \cong L_3(q)$.

Throughout this section take G to be a counter example to Theorem 2. Let $M = M_1$, $L = L_1$, $V = V_1$, $Z = V \cap V_2$, and X a Hall $2'$-group of $N_L(J)$.

(3.1) (1) $\mathscr{A}(J) = \{V, V_2\}$.

(2) *L splits over V.*

(3) *J is of type $L_3(q)$.*

Proof. Straightforward.

(3.2) $V = O_2(LT)$.

Proof. Let $Q = O_2(LT)$. Then $QJ = C_T(J/Z)$ and as L acts irreducibly on V, $Q = C_{QJ}(V)$. Hence $QJ = V_2 Q = V_2 C_{QJ}(V)$. By 3.1.1, $T \in \mathrm{Syl}_2(M_2)$, so by symmetry $QJ = VC_{QJ}(V_2)$. Therefore $QJ = JC_{QJ}(J)$. By a Frattini argument $QJ = JC_Q(XJ)$. Let t be an involution in L inverting X. Then $L = \langle J, t\rangle$ acts on $C_Q(XJ) = C_Q(X)$ so that $LQ = L \times C_Q(L)$ with $C_Q(L) = C_Q(XJ)$. By 3.1.1, $X \leq N(V_2) \leq M_2$ so X acts on $C_{QJ}(L_2)$ and centralizes QJ/J. Therefore $C_Q(L) = C_Q(L_2)$, so as $O_2(\langle LT, L_2\rangle) = 1$ we conclude $C_Q(L) = 1$.

(3.3) (1) *T is the split extension of J by cyclic group $F = N_T(X)$ inducing field automorphisms on L/V.*

(2) *If f is an involution in F then all involutions in fJ are fused to f in T. Moreover $C_L(f)$ is the split extension of $C_V(f)$ by $L_2(2^{e/2})$ acting naturally on $C_V(f)$.*

(3) $J = J(T)$.

Proof. Part (1) follows from 3.2 and a Frattini argument on X. An easy calculation supplies the remaining parts.

Let $S \in \mathrm{Syl}_2(G)$ with $T \leq S$.

(3.4) (1) $|S: T| \leq 2$ *and if* $s \in S - T$ *then* $V^s = V_2$.
(2) *If s is an involution in* $S - T$ *then* $C_J(s)$ *is of type* $L_2(q)$ *or* $U_3(q^{1/2})$.
(3) $J = J(S)$.

Proof. Let $R = N_S(T)$. As T is Sylow in $M = N(V)$, 3.3.3 and 3.1.1 imply $R = T\langle s\rangle$ where $V^s = V_2$. Assume s is an involution. s either inverts or centralizes a cyclic subgroup of $\mathrm{Aut}_G(Z)$ acting irreducibly on Z, so $|C_Z(s)| = q$ or $q^{1/2}$. Moreover $\langle s, J\rangle/Z$ is wreathed. Hence either $Z = C_J(s)$ is of type $L_2(q)$ or $C_Z(s) = \Omega_1(C_J(s))$ with $|C_J(s)| = q^{3/2}$, and we refer to this latter group as of type $U_3(q^{1/2})$.

From this information we conclude $J = J(R)$. Hence $R = S$.

(3.5) $q > 4$.

Proof. If $q = 4$ then by Theorem 3 in [4], G has sectional 2-rank 4.

(3.6) $Z^G \cap V = Z^M$.

Proof. By 3.4.3 and 3.1.1, $N(Z)$ is transitive on $V^G \cap C(Z)$, so $N(V)$ is transitive on $Z^G \cap C(V)$.

(3.7) $Z^G \cap S \subseteq V \cup V_2$.

Proof. Let $A = Z^g \in Z^G \cap S$. By 3.3 and 3.4, $m(S/J) \leq 2$, so as $q > 4$, $A \cap J \neq 1$. As each involution in J is in $V \cup V_2$ we may take $A \cap J = A \cap V$. Moreover $m(A \cap V) \geq e - 2$.

Suppose $a \in A$ induces a field automorphism on L/V. Set $B = \langle a\rangle(A \cap V)$. $C_V(a) = [V, a]$ so $N_V(B)$ is of index at most $|A: A \cap V| \leq 4$ in V. Let

$$N_V(B) \leq R \in \mathrm{Syl}_2(N(B)).$$

As $q > 4$ and a induces a field automorphism on L/V, $e \geq 4$. Hence by 3.3.2 and 3.4.2 every abelian subgroup of S of rank $2e - 2$ is contained in J. Therefore $N_V(B) \leq J(R) \leq C(B)$, a contradiction.

Therefore either $A \leq J$ or $|A: A \cap V| = 2$ and $a \in A - V$ induces a graph or graph-field automorphism on J. In the first case $A \leq V$ and we are done. So take $a \in A - V$. As $m(A \cap V) = e - 1$ and $A \cap V$ centralizes a, 3.4.2 implies $Z = C_J(a)$. Then $A \cap V \leq C_J(a) = Z$. Set $D = [J, a]$. a inverts D so that $Z = [D, a]$ and as $|Z: Z \cap A| = 2$,

$$m(N_{DA}(A)/C_{DA}(a)) = e - 1.$$

Let $N_{DA}(A) \leq R \in \mathrm{Syl}_2\,(N(A))$. Then

$$e - 1 = m(N_{DA}(A)/C_{DA}(A)) \leq m(C_R(A)/A) \leq 1,$$

a contradiction.

(3.8) *Z is a TI-set in G.*

Proof. Let $z \in Z^{\#}$ and $Q = O_2(C(z))$. Without loss we take $z \in Z(S)$. As G is of characteristic 2 type $C_S(Q) \leq Q$. Of course $Q \trianglelefteq S$. These two facts and the structure of S force $Z \leq Q$.

Suppose $z \in Z \cap Z^g$. As $Z \leq Q$, $\langle Z, Z^g\rangle$ is a 2-group. But then as Z is a TI-set in M, 3.6 and 3.7 imply $Z = Z^g$. Hence Z is a TI-set in G.

(3.9) *If a is an involution in S and $a^g \in J$, then $a \in J$.*

Proof. Assume $a \in S - J$. Then we may take $a = f$ or $a \in S - T$. Moreover we may take $b = a^g \in Z(C_J(a))$. Set $h = g^{-1}$.

Assume first that $[a, Z] \neq 1$. Then by 3.3 and 3.4, $C_J(a)$ is of type $L_3(q^{1/2})$ or $U_3(q^{1/2})$. Let $C_S(a) \leq R \in \mathrm{Syl}_2\,(C(a))$. The structure of S and $C_J(a)$ forces $b \in Z(J(R))$. But by 3.8, $Z(J(R)) = Z^h$ so $b \in Z \cap Z^h$, contradicting 3.8.

Hence $C_J(a) = Z$. As Z is a TI-set, $[Z, Z^h] = 1$. Set $Q = O_2(C(Z))$.

$$B = \langle a\rangle J \in \mathrm{Syl}_2\,(C(Z))$$

and as G is of characteristic 2 type, $C_B(Q) \leq Q$. This forces $Q = B$, J, or $\langle a^J\rangle$. Thus either $J \trianglelefteq C(Z)$ or $\Omega_1(J \cap Q) = Z$ and $a \in Q$. But as $Z^h \leq C(Z)$, 3.7 implies $a^x \in Z^{hx} \leq J$ for some $x \in C(Z)$, a contradiction.

(3.10) $J = T$.

Proof. If not then by 3.3 there is an involution $f \in F$. By 3.9, 3.3, and 3.4,

$$R = C_S(f) \in \mathrm{Syl}_2\,(C(f)).$$

As G is of characteristic 2 type, $Z(R) \leq Q = O_2(C(f))$. Then by 3.3, $C_V(f) = [Z(R), C_L(f)] \leq Q$. But $C_V(f) \not\leq O_2(L_2 \cap C(f))$, a contradiction.

(3.11) $J = S$.

Proof. Assume $S \neq J$. By 3.10 and 3.4, $|S\colon J| = 2$. An easy argument shows $S - J$ contains an involution. Now 3.9 and Thompson transfer implies $G \neq O^2(G)$. As $J \leq L \leq O^2(G)$, we get a contradiction by induction.

(3.12) $J \trianglelefteq N(Z)$.

Proof. J/Z is abelian so as G is of characteristic 2 type $J = O_2(C(Z)) \trianglelefteq N(Z)$.

(3.13) $F^*(G) \cong L_3(q)$.

Proof. By 3.8 and 3.12, $C(z)$ is 2-closed for each involution z in G. Now appeal to the main theorem of [6].

This completes the proof of Theorem 2.

4. $Sp_4(q)$

In this section we assume G to be a finite group of characteristic 2 type L_i, $i = 1, 2$, are distinct subgroups of G, $M_i = N_G(L_i)$, $T \in \mathrm{Syl}_2\,(M_i)$, $V_i = O_2(L_i T)$, $\Phi(V_i) = 1$, $L_i V_i/V_i \cong L_2(q)$, $q = 2^e > 2$, $V_i/(V_i \cap C(L_i))$ is the natural module for $L_2(q)$, $J = V_1 V_2 \in \mathrm{Syl}_2\,(L_i V_i)$ and $\{M_i\} = \mathscr{M}(L_i T)$.

THEOREM 3. *Under the hypothesis above either*

(1) $F^*(G) \cong L_3(q)$ *or* $Sp_4(q)$, *or*
(2) G *has sectional* 2-*rank* 4.

Throughout this section take G to be a counter example to Theorem 3. Let $M = M_1$, $L = L_1$, $V = V_1$, $Z = C_V(L)$ and Y_i a Hall $2'$-group of $L_i \cap N(J)$. $Z_2 = V_2 \cap C(L_2)$.

(4.1) (1) $\mathscr{A}(J) = \{V, V_2\}$.
(2) L *splits over* V.
(3) $|Z| = q$ *and* Y_1 *is transitive on* $Z_2^\#$.
(4) $Z_2 \leq [L, V]$.

Proof. $Z(J) = C_V(V_2) = V \cap V_2$ with $|V_i\colon V \cap V_2| = q$. So $|V| = |V_2|$. Moreover all involutions in J are in $V \cup V_2$, so (1) holds. There is a complement in V_2 to V, so (2) holds. By (1), $Y_1 \leq N(V_2) \leq M_2$. Hence Y_1 acts on Z_2. Also LT and L_2 act on $Z \cap Z_2$ so as $\mathscr{M}(LT) = \{M\}$, $Z \cap Z_2 = 1$. Finally $Z_2 \leq C(V) \cap V_2 = V \cap V_2$. Hence as Y_1 is transitive on $((V \cap V_2)/Z)^\#$, Y_1 is transitive on $Z_2^\#$, $Z_2 = [Z_2, Y_1] \leq [L, V]$ and $|Z_2| = q$ or 1. Now Theorem 2 completes part (3).

(4.2) (1) $V = [V, L]$.
(2) $[V, V_2] = V \cap V_2 = ZZ_2$.

Proof. Let $U = [V, V_2]$. By 4.1.3 either $Z \leq U$ or $Z \cap U = 1$. Assume the latter. Then $|U| = q = |[V_2/Z_2, V]$ so that $U \cap Z_2 = 1$. Let $h \in L - M_2$. Then $[V, L] = U \times U^h$ so that $[V, L] \cap V_2 = U$. But by 4.1.4, $Z_2 \leq V_2 \cap [V, L]$, a contradiction.

So $Z \leq U$. By symmetry $Z_2 \leq U$. As $|ZZ_2| = q^2 = |V \cap V_2|$, (2) holds. Also $Z \leq [V, L]$ so as $V/Z = [L, V/Z]$, (1) holds.

By 4.1, $Y_2 \leq M$ so we may choose Y_2 to normalize Y_1. By symmetry, Y_1 normalizes Y_2. Y_2 is regular on $Z^\#$ while $Y_1 \leq L \leq C(Z)$, so $Y_1 \cap Y = 1$. Hence for this choice of Y_i we have:

(4.3) $Y = Y_1 Y_2 \cong Y_1 \times Y_2 \cong Z_{q-1} \times Z_{q-1}$.

(4.4) *$Y = Y_1 \times C_Y(L/V)$ and YL/V acts naturally as $GL_2(q)$ on V/Z.*

Proof. Y acts on L/V and centralizes Y_1, so $YL/V \cong GL_2(q)$. As L/V acts irreducibly on V/Z, the ring D of endomorphisms of V/Z commuting with L/V is a division ring and then $C_Y(L/V)$ is a subfield isomorphic to $GF(q)$. Hence YL/V acts naturally on V/Z.

Set $F = N_T(Y)$ and let S be a Sylow 2-subgroup of G containing T.

(4.5) (1) *T is the split extension of J by F and F induces a group of field automorphisms on L/V.*

(2) *If f is an involution in F then all involutions in fJ fuse to f in T. Moreover $C_{LY}(f)$ is the split extension of $C_V(f)$ by $GL_2(q^{1/2})$ acting naturally on $C_V(f)/C_Z(f)$. $|C_Z(f)| = q^{1/2}$.*

Proof. Part (1) follows by a Frattini argument on Y. Then an easy calculation supplies (2).

(4.6) (1) *$J = J(S)$.*
(2) *$|S: T| \leq 2$.*
(3) *If s is an involution in $S - T$ then $C_J(s)$ is of type $Sz(q)$.*

Proof. From 4.5 we conclude $J = J(T)$. Let $R = N_S(T)$. Then $|R: T| \leq 2$ with $V^s = V_2$ for $s \in R - T$. Assume s is an involution. $Z^s = Z_2$ so $\langle s\rangle(V \cap V_2)$ and $R/(V \cap V_2)$ are wreathed. Hence $|C_J(s)| = q^2$ and $V \cap V_2 \cap C(s) = Z(C_J(s)) = \Omega_1(C_J(s))$ is of order q. We say such a 2-group is of type $Sz(q)$.

It follows that $J(R) \neq J$ and hence $S = R$.

(4.7) *Let $z \in Z^\#$, $z_2 \in Z_2^\#$, and $u = zz_2$. Then:*

(1) *All involutions in J are fused to z, z_2, or u in G.*
(2) *$u \notin z^G \cup z_2^G$.*

Proof. All involutions in V are fused to z, z_2 or u in LY. As all involutions in J are in $V \cup V_2$, (1) follows with the symmetry between V and V_2.

By 4.6.1, $\{V, V_2\}$ is weakly closed in S, so $S \cup M$ controls fusion in V, yielding (2).

(4.8) *Z is a TI-set.*

Proof. $\{M\} = \mathcal{M}(LT)$, so $C_G(z) \leq M$ for each $z \in Z \cap Z(T)$. As Y is transitive on Z^* and $Z \trianglelefteq M$ we conclude Z is a TI-set.

(4.9) *$Z^G \cap S \subseteq J$.*

Proof. Let $Z^g \leq S$. Then $1 \neq Z^g \cap T = Z^g \cap N(Z)$, so as $\langle Z, Z^g \rangle$ is a 2-group and Z is a TI-set, $Z^g \leq C_S(Z) = J$.

(4.10) $J \trianglelefteq C_G(u)$.

Proof. We may assume $u \in Z(S)$. Let $Q = O_2(C(u))$. As $J = J(S)$ it suffices to show $J \leq Q$. As G is of characteristic 2 type $C_S(Q) \leq Q$. Of course $Q \trianglelefteq S$. These two facts force $V \cap V_2 \leq Q$. Set $W = \langle (Z^G \cup Z_2^G) \cap C_Q(Z^G \cup Z_2^G) \rangle$. By 4.9, $W = V$, V_2 or $V \cap V_2$. If $W = V$ then $C(u) = C_M(u) \leq N(J)$, so take $W = V \cap V_2$. By 4.7, $Z^G \cap W = \{Z\}$ or $\{Z, Z_2\}$, so $C(u) = C_M(u)S \leq N(J)$.

(4.11) *If a is an involution in S and $a^g \in J$ then $a \in J$.*

Proof. Assume $a \in S - J$. Then we may take $a = f$ or $a \in S - T$. Thus by 4.5 and 4.6, $C_J(a)$ is of type $Sp_4(q^{1/2})$ or $Sz(q)$. Now we may take

$$b = a^g \in Z(C_J(a)) \quad \text{and} \quad C_J(a) \leq R \in \mathrm{Syl}_2\ (C(a)).$$

Next the structure of S and $C_J(a)$ force $b \in Z(J(R))$. But b is fused to z, z_2 or u, so $J \in \mathrm{Syl}_2\ (\langle J^{C(b)} \rangle)$ and hence is strongly closed in S with respect to $C(b)$. As $a \in S - J$ while $a \in J(R)$, this is a contradiction.

(4.12) $J = T$.

Proof. If not then by 4.5 there is an involution f in F. By 4.5, 4.6, and 4.11,

$$R = C_S(f) \in \mathrm{Syl}_2\ (C_G(f)).$$

As G is of characteristic 2 type, $Z(R) \leq Q = O_2(C(f))$. Then by 4.5,

$$C_V(f) = [Z(R), C_L(f)] \leq Q.$$

As $C_V(f) \not\leq O_2(L_2 \cap C(f))$, this is a contradiction.

(4.13) $J = S$.

Proof. Assume not. By 4.12 and 4.6, $|S: J| = 2$. Let $s \in S - J$. $V^s = V_2$ so $(S/(V \cap V_2))$ is wreathed and we may take $s^2 \in V \cap V_2$. Next $\langle s \rangle (V \cap V_2)$ is wreathed so we may take s to be an involution. Now 4.11 and Thompson transfer imply $G \neq O^2(G)$. As $J \leq L \leq O^2(G)$ we get a contradiction by induction.

At this point the 2-local structure of G is determined, so that any of a number of methods show $F^*(G)$ to be isomorphic to $Sp_4(q)$. For completeness we sketch a geometric proof of this fact.

As V_i is weakly closed in $S = T = J$ we get:

(4.14) $Z^G \cap V_i = Z^{M_i}$ *for $i = 1$ and 2.*

In particular:

(4.15) *Z is weakly closed in V so $Z_2 \notin Z^G$.*

Let $X = C_Y(L/O_2(L))$, $W = Y_1$, $K = C_L(X)$ and $A = T \cap K$. By 4.4, $C_V(X) = 1$, so by a Frattini argument:

(4.16) $K \cong L_2(q)$.

Next X acts on $Z_2 A \leq V_2$ and hence on $Z^G \cap Z_2 A = \{A_1\}$. Thus Z_2 and A_1 are the only X-invariant subgroups of $Z_2 A$ of order q, so that:

(4.17) $A \in Z^G$.

(4.18) $Z^G \cap M = \{Z\} \cup A^M$.

Proof. By 4.14, $Z^G \cap T = Z^G \cap V_2 = \{Z\} \cup A^V$.
Set $Z * A = \{Z\} \cup A^V$. Then

(4.19) *$\langle Z * A\rangle = V_2$ is abelian.*

(4.20) *For $h \in M$ either $A^h \in Z * A$ or $\langle A, A^h\rangle \in K^M$.*

Proof. $|A^M| = |L: V_2 W| = q(q+1)$ so there are $q^3(q+1)$ pairs (A^r, A^s) with $r, s \in M$ and $A^s \notin Z * A^r$. Also $|L^M| = |M: LZ| = q^2$, so there are $q^3(q+1)$ pairs (A^r, A^s) with $\langle A^r, A^s\rangle \in K^M$.
Set $I = O^{2'}(C(AW))$.

(4.21) *$I \in K^G$ and $W \in X^G$.*

Proof. Let $A = Z^g$. $W \leq N(A) \leq M^g$. Also W centralizes Z, so $[W, L^g] \leq V^g$. Hence $W \in X^G$ and $I \in K^G$.

(4.22) $I = O^{2'}(C(W))$.

Proof. $Z \in \mathrm{Syl}_2\,(C(W\langle z\rangle))$ for each $z \in Z^\#$, so the result follows from [6].

(4.23) $[I, K] = 1$.

Proof. Let $B \in A^K - \{A\}$. By 4.22, $O^{2'}(C(BW)) = O^{2'}(C(W)) = I$, so $K = \langle A, B\rangle \leq C(I)$.
Set $\Sigma(Z^g) = Z^G \cap M^g - \{Z^g\}$ and let $\mathscr{D}$ be the graph with vertex set Z^G and Z^g joined to the vertices in $\Sigma(Z^g)$.

(4.24) *$\mathscr{D}$ is connected.*

Proof. Let Γ be the connected component of $\mathscr{D}$ containing Z. With 4.7 and 4.10, $C_G(t) \leq N(\Gamma)$ for each $t \in T^\#$, so if $\Gamma \neq Z^G$ then $N(\Gamma)$ is strongly embedded in G. As G has more than one class of involutions, this is impossible.

(4.25) (1) *$\mathscr{D}$ has diameter 2.*
(2) *If $[Z, Z^g] \neq 1$ then $\langle Z, Z^g \rangle \in K^G$ and $|\Sigma(K^g) \cap Z * A| = 1$ for each $A \in \Sigma(Z)$.*

Proof. If $\langle Z, Z^g \rangle \in K^G$ then by 4.23, $|\Sigma(Z^g) \cap Z * A| = 1$ for each $A \in \Sigma(Z)$. Suppose $ZABZ^g$ is a chain in $\mathscr{D}$ from Z to Z^g. By 4.20, $\langle A, Z^g \rangle \in K^G$, so $Z * A \cap \Sigma(Z^g) = \{C\}$ and hence Z^g is of distance 2 from Z in $\mathscr{D}$. This yields (1). Now 4.20 completes the proof.

(4.26) $F^*(G) \cong Sp_4(q)$.

Proof. For $B \in Z^G$ set $B^\perp = \{B\} \cup \Sigma(B)$. Let $\mathscr{B}$ be the block design with point set Z^G, block set $\{B^\perp : B \in Z^G\}$, and incidence defined by inclusion. From 4.18 and 4.25 an easy calculation shows $\mathscr{B}$ is a symmetric block design with $k = q(q+1)+1$ and $l = q+1$. For $C \in Z^G - B^\perp$ define $B * C = Z^G \cap \langle B, C \rangle$. Then $B * C$ is defined for each pair of distinct points B and C, and one checks that $B * C$ is the line

$$\bigcap_{D \in B^\perp \cap C^\perp} D^\perp$$

through B and C in $\mathscr{B}$. Hence [2] implies $\mathscr{B}$ is 3-dimensional projective space over $GF(q)$. Moreover for $z \in Z^*$, z^G is the set of elations of $\mathscr{B}$ commuting with the symplectic polarity $B \leftrightarrow B^\perp$ of $\mathscr{B}$. Therefore $F^*(G) = \langle z^G \rangle \cong Sp_4(q)$.

This completes the proof of Theorem 3.

5. Theorem 4

In this section we assume the hypothesis of Theorem 1. Set

$$V = [H, \Omega_1(Z(O_2(H)))]$$

and assume some element of T induces a transvection on V. Assume $H/O_2(H) \cong S_5$.

THEOREM 4. *Under the hypothesis of this section either*

(1) $N_G(T) \leq M$,

or

(2) *G is of sectional 2-rank 4.*

Throughout this section G is a counterexample to Theorem 4. Set

$$Z = \Omega_1(Z(T)), \quad Y = O^2(C_H(Z)),$$

and let X be the preimage of the centralizer in $H/O_2(H)$ of a transposition in $T/O_2(H)$. $L = O^2(H)$. From 2.3 we conclude:

(5.1) (1) *V is the natural module for $O_4^-(2)$.*
(2) *$TY/O_2(YT) \cong S_3$ and $J(O_2(YT)) = J(O_2(H))$.*
(3) *$J(T \cap X) = J(T)$ and $X/O_2(X) \cong S_3$.*

(5.2) (1) *If* $1 \neq B \trianglelefteq H$ *then* $N_G(B) \leq M$.
(2) *If* $1 \neq B$ *is characteristic in* T *then* B *is not normal in* H.

Proof. $N(T) \not\leq M$ while $\{M\} = \mathscr{M}(H)$.
The next lemma is the key to Theorem 4 and is essentially due to G. Glauberman.

(5.3) $V = [O_2(H), L]$.

Proof. Set $R = T \cap X$. By 5.1, $|T: R| = 2$ and $J(R) = J(T)$. By 2.2 there is a normal subgroup A of X with $R \cap A = C_R(\tilde{Z}(R))$. Hence $R \cap A$ is characteristic in T, since $C_T(\tilde{Z}(T)) = C_R(\tilde{Z}(R))$. Moreover if B is a characteristic subgroup of $R \cap A$ normal in A then B is characteristic in T and $B \trianglelefteq \langle T, X \rangle = H$. Hence by 5.2, $B = 1$. We conclude from 2.1 that A has a unique noncentral chief factor in $O_2(A)$. As $A \trianglelefteq X$ the same holds for X. Thus if x is an element of order 3 in X then $[x, O_2(H)] = [x, O_2(X)] = [x, V]$. Hence $V = [O_2(H), L]$.

(5.4) $N_G(T) \cap N(Y) \leq M$.

Proof. $N(T) \cap N(Y) \leq N(YT) \leq N(J(O_2(YT)) = N(J(O_2(H)) \leq M$ by 5.1 and 5.2.
Set $Q = O_2(H)$, $D = C_Q(L)$, $F = C_M(L)$, $E = C_F(Z)$, and $g \in N(T) - M$.

(5.5) $Q = V \times D$.

Proof. Let $P \trianglelefteq H$ with $[P, L] = 1$, and subject to these conditions choose P maximal. Set $\bar{H} = H/P$ and assume $Q \neq 1$. Let $\bar{U}/\bar{V}$ be a subgroup of order 2 in $Z(\bar{T}/\bar{V}) \cap \bar{Q}/\bar{V}$. By 5.3, $U \trianglelefteq H$. As H acts irreducibly on V, $\Phi(\bar{U}) = 1$. Now by 2.5, $\bar{U} = \bar{V} \times C_{\bar{U}}(\bar{H})$, contradicting the maximality of P.
So $Q = VC_Q(L)$ and as $C_V(L) = 1$, the product is direct.

(5.6) $D \neq 1$.

Proof. If $D = 1$ then by Theorem 3 in [4], G has sectional 2-rank 4.

(5.7) $O_2(C_G(Z)) = (T \cap L)O_2(E)$.

Proof. As $D \neq 1$, $Z \cap D \neq 1$, so $C_G(Z) \leq C_G(Z \cap D) \leq M$. $M = TLF$ with $T \leq C(Z)$, so

$$C(Z) = C(Z) \cap TLF = TC_{LF}(Z).$$

$Y(T \cap L)F$ is a maximal subgroup of LF with $C_{LF/F}(Z) \leq Y(T \cap L)F/F$ and $Y(T \cap L) \leq C(Z)$, so

$$C_{LF}(Z) = Y(T \cap L)C_F(Z) = Y(T \cap L)E.$$

Hence $C_G(Z) = TYE$. Now $O_2(TYE/E) = (T \cap L)E/E$ and $T \cap L \leq O_2(TYE)$

so

$$O_2(C(Z)) = (T \cap L)O_2(E).$$

(5.7) $J(O_2(C(Z))) = VJ(O_2(E)) \trianglelefteq H.$

Proof. Let $A \in \mathscr{A}(O_2(C(Z)))$. If $A \not\leq VE$ then $m(A/A \cap E) < 4$. Hence

$$m(A) \geq m(V(A \cap E)) = 4 + m(A \cap E) > m(A),$$

a contradiction. Thus $A \leq VE = V \times E$, so AV is elementary abelian. Hence $V \leq A$ and then $A = V \times (A \cap E)$ with $A \cap E \in \mathscr{A}(O_2(E))$.

Set $J = J(O_2(C(Z)))$. By 5.7, $\langle H, g\rangle \leq N(J)$ contradicting $\{M\} = \mathscr{M}(H)$. This completes the proof of Theorem 4.

6. Graphs

In this section G is a transitive permutation group on a set Ω, $\alpha \in \Omega$, $H = G_\alpha$, and $\Delta = \Delta(\alpha)$ is an orbit of H on Ω. $\mathscr{G} = \mathscr{G}(\Delta)$ is a directed graph on Ω with edges (α^g, β^g), $g \in G$, $\beta \in \Delta$. Set $\Delta(\alpha^g) = \Delta^g$.

Most of the results in this section are due to Sims and come from [5].

The *connected component* of $\mathscr{G}$ containing α is the collection of vertices β for which there exists a path $\alpha = \alpha_0, \ldots, \alpha_n = \beta$ between α and β such that for each i either (α_i, α_{i+1}) or (α_{i+1}, α_i) is an edge.

(6.1) *Let Σ be the connected component of $\mathscr{G}$ containing α. Then Σ consists of those vertices β for which there exists a directed path $\alpha = \alpha_0, \ldots, \alpha_n = \beta$ from α to β with (α_i, α_{i+1}) and edge for each i.*

Proof. See 3.1 in [5].

(6.2) *If $G = \langle H, g\rangle$ for $\alpha^g \in \Delta$ then $\mathscr{G}$ is connected.*

Proof. Let Σ be the connected component of $\mathscr{G}$ containing α. Then Σ is the equivalence class of a G-invariant equivalence relation, so if $\alpha^x \in \Sigma$ then $x \in N(\Sigma)$. As $G = \langle H, g\rangle$ is transitive on Ω the lemma follows.

In the remainder of this section assume $\mathscr{G}$ is connected, $\beta \in \Delta$, and $D = G_{\alpha\beta}$.

(6.3) *If $A \leq D$ with $A^G \cap D \subseteq D_{\Delta(\beta)}$ then $A = 1$.*

Proof. Let $\gamma \in \Delta(\beta)$. Then there exists $g \in G$ with $(\alpha, \beta)^g = (\beta, \gamma)$. $A \leq D^g$ so by hypothesis $A \leq D^g_{\Delta(\gamma)}$. As $\mathscr{G}$ is connected and G is faithful on Ω we conclude $A = 1$.

Assume now that $\Gamma = \Gamma(\alpha, \beta)$ is a nontrivial orbit of D on $\Delta(\beta)$. Let Ω_s be the set of sequences $\alpha_0 \alpha_1 \cdots \alpha_s$ with $\alpha_0 \in \Omega$, $\alpha_1 \in \Delta(\alpha_0)$, and for $i > 1$,

$$\alpha_i \in \Gamma(\alpha_{i-2}, \alpha_{i-1}).$$

Ω_s is the set of *s-arcs*. A *subarc* of $\bar{\alpha} = \alpha_0\alpha_1 \cdots \alpha_s \in \Omega_s$ is a t-arc $\alpha_i\alpha_{i+1} \cdots \alpha_{i+t}$. A *successor* or *predeccessor* of $\bar{\alpha}$ is an s-arc

$$\alpha_1\alpha_2 \cdots \alpha_s\alpha_{s+1} \quad \text{or} \quad \alpha_{-1}\alpha_0 \cdots \alpha_{s-1},$$

respectively. Define the graph $\mathscr{G}_s$ with vertex set Ω_s and edges $(\bar{\alpha}, \bar{\beta})$ where $\bar{\beta}$ is a successor of $\bar{\alpha}$. Then $\mathscr{G} = \mathscr{G}_0$ and G acts on Ω_s.

(6.4) *Let $\bar{\alpha}$ and $\bar{\beta}$ be s-arcs with a common* 1*-subarc. Then $\bar{\alpha}$ and $\bar{\beta}$ are in the same connected component of $\mathscr{G}_s$.*

Proof. See 5.9 in [5].

(6.5) *Assume $\mathscr{G}_1$ is connected. Then $\mathscr{G}_s$ is connected for all $s \geq 0$.*

Proof. Assume $\mathscr{G}_s$ is not connected. Then $s > 1$. Let Σ be a connected component of $\mathscr{G}_s$ and θ the collection of 1-arcs which are subarcs of some s-arc in Σ. Claim $\theta = \Omega_1$. For if not then as $\mathscr{G}_1$ is connected there exists $\alpha\beta \in \theta$, and $\beta\gamma \in \Omega_1 - \theta$ with $\gamma \in \Gamma$. Let $\bar{\alpha} = \alpha_0\alpha_1 \cdots \alpha\beta \in \Sigma$. Then $\bar{\beta} = \alpha_1 \cdots \alpha\beta\gamma$ is a successor of $\bar{\alpha}$ and hence in Σ, a contradiction.

So $\theta = \Omega_1$. But now the lemma follows from 6.4.

(6.6) *Assume $\gamma\alpha\beta \in \Omega_2$ and $H = \langle D, G_{\gamma\alpha}\rangle$. Then $\mathscr{G}_s$ is connected for all $s \geq 0$.*

Proof. By 6.5 it suffices to show $\mathscr{G}_1$ is connected. Let Σ be the connected component of $\mathscr{G}_1$ containing $\alpha\beta$. Then $H = \langle D, G_{\gamma\alpha}\rangle \leq N(\Sigma)$. If $\Sigma \neq \Omega_1$ then as $\mathscr{G}$ is connected we may assume $\alpha\delta \in \Omega_1 - \Sigma$. But as $H \leq N(\Sigma)$ and $\alpha\delta$ is conjugate to $\alpha\beta$ under H, this is impossible.

(6.7) *Let $\bar{\alpha} = \alpha_0\alpha_1 \cdots \alpha_s \in \Omega_s$ and K the stabilizer in G of $\bar{\alpha}$. Assume G is transitive on Ω_s and $\mathscr{G}_i$ is connected for all $i \geq 0$. Let $A \leq K$ with $A^G \cap K \subseteq \Gamma(\alpha_{s-1}, \alpha_s)$. Then $A = 1$.*

Proof. As G is transitive on Ω_s this follows from 6.3 applied to the action of G on Ω_{s-1} with respect to the orbit of the stabilizer of $\alpha_0\alpha_1 \cdots \alpha_{s-1}$ on its successors.

7. Theorem 1

In this section we take G to be a counter example to Theorem 1. Most of the ideas in this section are due to Glauberman and Sims.

(7.1) (1) *If $1 \neq B \trianglelefteq H$ then $N_G(B) \leq M$.*
(2) *If $1 \neq B$ is characteristic in T then B is not normal in H.*

Proof. $\{M\} = \mathscr{M}(H)$ while $N(T) \nleq M$.

Set $V = \Omega_1(Z(O_2(H)))$. Let $L = H$ if $e = 1$ and $L = O^2(H)O_2(H)$ otherwise. Let $x \in N(T) - M$ and $X = \langle x, H\rangle$.

(7.2) $O_2(X) = 1$.

Proof. $\{M\} = \mathscr{M}(H)$.

Represent X on the collection Ω of cosets of H in X. By 7.2 this representation is faithful. Let $\alpha = H$, $\beta = Hx$, and $\Delta = \beta^H$. Adopt the notation of Section 6.

$T \leq D = G_{\alpha\beta} < H^x$, so as $N = N_G(L^x \cap T) \cap H^x$ is the unique maximal subgroup of H^x containing T, $D \leq N$. $T \leq D$, so N is transitive on its subgroups isomorphic to D. Hence:

(7.3) *D is the stabilizer in X of β and some point $\alpha' \in \Delta(\beta)$.*

Next $L^x \cap T$ has a complement C in D and C normalizes a second Sylow 2-subgroup $(L^x \cap T)^y$ of L^x for some $y \in N(C)$. Set $\Gamma = \Gamma(\alpha, \beta) = ((\alpha')^y)^D$ and define $\mathscr{G}_i$ with respect to Γ. Let $q = 2^e$. Notice:

(7.4) $|\Gamma| = q$.

$H^x = \langle D, D^y \rangle$ so by 6.6:

(7.5) *$\mathscr{G}_i$ is connected for each i.*

(7.6) *If $g \in X$ with $\alpha^g = \alpha^x$ then $X = \langle H, g \rangle$.*

Proof. Set $Y = \langle H, g \rangle$ and $\Sigma = \alpha^Y$. $\Delta = (\alpha^g)^H \subseteq \Sigma$ so as Y is transitive on Σ, Σ is the union of connected components of $\mathscr{G}$. Therefore by 7.5, $\Omega = \Sigma$. That is Y is transitive on Ω. Thus $X = YH = Y$.

Define

$$s = \max \{i\colon X \text{ is transitive on } \Omega_i\}$$

Let $\alpha_{-1}\alpha_0\alpha_1 \cdots \alpha_s \in \Omega_{s+1}$ with $\alpha = \alpha_0$ and $\beta = \alpha_1$. By definition of s there exists $g \in X$ with $\alpha_i^g = \alpha_{i-1}$, $0 \leq i \leq s$. Define $\alpha_{s+j} = \alpha_s^{g^{-j}}$ for each integer j. As

$$\alpha_s \in \Gamma(\alpha_{s-2}, \alpha_{s-1}), \quad \alpha_{s+i} \in \Gamma(\alpha_{s+i-2}, \alpha_{s+i-1}),$$

so $\alpha_j\alpha_{j+1} \cdots \alpha_k$ is a $k-j$ arc for each $k \geq j$. Define H_j to be the stabilizer in X of α_j, $D_j = H_{j-1} \cap H_j$, $K_j = D_j \cap H_{j+1}$, $V_j = \Omega_1(Z(O_2(H_j)))$ and $L_j = L^{g^{-j}}$. For $j \geq 0$ define $G_j = H_0 \cap \cdots \cap H_j$. Set $K = K_0$. Define

$$v = \max \{i\colon m(G_i) = m(T)\}$$

if this maximum exists and set $v = \infty$ otherwise.

(7.7) (1) *$\mathscr{A}(H) \subseteq L$ but $\mathscr{A}(H) \nsubseteq O_2(H)$.*
(2) *$\mathscr{A}(K) = \mathscr{A}(O_2(H))$.*
(3) *Let $A \in \mathscr{A}(D) - \mathscr{A}(K)$. Then $(A \cap K)V \in \mathscr{A}(K) \subseteq \mathscr{A}(H)$ and $D = AK$.*
(4) *$V/C_V(L)$ is the natural module for $L_2(q)$.*

Proof. By 7.1 neither $\Omega_1(Z(T))$ nor $J(T)$ is normal in H, so as $\Omega_1(Z(T)) \leq V$, $[L, V] \neq 1$ and $\mathscr{A}(H) \nsubseteq O_2(H)$. Therefore by 2.4 and Theorem

3, $V/C_V(L)$ is the natural module for $L_2(q)$ and if $A \in \mathscr{A}(H) - \mathscr{A}(O_2(H))$ then $AO_2(H) = T \cap L$. This yields (1) and (4).

Next as K fixes α_{-1}, α, and β, $K/O_2(H)$ is a complement of $(T \cap L)/O_2(H)$ in $D/O_2(H)$. Hence (2) is a consequence of (1) and (3). Finally

$$(A \cap K)V = (A \cap O_2(H))V \in \mathscr{A}(H)$$

as $L/O_2(L)$ acts naturally on $V/C_V(L)$. So $\mathscr{A}(K) \subseteq \mathscr{A}(H)$ and as $AO_2(H) = T \cap L$; $D = AK$.

(7.8) *Let* $1 \leq i \leq v$. *Then*

(1) $\mathscr{A}(G_i) - \mathscr{A}(G_{i+1})$ *is nonempty.*
(2) *Let* $A \in \mathscr{A}(G_i) - \mathscr{A}(G_{i+1})$. *Then* $G_i = AG_{i+1}$.
(3) G *is transitive on* Ω_{i+1}.

Proof. Let i be a minimal counter example. As $i \leq v$, $\mathscr{A}(G_i) \subseteq \mathscr{A}(D)$. If $\mathscr{A}(G_i) \subseteq \mathscr{A}(G_{i+1})$ then by 7.7.1, $\mathscr{A}(G_i) \subseteq G_{i+1} \cap L_i \subseteq O_2(H_i)$, and hence fixes $\Gamma(\alpha_{i-1}, \alpha_i)$ pointwise. But by minimality of i, G is transitive on Ω_i, so 6.7 yields a contradiction. So let $A \in \mathscr{A}(G_i) - \mathscr{A}(G_{i+1})$. Then $A \in \mathscr{A}(D_i) - \mathscr{A}(K_i)$, so $D_i = AK_i$ by 7.7.3. Thus

$$G_i = D_i \cap G_i = AK_i \cap G_i = A(K_i \cap G_i) = AG_{i+1}.$$

G is transitive on Ω_i and as $D_i = AK_i$, A is transitive on $\Gamma(\alpha_{i-1}, \alpha_i)$. So G is transitive on Ω_{i+1}.

(7.9) $v < s$.

Proof. 7.8.3.

(7.10) $G_{i+1}^g = K \cap G$ *for* $i \geq 0$.

(7.11) *Let* $1 \leq i \leq v$ *and* $Y = Y^g \trianglelefteq G_{i+1}$. *Then:*

(1) $\mathscr{A}(K \cap G_i) \subseteq \mathscr{A}(G_i)$.
(2) $\mathscr{A}(K \cap G_i) \nsubseteq \mathscr{A}(G_{i+1})$.
(3) $Y = 1$.

Proof. As $i < v$, $\mathscr{A}(G_{i+1}) \subseteq \mathscr{A}(G_i)$. Thus part (1) is a consequence of 7.10. Suppose $\mathscr{A}(K \cap G_i) = \mathscr{A} \subseteq \mathscr{A}(G_{i+1})$. Then by 7.10, $\mathscr{A} = \mathscr{A}(G_{i+1})$ and hence $\mathscr{A} = \mathscr{A}^g$. But $\mathscr{A} \subseteq K$ so $\mathscr{A} = \mathscr{A}(K \cap G_{i+1})$ and then $\mathscr{A}^g = \mathscr{A} = \mathscr{A}(G_{i+2})$, contradicting 7.8.1. Finally $Y = Y^g$ is normal in G_{i+1} and $K \cap G_i$ by 7.10. By (2) and 7.8, $G_i = (K \cap G_i)G_{i+1}$, so $Y \trianglelefteq G_i$. Now by induction, $Y \trianglelefteq G_1$. Then $Y = Y^g \trianglelefteq \langle G_1, g\rangle = \langle H, g\rangle = G$, so as G is faithful on Ω, $Y = 1$.

(7.12) *Let* $1 \leq i < v$. *Then* $V \leq \tilde{Z}(K \cap G_i)$ *and for* $A \in \mathscr{A}(G_i) -$

$\mathscr{A}(K \cap G_i)$,

$$(A \cap K)V \in \mathscr{A}(K \cap G_i) \quad \textit{and} \quad A(K \cap G_i) = G_i.$$

Proof. By 7.11, $\mathscr{A}(K \cap G_i) \subseteq \mathscr{A}(G_i)$. By 7.7, $V \leq \tilde{Z}(K)$, so $V \leq \tilde{Z}(K \cap G_i)$. By 7.8 and 7.10 there exists $A \in \mathscr{A}(G_i) - \mathscr{A}(K \cap G_i)$. By 7.7,

$$D = AK \quad \text{and} \quad V(A \cap K) \in \mathscr{A}(K).$$

$V(A \cap K) \leq K \cap G_i$ so $V(A \cap K) \in \mathscr{A}(K \cap G_i)$. Also $G_i = G_i \cap D = G_i \cap AK = A(G_i \cap K)$.

(7.13) *$V_{v-1} \nleq K$ and $G_{v-1} = V_{v-1}(K \cap G_{v-1})$ with $V_{v-1} \leq A \in \mathscr{A}(G_{v-1})$.*

Proof. Set $P = G_{v-1}$, $Q = K \cap P$, $R = G_v$, and $U = V_{v-1}$. By 7.11 there exists $A \in \mathscr{A}(Q) - \mathscr{A}(R)$. Set $B = (A \cap R)U$. By 7.7, $B \in \mathscr{A}(R)$. By definition of v, $m(G_{v+1}) < m(B)$, so by 7.12, $P = BQ = U(A \cap R\ Q = UQ$. So $U \nleq Q$ and hence $U \nleq K$.

(7.14) (1) *$\mathscr{A}(G_v) = \{A\}$ and $\mathscr{A}(G_{v-1}) = \{A, A^g\}$ with $V \leq A^g$.*
(2) *If $v = 2$ then $O_2(H) = V$, $\mathscr{A}(T) = \{V, V^x\}$ and $J(T) \in \mathrm{Syl}_2\ (L)$.*

Proof. Let $T \cap G_{v-1} \leq S \in \mathrm{Syl}_2\ (G_{v-1})$ and $U = V_{v-1}$. By 7.13 and 7.7, $S \cap L = UC_S(V)$. Let $s \in S \cap L$. Then $s = ut$, $u \in U$, $t \in C_S(V)$. By symmetry $S \cap L_{v-1} = VC_S(U)$, so t centralizes $(S \cap L_{v-1})/C_S(U)$. Hence $t \in L_{v-1}$. Of course $u \in U \leq L_{v-1}$, so $S \cap L = S \cap L_{v-1}$.

Let $P = UV$, $Q = S \cap L$. Then $Q = UC_Q(V) = VC_Q(U)$, so

$$C_Q(V) = C_Q(V) \cap VC_Q(U) = VC_Q(P).$$

Hence $Q = PC_Q(P)$. Now $\Phi(C_Q(V)) = \Phi(C_Q(P)) = \Phi(C_Q(U))$.

Suppose $v = 2$. Then $C_Q(V) = O_2(H)$ and $C_Q(U) = O_2(H^x)$. Hence $\Phi(C_Q(P)) \trianglelefteq \langle H, x\rangle = G$, and therefore $\Phi(C_Q(P)) = 1$. Thus $O_2(H) = C_Q(V) = VC_Q(P)$ is elementary abelian, so $O_2(H) = V$. Also $Q \in \mathrm{Syl}_2\ (L)$ and $Q = UV$ with $\mathscr{A}(Q) = \{U, V\}$. Then by 7.7, $Q = J(T)$. The proof of (2) is complete.

Let $Y = J(G_{v-1})$. By 7.13, $U \leq A \in \mathscr{A}(Y)$ and by symmetry between H and H_{v-1}, $V \leq Y$. By 7.7, $Y \leq Q$. Hence $Y = PC_Y(P)$. $J(G_v) \leq Y$, so $J(G_v) = UJ(C_Y(P))$. Thus $\Phi(J(G_v)) = \Phi(J(C_Y(P)))$. $G_v^g = K \cap G_{v-1}$, so a similar argument shows $\Phi(J(G_v^g)) = \Phi(J(C_Y(P)))$. Now by 7.11.3, $J(C_Y(P))$ is elementary abelian. Thus $J(G_v) = A$, $J(K \cap G_{v-1}) = B$. Also if $C \in \mathscr{A}(Y)$ then $C \leq UC_Y(P) \leq G_v$ or $VC_Y(P) \leq K \cap G_{v-1}$, so $\mathscr{A}(Y) = \{A, B\}$. The proof is complete.

(7.15) *Assume $v > 2$. Then $A \trianglelefteq G_{v-2}$.*

Proof. $A = O_2(L_{v-1}) \cap \mathscr{A}(G_{v-1})$ is normalized by G_{v-1}. Hence as $\mathscr{A}(G_{v-1}) = \{A, A^g\}$, G_{v-1} normalizes A^g. As $v > 2$, $G_{v-2} = G_{v-1}G_{v-1}^g$ by 7.8.2.

Hence $A^g \trianglelefteq G_{v-2}$. Also

$$\mathscr{A}(G_{v-1}) = \mathscr{A}(G_{v-2}) \cap O_2(L_{v-2})$$

is G_{v-2} invariant so as $\mathscr{A}(G_{v-1}) = \{A, A^g\}$, $A \trianglelefteq G_{v-2}$.

Set $Z = \Omega_1(Z(T))$ and $W = C_V(H)$. As $H = \langle T, T^g \rangle = \langle V_{v-1}, T^g \rangle$ by 7.13, we conclude:

(7.16) $Z^g \leq V$ *and* $Z^g \cap C(V_{v-1}) = W$.

(7.17) $O_2(H) = V$, $\mathscr{A}(T) = \{V, V^x\}$ *and* $J(T) \in \mathrm{Syl}_2\ (L)$.

Proof. By 7.14 we may take $v > 2$. Hence $[Z^g, V_{v-1}] \leq V_{v-1} \leq C(V_v)$. So

$$W_0 = [Z^g, V_{v-1}] \cap Z \leq C(V_v) \cap Z = W^{g^{-1}}$$

by 7.16. $T = C_T(V)Y$ where $Y = T \cap T^g$. Let $U \leq V_{v-1}$ with $UC_T(V)/C_T(V) = Z(T/C_T(V))$. Then

$$|Z\colon W| = |UC_T(V)\colon C_T(V)| \leq |[Z^g, u]|.$$

Also $[Y, Z^g] = 1$ and $[U, Y] \leq C_T(V) \leq C(Z^g)$, so by the 3-subgroup lemma, $[Z^g, U, Y] = 1$. Thus $[Z^g, U] \leq W_0$, so $|W_0| \geq |Z\colon W|$. But $\langle H, H^x \rangle$ centralizes $W \cap W^x$, so as $\{M\} = \mathscr{M}(H)$, $W \cap W^x = 1$. Hence $|Z\colon W| \geq |W| \geq |W_0| \geq |Z\colon W|$. We conclude $W^{g^{-1}} = W^x = W_0$ is of order $|Z\colon W|$.

Assume $v \geq 4$. Then $W_0 \leq V_{v-1} \leq C(V_{v+1})$. But $Z^{g^{-1}} \cap C(V_{v+1}) = W_0^{g^{-1}}$, whereas $W_0^{g^{-1}} \neq W_0 \leq Z^{g^{-1}}$, a contradiction.

Hence $v = 3$. Then by 7.14, $\mathscr{A}(G_2) = \{A, A^g\}$. But $\mathscr{A}(G_2) = \mathscr{A}(O_2(H_1))$ and $A \trianglelefteq G_1$ by 7.15. Therefore, $A \trianglelefteq O^2(H_1)G_1 = H_1$. Thus A^g is also normal in H_1. But now $A^g \trianglelefteq \langle H_1, H_1^g \rangle = \langle H, H^x \rangle$ a contradiction.

(7.18) $q = 2$.

Proof. If $q > 2$ apply Theorem 3 to L, L^x, using 7.17. We conclude $F^*(G) \cong L_3(q)$ or $Sp_4(q)$. Now we may choose x to induce an involutory outer automorphism on $F^*(G)$. But then $F^*(C_G(x)) \neq O_2(C(x))$, a contradiction.

(7.19) $q > 2$.

Proof. Assume $q = 2$. Then $Z(H)$ is a hyperplane of $Z(T)$ such that $\mathscr{M}(C(v)) \subseteq \mathscr{M}(H) = \{M\}$ for each $v \in Z(H)^\#$. Therefore $Z(H) \cap Z(H)^x = 1$, so $|Z(T)| \leq 4$. As $Z(T)$ is a hyperplane of V, $|V| \leq 8$. Of course $C_G(V) = C_M(V) = V$. Hence by Theorem 2 in [4], G has sectional 2-rank at most 4.

Notice 7.18 and 7.19 complete the proof of Theorem 1.

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CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA