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REAL ZEROES OF EPSTEIN'S ZETA FUNCTION FOR TERNARY POSITIVE DEFINITE QUADRATIC FORMS

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1. Introduction

Let S be the $n\times n$ matrix of a positive definite (real) quadratic form, and let ρ be a complex variable with $\operatorname{Re}\rho > n/2$. Then Epsteins zeta function $Z_n(S,\rho)$ is defined by

$$Z_n(S,\rho)=\tfrac{1}{2}\sum_{a\in\mathbf{Z}^n-0} S[a]^{-\rho}, \tag{1.1}$$

where the sum is over all column vectors with integral entries, not all of which are zero. Here $S[A]={}^tASA$, with ${}^tA$ being the transpose of the matrix A. We shall be concerned with the sign of $Z_3(S, 1)$. The value is, of course, not obtained by substituting $\rho=1$ in formula (1.1), but by analytic continuation using theta functions or using the Selberg-Chowla formula generalizations in [7, p. 480].

Let us first recall the analogous result obtained for binary forms and $Z_2(S,\frac{1}{2})$ by Bateman and Grosswald [2, p. 367]. We need to recall also what Siegel [6, p. 29] calls the Jacobi transformation of a positive definite symmetric matrix S:

$$S=\begin{pmatrix} s_{11} & s_{12}\\ s_{12} & s_{22}\end{pmatrix}=\begin{pmatrix} u & 0\\ 0 & s_{22}\end{pmatrix}\begin{bmatrix} 1 & 0\\ r & 1\end{bmatrix}, \tag{1.2}$$

where $r=s_{22}^{-1}s_{12}$ and $u=s_{11}-s_{22}^{-1}s_{12}^2$. This is releated to the Iwasawa decomposition of $SL(2,\mathbf{R})$. The reduction theory of binary positive definite quadratic forms [6, p. 70] shows that for every matrix T of a binary positive definite quadratic form, there exists a matrix U which has integer entries and determinant ±1, such that $S=T[U]$ has Jacobi transformation as in (1.2) with

$$s_{22}\le\tfrac{4}{3}u,\quad |r|\le\tfrac{1}{2}. \tag{1.3}$$

Since $Z_2(S,\rho)=Z_2(T,\rho)$, if S and T are related as above by an integer matrix of determinant ±1, it follows that we may assume inequalities (1.3) in discussing the sign of $Z_2(S,\rho)$.

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Bateman and Grosswald [2, p. 367] prove that

$$
\begin{aligned}
Z_2(S, \tfrac{1}{2}) &> 0 \quad \text{if } \sqrt{\frac{u}{s_{22}}} \geq 7.0556, \\
Z_2(S, \tfrac{1}{2}) &< 0 \quad \text{if } \frac{\sqrt{3}}{2} \leq \sqrt{\frac{u}{s_{22}}} \leq 7.0554.
\end{aligned} \tag{1.4}
$$

The proof is carried out via the Selberg–Chowla formula [2, p. 366, Theorem 1], which is essentially the calculation of the Fourier coefficients of the non-analytic automorphic form $Z_2(S, \rho)$—the result being a rapidly convergent expansion involving Bessel functions.

Since $Z_2(S, \rho)$ has a simple pole at $\rho = 1$ with a positive residue, it follows that if S is such that $\sqrt{u/s_{22}} \geq 7.0556$ then $Z_2(S, \rho)$ must vanish for some ρ in the interval (1/2, 1). It is perhaps surprising, given the well known relation between the Epstein and Dedekind zeta functions of imaginary quadratic fields, that this discovery of zeroes of Epstein's zeta function does not lead to zeroes of the Dedekind zeta function for imaginary quadratic fields. In fact, Low [3] and Purdy [4] have used similar methods and a computer to show that the Dedekind zeta function of any imaginary quadratic field with discriminant between -3 and -800000 is negative throughout the interval (0, 1)—with three discriminants for which the Dedekind zeta function is very close to zero.

Our aim here is to prove a result similar to (1.4) for $Z_3(S, 1)$. Write the Jacobi transformation of S as

$$
S = \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{12} & s_{22} & s_{23} \\ s_{13} & s_{23} & s_{33} \end{pmatrix} = \begin{pmatrix} t & 0 & 0 \\ 0 & w & 0 \\ 0 & 0 & s_{33} \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 \\ r_1 & 1 & 0 \\ r_2 & r_3 & 1 \end{bmatrix}. \tag{1.5}
$$

Then the three dimensional case of Raghunathan's theorem [5, p. 160] says that for any matrix T of a ternary positive definite form there exists a matrix U with integer entries and determinant ± 1 such that $S = T[U]$ has Jacobi transformation as in (1.5) with

$$
s_{33} \leq \tfrac{4}{3} w \leq \tfrac{16}{9} t \quad \text{and} \quad r_1, r_2, r_3 \in [-\tfrac{1}{2}, \tfrac{1}{2}]. \tag{1.6}
$$

Raghunathan's assumption that the determinant of S be 1 is easily removed. There are many versions of the reduction theory of positive definite quadratic forms, but Raghunathan's seems the most convenient for our purposes. Again, it is no loss of generality to assume (1.6) since $Z_3(S, \rho) = Z_3(T, \rho)$ if S and T are related as above by an integer matrix of determinant ± 1.

Our main result is:

$$
\begin{aligned}
Z_3(S, 1) &> 0 \quad \text{if } \frac{\pi}{6} \sqrt{\frac{w}{s_{33}}} + \log \sqrt{\frac{t}{w}} \geq 1.9633 \text{ and (1.6) holds,} \\
Z_3(S, 1) &< 0 \quad \text{if } \frac{\pi}{6} \sqrt{\frac{w}{s_{33}}} + \log \sqrt{\frac{t}{w}} \leq 1.9443 \text{ and (1.6) holds.}
\end{aligned} \tag{1.7}
$$

The cut-off band here has a width of .019 which does not compare well with the width of .002 in (1.4). It does not seem to be easy to narrow the band appreciably in (1.7). Perhaps a computer could improve the situation. We used only an *HP*-45 in the following calculations. The ALGOL programs required for such calculations have been developed by R. Terras, cf. [11].

The proof will be carried out via the generalization of the Selberg-Chowla formula to be found in [7, p. 480]. Section 2 contains the preliminary formulas and estimates. The strip where the sign of $Z_3(S, \rho)$ is undetermined is first established roughly (in Section 3) and then narrowed by use of better estimates over the rough strip (in Section 4).

Since $Z_3(S, \rho)$ has a simple pole at $\rho = 3/2$ with positive residue, it follows from (1.7) that if S is such that

$$\frac{\pi}{6}\sqrt{\frac{w}{s_{33}}} + \log\sqrt{\frac{t}{w}} > 1.9633,$$

then $Z_3(S, \rho)$ has a zero in the interval $(1, 3/2)$. Consequences for the Dedekind zeta function are yet to be considered.

One might ask whether similar results hold for n-ary quadratic forms. The answer to this question is "yes". For example, we show in [10, Theorem 1] that given ρ in the interval $(0, n/2)$, there exist positive n-ary quadratic forms S such that $Z_n(S, \rho) > 0$. There are also S with $Z_n(S, \rho) < 0$ or $= 0$. The results of [8] and [9] are more precise. They involve m_S defined by $m_S = \min\{S[a] \mid a \in \mathbf{Z}^n - 0\}$. For example, if $\det S = 1$, $\rho \in (0, n/2)$, and both

$$m_S < \frac{1}{\pi}\left(\frac{\Gamma(\rho+1)}{2}\right)^{1/\rho} \quad \text{and} \quad m_{S^{-1}} < \frac{1}{\pi}\left(\frac{\Gamma(n/2-\rho+1)}{2}\right)^{1/(n/2-\rho)},$$

then $Z_n(S, \rho) > 0$. Such results are still weaker then (1.7) for $n = 3$. But it does not appear to be easy to generalize the proof of (1.7) to arbitrary n. Again a computer might help for small n.

2. The Bessel series expansion of Epstein's zeta function

Let us first recall some notation from [7, p. 479]. If S is a 3×3 positive definite symmetric matrix, we write the block "Jacobi transformation"

$$S = \begin{pmatrix} s_{11} & S_{12} \\ {}^tS_{12} & S_2 \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & S_2 \end{pmatrix}\begin{bmatrix} 1 & 0 \\ Q & I \end{bmatrix} \tag{2.1}$$

Here

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

S_2 is a 2×2 positive definite symmetric matrix,

$$Q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = S_2^{-1}\,{}^tS_{12}, \quad t = s_{11} - S_2^{-1}[{}^tS_{12}].$$

And we write the Jacobi transformation of S_2 as

$$S_2 = \begin{pmatrix} s_{22} & s_{23} \\ s_{23} & s_{33} \end{pmatrix} = \begin{pmatrix} w & 0 \\ 0 & s_{33} \end{pmatrix} \begin{bmatrix} 1 & 0 \\ p & 1 \end{bmatrix}. \tag{2.2}$$

with $p = s_{33}^{-1} s_{23}$ and $w = s_{22} - s_{33}^{-1} s_{23}^2$. Finally, the Jacobi transformation of S is

$$S = \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{12} & s_{22} & s_{23} \\ s_{13} & s_{23} & s_{33} \end{pmatrix} = \begin{pmatrix} t & 0 & 0 \\ 0 & w & 0 \\ 0 & 0 & s_{33} \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 \\ q_1 & 1 & 0 \\ pq_1 + q_2 & p & 1 \end{bmatrix}. \tag{2.3}$$

It follows from (1.6) that, upon replacing S by $S[U]$, with a suitable integer matrix U of determinant ± 1, we may assume

$$s_{33} \le 4w/3 \le 16t/9 \quad \text{and} \quad p,\ pq_1 + q_2,\ q_1 \in [-\tfrac{1}{2}, \tfrac{1}{2}], \tag{2.4}$$

without changing the value of $Z_3(S, 1)$.

We can now recall the necessary results from [7]. Let $\zeta(s)$ be the Riemann zeta function, $\Gamma(s)$ be the gamma function, $K_0(s)$ be the modified Bessel function of the second kind. Set

$$\Lambda_n(S, \rho) = \pi^{-\rho} \Gamma(\rho) Z_n(S, \rho). \tag{2.5}$$

Then formulas (2.2) and (2.3) of [7, pages 479–480] yield

$$\Lambda_3(S, \rho) = \Lambda_2(S_2, \rho) + |S_2|^{-1/2} \Lambda_1(t, \rho - 1) + H_{1,2}(S, \rho), \tag{2.6}$$

where $|S_2|$ denotes the determinant of S_2 and

$$H_{1,2}(S, \rho) = |S_2^{-1/2}| \sum_{\substack{a \in \mathbf{Z}^2 - 0 \\ (b,c) \in \mathbf{Z}^2 - 0}} \exp[2\pi i (bq_1 + cq_2) a] \times \left(\frac{ta^2}{S_2^{-1}\begin{bmatrix} b \\ c \end{bmatrix}} \right)^{(1-\rho)/2} K_{1-\rho}\left(2\pi \sqrt{t}\, |a| \sqrt{S_2^{-1}\begin{bmatrix} b \\ c \end{bmatrix}} \right).$$

It is easy to see that the right-hand side of (2.6) has a removable singularity at $\rho = 1$ and a simple pole at $\rho = 3/2$. This is analogous to the situation for $Z_2(T, \rho)$ at $\rho = 1/2$ and $\rho = 1$ [2, p. 366].

The next step is the evaluation of the sum of the first two terms on the right-hand side of (2.6) at $\rho = 1$. Suppose that $f(\rho)$ is a meromorphic function with a pole at $\rho = 1$. Define $k(f(\rho))$ to be the constant term in the Laurent expansion of $f(\rho)$ about $\rho = 1$. Then by (2.6) we have

$$\Lambda_3(S, 1) = \pi^{-1} Z_3(S, 1) = k(\Lambda_2(S_2, \rho)) + |S_2|^{-1/2} k(\Lambda_1(t, \rho - 1)) + H_{1,2}(S, 1). \tag{2.8}$$

And using [7, pages 479–480], one obtains

$$\Lambda_2(S_2, \rho) = \Lambda_1(s_{33}, \rho) + s_{33}^{-1/2} \Lambda_1(w, \rho - 1/2) + H_{1,1}(S_2, \rho).$$

Thus we have

$$k(\Lambda_2(S_2,\rho)) = (\pi s_{33}^{-1})\frac{\pi^2}{6} + s_{33}^{-1/2} k(\Lambda_1(w,\rho-1/2)) + H_{1,1}(S_2,\rho),$$

where

$$H_{1,1}(S_2,\rho) = s_{33}^{-1/2} \sum_{\substack{a\in\mathbf{Z}-0\\ b\in\mathbf{Z}-0}} (ws_{33})^{1/4-\rho/2} \times (ab^{-1})^{1/2-\rho} \exp(2\pi i pab) K_{1/2-\rho}\left(2\pi\sqrt{\frac{w}{s_{33}}}|ab|\right).$$

Using facts about $\zeta(s)$ and $\Gamma(s)$ to be found in Abramowitz and Stegun [1, pages 258 and 807], one sees that since $\Lambda_1(w,\rho) = (\pi w)^{-\rho}\Gamma(\rho)\zeta(2\rho)$, we have

$$k(\Lambda_1(w,\rho-1/2)) = w^{-1/2}(\gamma/2 - \log(2\sqrt{\pi w})),$$

where γ is Euler's constant. Therefore

$$k(\Lambda_2(S_2,\rho)) = s_{33}^{-1}\pi/6 + |S_2|^{-1/2}[\gamma/2 - \log(2\sqrt{\pi w})] + H_{1,1}(S_2, 1). \quad (2.9)$$

Again facts about zeta and gamma on the above-mentioned pages of [1] show that

$$k(\Lambda_1(t,\rho-1) = \gamma/2 + \log(\sqrt{t}/2\pi). \quad (2.10)$$

Finally combining the last six formulas yields

$$\pi^{-1}Z_3(S,1) = |S_2|^{-1/2}\left\{\gamma - \log 4\pi + \frac{\pi}{6}\sqrt{\frac{w}{s_{33}}} + \log\sqrt{\frac{t}{w}}\right\} + H_{1,1}(S_2,1) + H_{1,2}(S,1). \quad (2.11)$$

We have also used the fact that $|S_2| = ws_{33}$.

Our goal is a result on the sign of $Z_3(S,1)$ and the main tool used will be (2.11). One has

$$\gamma - \log 4\pi \cong -1.953808582. \quad (2.12)$$

It remains to estimate $|S_2|^{1/2} H_{1,1}(S_2,1)$ and $|S_2|^{1/2} H_{1,2}(S,1)$. An estimate for the former can be derived as in [2, p. 367, Theorem 3] or [3, p. 130, Theorem 4]. We include a slightly different proof in this section for completeness and as an introduction to the methods which will be used in the next section to estimate $|S_2|^{1/2} H_{1,2}(S,1)$. The estimates we use are somewhat easier to obtain, involving only the Bessel function $K_{1/2}(z)$, which is essentially an exponential function, rather than the Bessel function $K_0(z)$.

Proceeding with the estimates for $|S_2|^{1/2} H_{1,1}(S_2,1)$, first note that

[1, p. 444, formula 10.2.17] gives

$$K_{1/2}(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z}. \tag{2.13}$$

Thus, by (2.9),

$$|S_2|^{1/2}\, H_{1,1}(S_2, 1) = 2 \sum_{a,b \geq 1} a^{-1} \cos(2\pi pab) \exp\left(-2\pi \sqrt{\frac{w}{s_{33}}}\, ab\right). \tag{2.14}$$

Then, using (2.4) and the formula for the sum of the geometric series,

$$|S_2|^{1/2}\, |H_{1,1}(S_2, 1)| \leq 2 \sum_{a \geq 1} a^{-1}[\exp(\pi\sqrt{3}a) - 1]^{-1}.$$

Now

$$\begin{aligned} 2 \sum_{a \geq 5} [\exp(\pi\sqrt{3}a) - 1]^{-1} &< 2\{1 + [\exp(5\pi\sqrt{3}) - 1]^{-1}\} \exp(5\pi\sqrt{3}) \\ &\quad \times [1 - \exp(-\pi\sqrt{3})]^{-1} \\ &= 2[\exp(5\pi\sqrt{3}) - 1]^{-1}[1 - \exp(-\pi\sqrt{3})]^{-1}, \end{aligned}$$

which is a negligible quantity. And the sum of the remaining four terms yields the estimate

$$|S_2|^{1/2}\, |H_{1,1}(S_2, 1)| < .0087313. \tag{2.15}$$

Using an idea of Low we can get an even better upper bound for $|S_2|^{1/2} H_{1,1}(S_2, 1)$. One considers two cases separately. The first case is that in which $1/4 \leq |p| \leq 1/2$. This implies $\cos(2\pi p) < 0$.

Therefore

$$|S_2|^{1/2}\, H_{1,1}(S_2, 1) \leq 2 \sum_{\substack{a \geq 2 \\ b \geq 1}} a^{-1} \exp\left(-2\pi \sqrt{\frac{w}{s_{33}}}\, ab\right) + 2 \sum_{b \geq 2} \exp\left(-2\pi \sqrt{\frac{w}{s_{33}}}\, b\right).$$

Methods similar to the proceeding yield

$$|S_2|^{1/2} H_{1,1}(S_2, 1) < .0000755 \quad \text{if} \quad 1/4 \leq |p| \leq 1/2. \tag{2.16}$$

The remaining case concerns p such that $|p| < 1/4$. By (2.2), in this case $w > 15s_{33}/16$. Thus the methods used to obtain (2.15) give the better result

$$|S_2|^{1/2}\, H_{1,1}(S_2, 1) < .0045753. \tag{2.17}$$

The last estimate holds for all p.

One has, in general,

$$|S_2|^{1/2}\,|H_{1,1}(S_2,1)|\le 2\sum_{a,b\ge 1} a^{-1}\exp\left(-2\pi\sqrt{\frac{w}{s_{33}}}\,ab\right)$$

$$=2\sum_{a\ge 1} a^{-1}\left[\exp\left(2\pi\sqrt{\frac{w}{s_{33}}}\,a\right)-1\right]^{-1}$$

$$<2\left\{1+\left[\exp\left(2\pi\sqrt{\frac{w}{s_{33}}}\right)-1\right]^{-1}\right\}\sum_{a\ge 1}\exp\left(-2\pi\sqrt{\frac{w}{s_{33}}}\,a\right),$$

$$|S_2|^{1/2}\,|H_{1,1}(S_2,1)|<2\exp\left(2\pi\sqrt{\frac{w}{s_{33}}}\right)\left[\exp\left(2\pi\sqrt{\frac{w}{s_{33}}}\right)-1\right]^{-2}. \tag{2.18}$$

It follows, for example, that

$$|S_2|^{1/2}\,|H_{1,1}(S_2,1)|<.0000070 \quad \text{if } w/s_{33}\ge 4. \tag{2.19}$$

This last estimate will be of use in Section 4.

3. The estimate for $H_{1,2}(S,1)$ and a weak version of the main theorem

It only remains to estimate $|S_2|^{1/2}\,|H_{1,2}(S,1)|$, in order to use (2.11) to examine the sign of $Z_3(S,1)$. From (2.6) it follows that

$$|S_2|^{1/2}\,|H_{1,2}(S,1)|\le 2\sum_{\substack{a\ge 1\\ (b,c)\ne(0,0)}} K_0\left(2\pi t^{1/2}a\sqrt{S_2^{-1}\begin{bmatrix} b\\ c\end{bmatrix}}\right). \tag{3.1}$$

Now

$$S_2=\begin{pmatrix} w & 0\\ 0 & s_{33}\end{pmatrix}\begin{bmatrix} 1 & 0\\ p & 1\end{bmatrix},$$

so that

$$S_2^{-1}=\begin{pmatrix} w^{-1} & 0\\ 0 & s_{33}^{-1}\end{pmatrix}\begin{bmatrix} 1 & -p\\ 0 & 1\end{bmatrix}.$$

Thus

$$tS_2^{-1}\begin{bmatrix} b\\ c\end{bmatrix}=\frac{t}{w}(b-pc)^2+\frac{t}{s_{33}}c^2. \tag{3.2}$$

And it follows from the integral formula for the modified Bessel function of the second kind [7, p. 478, formula (1.4)] that

$$K_0(z)\le K_{1/2}(z)=\sqrt{\frac{\pi}{2z}}\,e^{-z} \quad \text{for } z \text{ positive.} \tag{3.3}$$

Combining the above three results gives

$$|S_2|^{1/2}\,|H_{1,2}(S,1)|\le \sum_{\substack{a\ge 1\\ (b,c)\ne(0,0)}} a^{-1/2}(f(b,c))^{-1/4}\exp\left(-2\pi a\sqrt{f(b,c)}\right), \quad (3.4)$$

where

$$f(b,c)=\frac{t}{w}(b-pc)^2+\frac{t}{s_{33}}c^2.$$

Ignoring the $a^{-1/2}$, and using the formula for the sum of the geometric series yields

$$|S_2|^{1/2}\,|H_{1,2}(S,1)|\le \sum_{(b,c)\ne(0,0)} (f(b,c))^{-1/4}\left(\exp\left(2\pi\sqrt{f(b,c)}\right)-1\right)^{-1} \quad (3.5)$$

It might be worthwhile to retain $a^{-1/2}$ in the estimate by using the actual first few terms in the sum over a. However comparison of the first few terms of the series (3.4) and our final result Proposition 1 convinces one that the error made is negligible.

To estimate (3.5) we split the sum into four parts:

$$\sum_1=\sum_{\substack{b\ne0,c\ne0\\ |b|+|c|\ge7}},\quad \sum_2=\sum_{\substack{b\ne0,c\ne0\\ |b|+|c|<7}},\quad \sum_3=\sum_{b=0},\quad \sum_4=\sum_{c=0}.$$

To estimate $\sum_1$, first note that the minimum of $f(b,c)$ on the line segment

$$\{(b,c)\in\mathbf{R}^2 \mid b+c=k,\ b>0,\ c>0\}$$

is

$$k^2\left(\frac{w}{t}+\frac{s_{33}}{t}(p+1)^2\right)^{-1},$$

as is easily verified by freshman calculus. Using inequalities (2.4), it follows that the minimum of $f(b,c)$ on the line segment $\{(b,c)\in\mathbf{R}^2 \mid b+c=k,\ b>0,\ c>0\}$ is $\ge 3k^2/16$. The same result holds for $f(b,-c)$ on the same line segment, upon replacing p by $-p$.

It follows from the above considerations that

$$\begin{aligned}\sum_1 &\le 8(3)^{-1/4}\sum_{k\ge7} k^{1/2}\left[\exp\left(\frac{\pi\sqrt{3}}{2}k\right)-1\right]^{-1}\\ &< 8(3)^{-1/4}\left\{1+\left[\exp\left(\frac{7\pi\sqrt{3}}{2}\right)-1\right]^{-1}\right\}\sum_{k\ge7}\exp\left[\left(\frac{2}{7}-\pi\frac{\sqrt{3}}{2}\right)k\right]\end{aligned}$$

since

$$k^{1/2}\le 2k/7+\frac{1}{2}\left(\frac{2k}{7}\right)^2\le \exp(2k/7).$$

Thus

$$\sum_1 < 8e^2 3^{-1/4}\left[\exp\left(\frac{7\pi\sqrt{3}}{2}\right)-1\right]^{-1}\left[1-\exp\left(\frac{2}{7}-\frac{\pi\sqrt{3}}{2}\right)\right]^{-1} < .0000003. \quad (3.6)$$

Next we estimate $\sum_2$. First note that if $b > c/2 > 0$, then $|b-pc| \geq b - c/2$. Thus, by (2.4), if $b > c/2 > 0$,

$$f(b, c) = \frac{t}{w}(b-pc)^2 + \frac{t}{s_{33}}c^2 \geq \frac{3}{4}\left(b-\frac{c}{2}\right)^2 + \frac{9}{16}c^2 = \frac{3}{4}(b^2 - bc + c^2).$$

And if $0 < b \leq c/2$, then $f(b, c) \geq 9c^2/16$. Thus one obtains, upon calculation,

$$\sum_2 < .0192982. \quad (3.7)$$

We turn to the estimation of $\sum_3$. This is much easier:

$$\begin{aligned}\sum_3 &\leq 4(3)^{-1/2}\sum_{c\geq 1} c^{-1/2}[\exp(\tfrac{3}{2}\pi c)-1]^{-1} \\ &< 4(3)^{-1/2}[\exp(9\pi)-1]^{-1}\left[1-\exp\left(\frac{-3\pi}{2}\right)\right]^{-1}\end{aligned}$$

which is negligible. One gets the estimate

$$\sum_3 < .0210669. \quad (3.8)$$

Only the estimation of $\sum_4$ remains. Now

$$\sum_4 \leq 2\sqrt{2}\, 3^{-1/4}\sum_{b\leq 1} b^{-1/2}[\exp(\pi\sqrt{3}\, b)-1]^{-1}.$$

Again the error made by considering only the first five terms is negligible. And we thus obtain the estimate

$$\sum_4 < .0093823. \quad (3.9)$$

Adding the estimates (3.6) through (3.9) yields:

PROPOSITION 1. $|S_2|^{1/2}\,|H_{1,2}(S, 1)| < .0497476.$

This proposition leads to a weak version of the main result (1.7).

PROPOSITION 2.

$$Z_3(S, 1) < 0 \quad \textit{if } S \textit{ satisfies } (2.4) \textit{ and } \frac{\pi}{6}\sqrt{\frac{w}{s_{33}}} + \log\sqrt{\frac{t}{w}} \leq 1.899485.$$

$$Z_3(S, 1) > 0 \quad \textit{if } S \textit{ satisfies } (2.4) \textit{ and } \frac{\pi}{6}\sqrt{\frac{w}{s_{33}}} + \log\sqrt{\frac{t}{w}} \geq 2.012288.$$

Proof. From (2.11), (2.12), (2.17) and Proposition 1, it follows that

$$Z_3(S,1) < \pi\,|S_2|^{-1/2}\left\{-1.899485+\frac{\pi}{6}\sqrt{\frac{w}{s_{33}}}+\log\sqrt{\frac{t}{w}}\right\}.$$

And it follows from (2.11), (2.12), (2.15) and Proposition 1 that

$$Z_3(S,1) > \pi\,|S_2|^{-1/2}\left\{-2.012288+\frac{\pi}{6}\sqrt{\frac{w}{s_{33}}}+\log\sqrt{\frac{t}{w}}\right\}.$$

This concludes the proof.

It is possible to use methods similar to those preceding to obtain a much simpler estimate for $|S_2|^{1/2}\,|H_{1,2}(S,1)|$. Unfortunately this estimate seems to be too far off for our purposes in this section and the next.

PROPOSITION 3.

$$|S_2|^{1/2}\,|H_{1,2}(S,1)| < 4M^{-1/2}[\exp(2\pi M)-1]^{-1}[1-\exp(-2\pi M)]^{-2}$$

$$\text{where } M = t^{1/2}(w+9s_{33}/4)^{-1/2}.$$

Proof. By the methods used to estimate $\sum_1$, we obtain

$$\begin{aligned}|S_2|^{1/2}\,|H_{1,2}(S,1)| &\le 4M^{-1/2}\sum_{k\ge1} k^{1/2}[\exp(2\pi Mk)-1]^{-1}\\ &< 4M^{-1/2}\{1+[\exp(2\pi M)-1]^{-1}\}\sum_{k\ge1} k\exp(-2\pi Mk)\\ &= 4M^{-1/2}[\exp(2\pi M)-1]^{-1}\\ &\quad\times\exp(2\pi M)\exp(-2\pi M)[1-\exp(-2\pi M)]^{-2}\end{aligned}$$

by the formula for the derivative of the geometric series. This proves the proposition.

We make one final comment on the estimates of this section. One might question how big the error is due to replacing K_0 by $K_{1/2}$ via (3.3) Comparing values of the first few terms of series (3.1) and series (3.4) soon convinces one that not too much is lost. However, the combination of this error with that coming from ignoring the $a^{-1/2}$ in (3.4) no doubt adds up.

4. Proof of the main result — formula (1.7)

We wish to improve Proposition 2 of the last section by narrowing the band of indecision between 1.899485 and 2.012288. We expect the cut-off point to be γ-log 4π which is approximately 1.953808582. This was the case for $Z_2(\mathrm{T}, 1/2)$.

Our first result is:

THEOREM 1. *If S satisfies* (2.4) *and*

$$\frac{\pi}{6}\sqrt{\frac{w}{s_{33}}}+\log\sqrt{\frac{t}{w}}\geq 1.9633,$$

then $Z_3(S, 1)>0$.

Proof. We shall consider two cases. The first is that in which S satisfies the hypotheses of the theorem and $w/s_{33}\geq 4$. In this case we have a good estimate (2.19), for $|S_2|^{1/2}|H_{1,1}(S_2, 1)|$. We need a better estimate for $|S_2|^{1/2}|H_{1,2}(S, 1)|$, in the event that $w/s_{33}\geq 4$ and thus $t/s_{33}\geq 3$ (by 2.4)). One proceeds as in Section 3, using (3.5) and breaking up the sum in (3.5) into four parts:

$$\sum_1=\sum_{\substack{b\neq 0, c\neq 0\\ |b|+|c|\geq 6}},\quad \sum_2=\sum_{\substack{b\neq 0, c\neq 0\\ |b|+|c|<6}},\quad \sum_3=\sum_{b=0},\quad \sum_4=\sum_{c=0}.$$

Then, as in the proof of formula (3.6) one obtains

$$\sum_1\leq 4\left(\frac{25}{12}\right)^{1/4}\sum_{k\geq 6}k^{1/2}\left[\exp\left(2\pi k\sqrt{\frac{12}{25}}\right)-1\right]^{-1}$$
$$<4\frac{(25)^{1/4}}{12}\left[\exp\left(12\pi\sqrt{\frac{12}{25}}\right)-1\right]^{-1}e^2\left[1-\exp\left(\frac{1}{3}-2\pi\sqrt{\frac{12}{25}}\right)\right]^{-1},$$

which is negligible.

Next we estimate $\sum_2$. This time, in analogy to the proof of (3.7), if $b>c/2>0$, we have

$$f(b, c)=\frac{t}{w}(b-pc)^2+\frac{tc^2}{s_{33}}\geq\tfrac{3}{4}\left(b-\frac{c}{2}\right)^2+3c^2=\tfrac{3}{4}(b^2-bc+\tfrac{17}{4}c^2).$$

And if $0<b\leq c/2$, then $f(b, c)\geq 3c^2$. Upon calculation, one finds that $\sum_2<.000044$.

The new estimate for $\sum_3$ is obtained similarly to (3.7):

$$\sum_3\leq 2(3)^{-1/4}\sum_{c\geq 1}c^{-1/2}[\exp(2\pi\sqrt{3}c)-1]^{-1}<.0000286.$$

The estimate for $\sum_4$ is still (3.9).

Adding up the estimates for $\sum_1$ through $\sum_4$ yields

$$|S_2|^{1/2}|H_{1,2}(S, 1)|<.0094545\quad\text{if } w/s_{33}\geq 4. \tag{4.1}$$

It follows from (2.11), (2.12), (2.19), and (4.1) that

$$Z_3(S, 1)>\pi|S_2|^{-1/2}\left\{-1.9632701+\frac{\pi}{6}\sqrt{\frac{w}{s_{33}}}+\log\sqrt{\frac{t}{w}}\right\},$$

if $w/s_{33}\geq 4$. This proves the theorem in case 1.

Next we consider the case that S satisfies the hypotheses of the theorem and in addition $w/s_{33}<4$. This implies that $t/w>6.2472725$.

In this case the only estimate we can use for $|S_2|^{1/2}|H_{1,1}(S_2, 1)|$ is (2.15). We get the estimate for $|S_2|^{1/2}|H_{1,2}(S, 1)|$ in the usual way, using (3.5), and breaking up the sum in (3.5) into four parts:

$$\sum_1=\sum_{\substack{b\neq 0, c\neq 0\\ |b|+|c|\geq 5}}, \quad \sum_2=\sum_{\substack{b\neq 0, c\neq 0\\ |b|+|c|<5}}, \quad \sum_3=\sum_{b=0}, \quad \sum_4=\sum_{c=0}.$$

The same reasoning that led to (3.6) gives

$$\sum_1\leq 4\,(1.561818)^{-1/4}\sum_{k\geq 5} k^{1/2}\,[\exp(2\pi k\sqrt{1.561818})-1]^{-1},$$

which is negligible.

To estimate $\sum_2$, note that if $b>c/2>0$, then

$$f(b, c)=\frac{t}{w}(b-pc)^2+\frac{t}{s_{33}}c^2\geq(6.2472725)(b^2-bc+c^2).$$

And if $0<b\leq c/2$, then $f(b, c)\geq\frac{3}{4}(6.2472725)c^2$. It follows that in this case $\sum_2<.0000004$.

One finds that

$$\sum_3\leq 2(\tfrac{3}{4})^{-1/4}(6.2472725)^{-1/4}\sum_{c\geq 1} c^{-1/2}[\exp(\pi\sqrt{18.7418175}c)-1]^{-1}$$
$$<.0000017.$$

Also

$$\sum_4\leq 2(6.2472725)^{-1/4}\sum_{c\geq 1} c^{-1/2}[\exp(2\pi\sqrt{6.2472725}c)-1]^{-1}<.0000002.$$

Adding up the estimates for $\sum_1$ through $\sum_4$ gives:

(4.2) $|S_2|^{1/2}|H_{1,2}(S, 1)|<.0000023$ if S satisfies the hypotheses of the theorem and $w/s_{33}<4$.

It follows from (2.11), (2.12), (2.15) and (4.2) that

$$Z_3(S, 1)>\pi\,|S_2|^{-1/2}\left\{-1.96255+\frac{\pi}{6}\sqrt{\frac{w}{s_{33}}}+\log\sqrt{\frac{t}{w}}\right\}$$

if S satisfies the hypotheses of the theorem and $w/s_{33}<4$. This concludes the proof of the theorem.

Our next problem is to consider the negativity question. We shall prove:

THEOREM 2. *If S satisfies* (2.4) *and*

$$\frac{\pi}{6}\sqrt{\frac{w}{s_{33}}}+\log\sqrt{\frac{t}{w}}\leq 1.9443,$$

then $Z_3(S, 1)<0$.

Proof. As in the proof of Theorem 1, we consider two cases, the first being that in which S satisfies the hypotheses of Theorem 2 and $w/s_{33} \geq 4$. We also note that, by Proposition 2, it suffices to assume

$$\frac{\pi}{6}\sqrt{\frac{w}{s_{33}}}+\log\sqrt{\frac{t}{w}}$$

lies in the interval (1.899485, 1.9443).

In the case that $w/s_{33} \geq 4$ we can use (2.19) and (4.1) again. It follows from these inequalities combined with (2.11) and (2.12) that

$$Z_3(S, 1) < \pi\,|S_2|^{-1/2}\left\{-1.9443471+\frac{\pi}{6}\sqrt{\frac{w}{s_{33}}}+\log\sqrt{\frac{t}{w}}\right\} \quad \text{if } \frac{w}{s_{33}} \geq 4.$$

This completes the proof of Theorem 2 in case 1.

Next we consider the case $w/s_{33} < 4$ and now we need to use the added assumption that

$$\frac{\pi}{6}\sqrt{\frac{w}{s_{33}}}+\log\sqrt{\frac{t}{w}} > 1.899485.$$

This implies that $t/w > 5.4990475$.

In this case the only estimate we can use for $|S_2|^{1/2}\, H_{1,1}(S_2, 1)$ is (2.17). We obtain the estimate for $|S_2|^{1/2}\, |H_{1,2}(S, 1)|$ in the usual way, as in the proof of (4.2). The result is

$$|S_2|^{1/2}\,|H_{1,2}(S, 1)| < .0000056$$

$$\text{if } \frac{w}{s_{33}} < 4 \quad \text{and} \quad \frac{\pi}{6}\sqrt{\frac{w}{s_{33}}}+\log\sqrt{\frac{t}{w}} > 1.899486. \tag{4.3}$$

It follows from (2.11), (2.12), (2.17) and (4.3) that

$$Z_3(S, 1) < \pi\,|S_2|^{-1/2}\left\{-1.949227+\frac{\pi}{6}\sqrt{\frac{w}{s_{33}}}+\log\sqrt{\frac{t}{w}}\right\}$$

$$\text{if } \frac{w}{s_{33}} < 4 \quad \text{and} \quad \frac{\pi}{6}\sqrt{\frac{w}{s_{33}}}+\log\sqrt{\frac{t}{w}} > 1.899486.$$

This completes the proof of Theorem 2.

References

1. M. Abramowitz and I. A. Stegun (editors), *Handbook of mathematical functions, with formulas, graphs and mathematical tables,* 3rd printing, with corrections, National Bureau of Standards Applied Math. Series 55, Superintendent of Documents, U.S. Printing Office, Washington, D.C., 1965.
2. P. T. Bateman and E. Grosswald, *On Epstein's zeta function,* Acta Arithmetica, vol. 9 (1964), pp. 365–373.

3. M. E. Low, *Real zeroes of the Dedekind zeta function of an imaginary quadratic field*, Acta Arithmetica, vol. 14 (1969), pp. 117–140.
4. G. B. Purdy, *Some extremal problems in geometry and analysis*, CAC Document Number 52, Center for Advanced Computation, University of Illinois, Urbana, Illinois.
5. M. S. Raghunathan, *Discrete subgroups of Lie groups*, Springer Verlag, New York, 1972.
6. C. L. Siegel, *Lectures on quadratic forms*, Tata Institute, Bombay, India, 1957.
7. A. A. Terras, *Bessel series expansions of the Epstein zeta functions and the functional equation*, Trans. Amer. Math. Soc., vol. 183 (1973), pp. 477–486.
8. ———, *On a relation between the size of the minima of quadratic forms and the behavior of Epstein and Dedekind zeta functions in the real part of the critical strip*, preprint.
9. ———, *Applications of special functions for the general linear group to number theory*, Seminaire Delange-Pisot-Poitou, 1977, exp. 23.
10. ———, *Integral formulas for the symmetric space of the general linear group and applications to automorphic forms*, preprint
11. A. A. Terras, R. Terras, G. Purdy, and H. Williams, *Graphing L-functions of Kronecker symbols in the real part of the critical strip*, preprint.

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