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# PLANAR SURFACE IMMERSIONS

BY
LOUIS H. KAUFFMAN

## Introduction

In this paper immersions of surfaces with boundary into the plane, $\mathbf{R}^2$, will be classified up to an equivalence relation called image homotopy. When two immersions are image homotopic, there is a smooth deformation through immersion images of one image to the other. This deformation may be drawn or visualized. It gives the appearance of a motion of the immersion in time. Before proceeding further, the reader might enjoy viewing the long image homotopy shown in Figure 3. This image homotopy is an example of mod-2 planar phenomena that we shall deal with in greater detail.

Planar surface immersions are a mixture of integral and mod-2 phenomena. For example, there are infinitely many image homotopy classes of immersions of a once punctured torus, but only *two* image homotopy classes for a surface of genus greater than one having a single boundary component. In the latter case, these two immersions are distinguished by a mod-2 quadratic form just as in [KB]. In fact, our results are quite similar to those of [KB], where immersions into the sphere, $S^2$, are classified up to image homotopy. Except for the use of quadratic forms, we do not assume familiarity with [KB]. The paper is organized as follows:

In Section 1 image homotopy is discussed and defined. Proposition 1.6 shows that $\mathcal{I}(N)\simeq\mathcal{R}(N)/\mathcal{M}(N)$ where $\mathcal{I}(N)$ denotes image homotopy classes of immersions of $N$, $\mathcal{R}$ denotes regular homotopy, and $\mathcal{M}(N)$ is the mapping class group of $N$ (acting on $\mathcal{R}(N)$ by composition).

Section 2 discusses the role of curves on the surface. The Whitney–Graustein Theorem [W] is recalled and used to compute $\mathcal{R}(N)$. A boundary invariant, $B(f)$, of an immersion $f: N\to \mathbf{R}^2$ is defined in terms of the boundary curves of $N$. Proposition 2.3 computes $\mathcal{I}(N)$ for a $k$-holed disk in terms of the boundary invariant.

Section 3 discusses the generators of the mapping class group $\mathcal{M}(N)$ and then considers three important examples: (1) If $N=T$, a punctured torus, then $\mathcal{M}(N)\cong SL(2,\mathbf{Z})$ and $\mathcal{I}(N)\simeq\mathbf{Z}^+$. (2) If $N=T\# A$ (a torus with two holes), then the extra boundary component acts as a catalyst to reduce the toral part of the immersion modulo two. (3) If $N=T\# T$, a once-punctured double torus, then $\mathcal{I}(N)$ contains no more than two elements. These examples reflect the way particular sorts of diffeomorphisms of $N$ act on

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$\mathcal{R}(N)$. In each case, the diffeomorphisms are expressed by visualizable handle-sliding moves.

The mod-2 quadratic form is introduced in Section 4. An image homotopy classification (Theorem 4.2) is obtained by combining the reductions, exemplified in Section 3, with obstructions provided by the quadratic form and boundary invariant.

Throughout the paper the symbol $\cong$ stands for isotopy, isomorphism or diffeomorphism, while $\approx$ stands for image homotopy, and $\asymp$ stands for one-to-one correspondence. All surfaces are compact and differentiable.

## 1. Image homotopy

Let $N$ be an orientable surface with boundary. A differentiable mapping $f: N \to \mathbf{R}^2$ is said to be an *immersion* if the differential, $df$, is non-singular for each point of $N$. Thus $f$ is locally one-to-one, but a point in $\mathbf{R}^2$ may have many points in its pre-image. It is often convenient to deal directly with the image $f(N) \subset \mathbf{R}^2$.

One draws a picture of an immersion by illustrating $f(N) \subset \mathbf{R}^2$. The overlapping parts of the image are indicated by shading, or tacit convention, so that there is no ambiguity about what surface is being immersed. Nevertheless, a picture does not allow a unique reconstruction of the mapping $f: N \to \mathbf{R}^2$. For example, a change in parametrization on $N$ is given by a diffeomorphism $h: N \to N$; but $f$ and $f \circ h$ have identical images. This fact and the considerations below will lead us to divide out by the diffeomorphisms of the surface.

We would like to think of an immersion as a self-overlapping rubber sheet surface on the plane. Deformations of the immersion should correspond to stretchings of this idealized elastic sheet that are confined to the two dimensions of motion in the plane. This imaginary sheet needs also the property of being able to pass through itself without damage, in order to avoid entangling the overlappings. Such a deformation will be called an *image homotopy* of the immersion. Figure 1 (bottom line) illustrates an image homotopy of a punctured torus. Figure 3 gives twenty-four stages in the history of a more complex image homotopy.

It is fascinating to explore image homotopies of surfaces, but a precise definition is called for. I shall give a preliminary definition that is close to the intuition and then abstract this to a more general definition that is easier to handle.

The first thing to notice about the elastic sheet deformation is that it really does not happen *in* the plane! If it did, then overlapping parts of the surface would have to move in unison, while we certainly want to be able to slide them separately. One way to clarify this is to see the image homotopy as the projection of an ambient isotopy of surfaces embedded in $\mathbf{R}^3$.

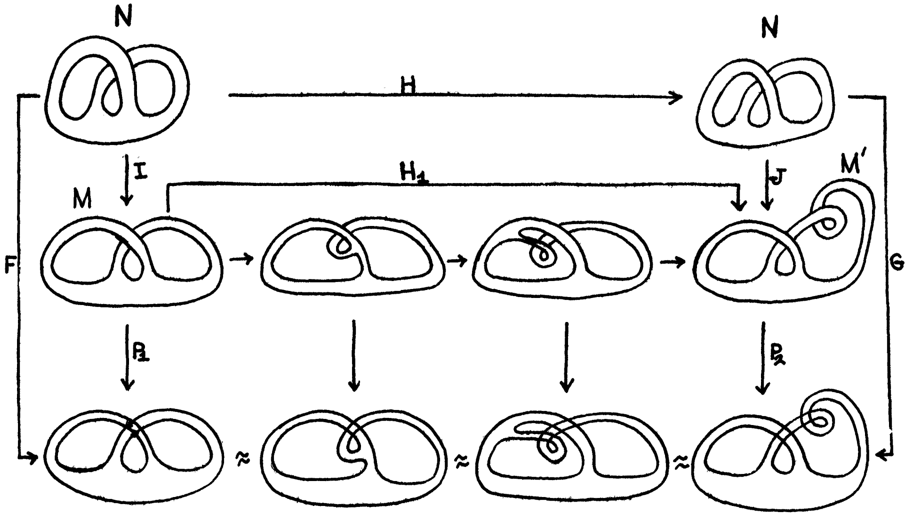


Figure 1

Recall that if $M$ and $M'$ are surfaces embedded in $\mathbf{R}^3$, then $M$ is said to be *ambient isotopic* to $M'$ if there is a family of diffeomorphisms $h_t:\mathbf{R}^3 \to \mathbf{R}^3$, $0 \leq t \leq 1$, so that $h_0$ is the identity and $h_1(M) = M'$, with $h_t$ and $dh_t$ varying continuously with $t$. In Figure 1 we show an ambient isotopy of surfaces in $\mathbf{R}^3$ so that each stage of the isotopy projects to an immersion in $\mathbf{R}^3$.

DEFINITION 1.1. Let $p:\mathbf{R}^3 \to \mathbf{R}^2$ be the projection defined by $p(x, y, z) = (x, y)$. Let $M, M' \subset \mathbf{R}^3$ be embedded surfaces so that $p_1 = p \mid M$ and $p_2 = p \mid M'$ are immersions. We say that $p_1$ and $p_2$ are *image homotopic* if there is an ambient isotopy $h_t:\mathbf{R}^3 \to \mathbf{R}^3$ from $M$ to $M'$ so that $p \mid h_t(M)$ is an immersion for each $t$, $0 \leq t \leq 1$.

This definition is difficult to work with. It is not always obvious what embedding should be chosen. For example, in Figure 2 we have covered an image homotopy of an annulus with an ambient isotopy in 2A; if we had chosen the embedding of 2B then no such isotopy could have been produced (the boundary curves of the second embedding are linked). The final definition provides a way out. Recall that two immersions $f, g: N \to \mathbf{R}^2$ are *regularly homotopic* if there is a family of immersions $f_t: N \to \mathbf{R}^2$, $0 \leq t \leq 1$, so that $f_0 = f$, $f_1 = g$ and $f_t$ and $df_t$ vary continuously with $t$.

DEFINITION 1.2. Let $N$ be an oriented surface with boundary, and let $f, g: N \to \mathbf{R}^2$ be two orientation-preserving immersions of $N$ into the plane. We say that $f$ and $g$ are *image homotopic* ($f \approx g$) if there exists an orientation-preserving diffeomorphism $h: N \to N$ such that $f$ is regularly homotopic to $g \circ h$.

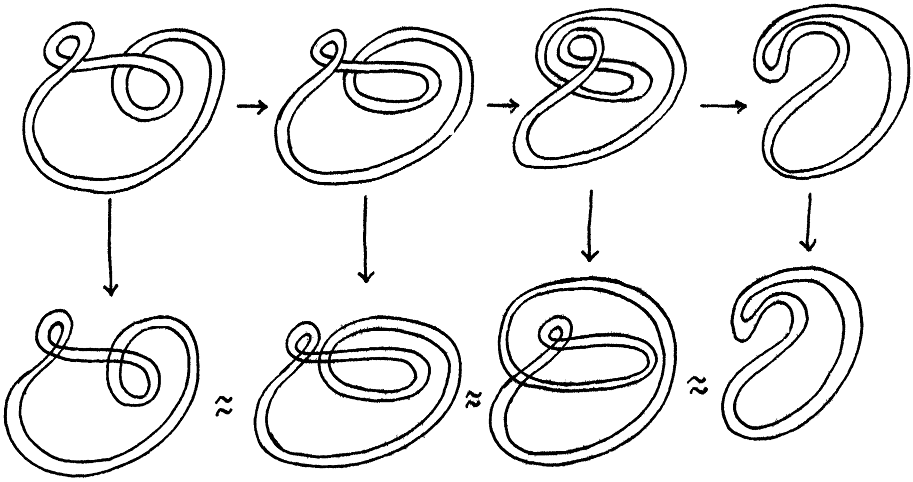

Figure 2A

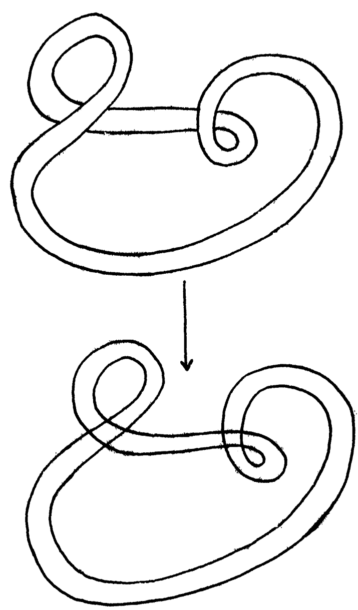

Figure 2B

Definition 1.1 is a special case of Definition 1.2. The diffeomorphism $h$ arises naturally from the ambient isotopy.

PROPOSITION 1.3. *Let $M$, $M' \subset \mathbf{R}^3$ be ambient isotopic embeddings of a surface $N$ into $\mathbf{R}^3$. Let $i: N \to \mathbf{R}^3$ and $j: N \to \mathbf{R}^3$ represent these embeddings so that $i(N) = M$ and $j(N) = M'$. Suppose that $M$ and $M'$ project to image*

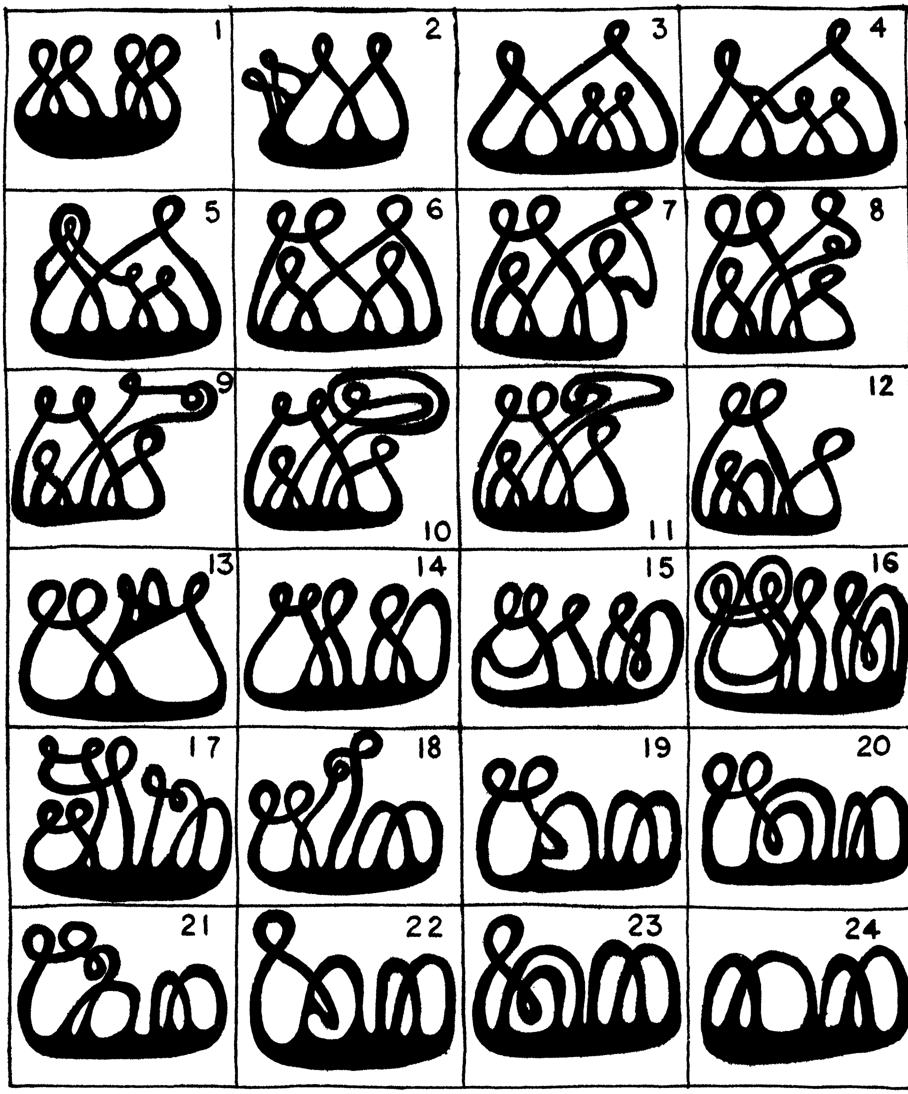


Figure 3

*homotopic immersions in the sense of* 1.1. *Let* $p_1, p_2, h_t$ *be as in* 1.1. *Let*

$$f = p_1 \circ i, \quad g = p_2 \circ j, \quad h = j^{-1} \circ h_1 \circ i$$

*and assume that f and g preserve orientation. Then f is regularly homotopic to* $g \circ h$. *Hence* $f, g: N \to \mathbf{R}^2$ *are image homotopic in the sense of* 1.2.

*Proof.* Define $f_t: N \to \mathbf{R}^2$ by the equation $f_t = p \circ h_t \circ i$. Then $f_0 = f$ while

$$f_1 = (p \circ j) \circ (j^{-1} \circ h_1 \circ i) = g \circ h.$$

Thus $f$ and $g \circ h$ are regularly homotopic as desired.

The maps in Proposition 1.3 are illustrated in Figure 1.

By using the second definition (1.2), immersions can be classified up to image homotopy by looking at the action of the diffeomorphisms of $N$ on the regular homotopy classes. We shall formulate this in terms of the mapping class group of $N$ (see [B], [L]).

DEFINITION 1.4. Let $N$ be an oriented surface. Two diffeomorphisms $h, h': N \to N$ are *isotopic* if there is a family of diffeomorphisms $h_t: N \to N$, $0 \leq t \leq 1$, with $h_t$ and $dh_t$ varying continuously with $t$, and $h_0 = h$, $h_1 = h'$. The *mapping class group of N*, $\mathcal{M}(N)$, is the set of isotopy classes of orientation-preserving diffeomorphisms of $N$.

DEFINITION 1.5. Given an oriented surface with boundary $N$, let $\mathcal{R}(N)$ denote the set of regular homotopy classes of orientation-preserving immersions $f: N \to \mathbf{R}^2$. Let $\mathcal{I}(N)$ denote the set of image homotopy classes of orientation-preserving immersions into $\mathbf{R}^2$.

Note that $\mathcal{M}(N)$ acts on $\mathcal{R}(N)$ by composition. That is, if $[h] \in \mathcal{M}(N)$ and $[f] \in \mathcal{R}(N)$ are represented by $h: N \to N$ and $f: N \to \mathbf{R}^2$ respectively, then we define $[h] \cdot [f]$ by the equation $[h] \cdot [f] = [f \circ h]$.

PROPOSITION 1.6. *If $N$ is an oriented surface with boundary, then* $\mathcal{I}(N) \simeq \mathcal{R}(N)/\mathcal{M}(N)$.

*Proof.* This follows at once from the definitions.

## 2. Curves on the surface

In order to study immersions of a surface it is necessary to first understand immersions of closed curves. Given an immersion $\alpha: S^1 \to \mathbf{R}^2$ of an oriented circle into the plane, Whitney [W] defined a *degree*, $D(\alpha) \in \mathbf{Z}$. The degree measures the total number of times (counted with sign) that the image unit tangent vector turns through $2\pi$ as the curve is traversed. Our convention gives a counterclockwise circle about the origin degree $+1$. Whitney and Graustein proved the fundamental result: Two curves $\alpha, \beta: S^1 \to \mathbf{R}^2$ are regularly homotopic if and only if $D(\alpha) = D(\beta)$.

An oriented surface with boundary $N$ has normal form as depicted in Figure 4. A collection of canonical curves

$$K(N) = \{a_1, b_1, a_2, b_2, \ldots, a_g, b_g, c_1, \ldots, c_k\}$$

is obtained by choosing an oriented core curve for each band from the torus groups, and oriented boundary curves $c_1, \ldots, c_k$. The surface is given with $k+1$ boundary components and an orientation. Hence each boundary component inherits an orientation from the surface. Note that in this normal form, the curves $c_1, \ldots, c_k$ actually also correspond to cores of the

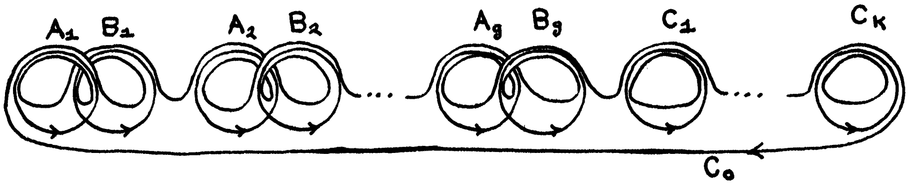


Figure 4

extra bands. We shall refer to the remaining boundary curve $c_0$ as the "outer boundary".

With orientations chosen as in Figure 4, we call immersions that look like direct projections of the disk part of Figure 4 with modifications on the bands, orientation preserving. Thus the immersion indicated in Figure 8 is orientation preserving. With these conventions it is easy to read the degrees of the canonical curves directly from the picture. For example, in Figure 8, $K(N)=\{a_1, b_1; c_1, c_2\}$ and $D(a_1)=0$, $D(b_1)=1$, $D(c_1)=2$, $D(c_2)=-1$.

Let $\mathscr{C}(N)$ denote the set of oriented curves embedded in $N$. Elements of $\mathscr{C}(N)$ are represented by maps $\alpha: S^1 \to N$. Given an immersion $f: N \to \mathbf{R}^2$ and $\alpha \in \mathscr{C}(N)$, we obtain a curve $f \circ \alpha: S^1 \to \mathbf{R}^2$ and we let $D_f(\alpha) = D(f \circ \alpha)$ denote the degree of this immersion.

PROPOSITION 2.1. *Let $N$ be an oriented surface with boundary, having canonical curves $K(N)$. Let $f, g: N \to \mathbf{R}^2$ be two orientation-preserving imersions. Then $f$ is regularly homotopic to $g$ if and only if $D_f(\alpha) = D_g(\alpha)$ for all $\alpha \in K(N)$.*

The proof of this result is sketched in [KB]. It follows that the regular homotopy classes $\mathscr{R}(N)$ are in one-to-one correspondence with $(2g+k)$-tuples of integers

$$(\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g; \gamma_1, \ldots, \gamma_k)$$

where, for a representative immersion $f$,

$$\alpha_i = D_f(a_i),\ \beta_i = D_f(b_i),\ \gamma_i = D_f(c_i).$$

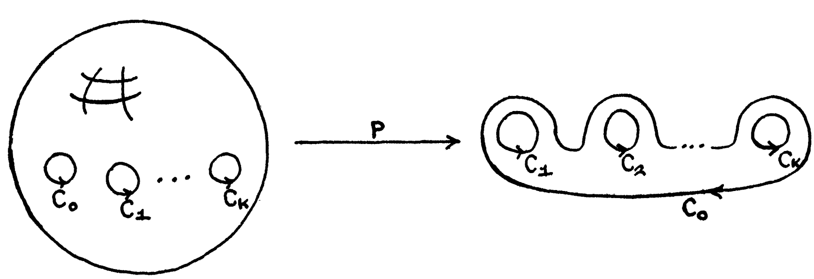


Figure 5

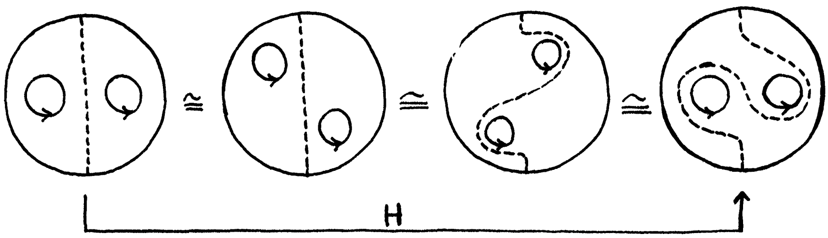


Figure 6

For ease in handling this correspondence we shall use the notation

$$x = [\alpha_1, \beta_1 | \alpha_2, \beta_2 | \cdots | \alpha_g, \beta_g | \gamma_1, \ldots, \gamma_k].$$

When two such vectors represent image homotopic immersions, we shall write $x \approx y$.

As a first example, we shall calculate $\mathcal{I}(N)$ when $N$ is a disk with $k$-holes. We view $N$ as an oriented sphere with $(k+1)$-holes and oriented boundary curves $c_0, c_1, \ldots, c_k$ as in Figure 5. The mapping class group is generated by "braiding" these holes (see [B]). That is, one can choose any deformation of the surface that moves the holes around, finally returning them (possibly permuted) to their original configuration. (See Figure 6.) On the band representation, the result of such a diffeomorphism is to permute the $c_1, \ldots, c_k$ among themselves, or to interchange the outer boundary $c_0$ with one of $c_1, \ldots, c_k$ as in Figure 7.

The next lemma gives a key relationship.

LEMMA 2.2 *Let $N$ be an oriented surface with oriented boundary curves $c_0, c_1, \ldots, c_k$. Let $f: N \to \mathbf{R}$ be an orientation-preserving immersion, and let $\gamma_i = D_f(c_i)$ for $i = 0, 1, \ldots, k$. Then $\gamma_0 + \gamma_1 + \cdots + \gamma_k = (k-1)$.*

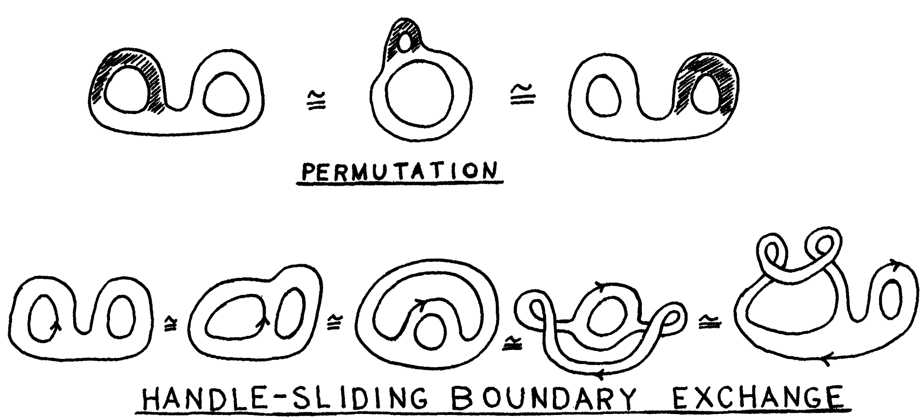


Figure 7

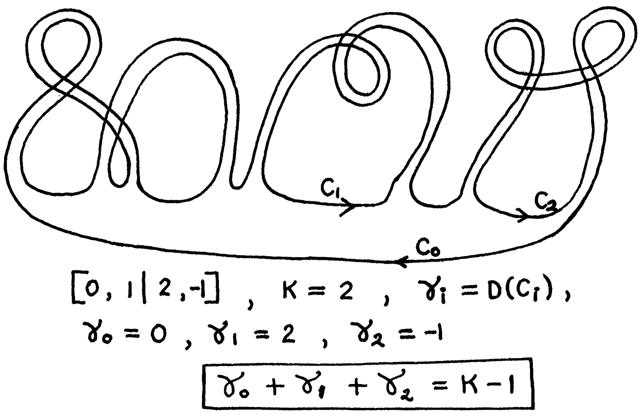


Figure 8

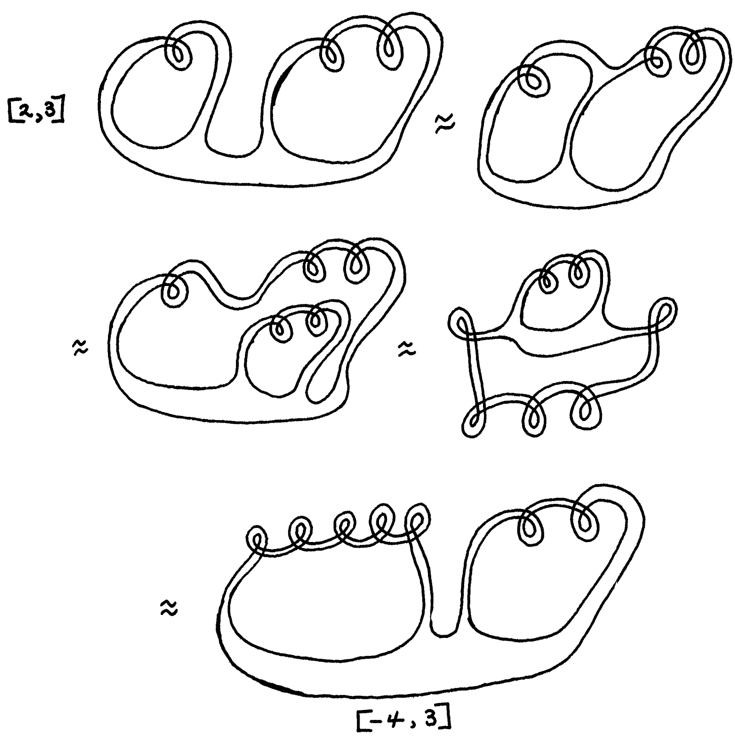


Figure 9

We shall omit the proof of this lemma. The result is illustrated in Figure 8.

DEFINITION 2.3. Let $B_k$ denote the collection of sets of integers $\{n_0, n_1, \ldots, n_k\}$ satisfying $\sum_{l=0}^{k} n_l = (k-1)$. Given an oriented surface $N$ with oriented boundary components $c_0, c_1, \ldots, c_k$ define the *boundary invariant of an immersion* $f\colon N \to \mathbf{R}^2$ by the equation $B(f) = \{D_f(c_0), \ldots, D_f(c_k)\} \in B_k$.

Note that if $f$ corresponds to $[\alpha_1, \beta_1 \mid \cdots \mid \alpha_g, \beta_g \mid \gamma_1, \ldots, \gamma_k] \in \mathcal{R}(N)$, then

$$B(f) = \{\gamma_0, \gamma_1, \ldots, \gamma_k\} \quad \text{where} \quad \gamma_0 = (k-1) - \sum_{l=1}^{k} \gamma_l.$$

This follows from Lemma 2.2. The boundary invariant is obviously an invariant of image homotopy.

PROPOSITION 2.3. *Let $N$ be an oriented $k$-holed disk. Then two immersions $f, g\colon N \to \mathbf{R}^2$ are image homotopic if and only if $B(f) = B(g)$. Hence $\mathcal{I}(N) \simeq B_k$.*

*Proof.* This follows at once from the remark that braiding diffeomorphisms can accomplish any permutation of the boundary components.

*Remark.* When $N = A$, an annulus, then $k = 1$ and

$$B_1 = \{\{n, -n\}\} \simeq \{n \mid n \geq 0, n \in \mathbf{Z}\} = \mathbf{Z}^+.$$

Thus $\mathcal{I}(A) = \mathbf{Z}^+$.

## 3. The mapping class group

We shall describe generators of $\mathcal{M}(N)$ for $N$ any oriented surface with boundary. This can be done in terms of braiding diffeomorphisms (as in the last section) plus twist mappings along curves in $\mathcal{C}(N)$. These will be described in some detail and related to handle-sliding moves.

Suppose that $N$ is modelled as in Figure 10, a sphere with g torus handles and $k+1$ holes. The braiding diffeomorphisms include those described in section 2 plus a new set obtained by moving a hole along any closed curve and returning it to its original position. For example, in Figure 11 we illustrate the handle-sliding move that corresponds to such a braiding diffeomorphism on a doubly punctured torus.

The twist mappings were invented by Dehn and later re-discovered by Lickorish (see [B] and [L]). Let $A = S^1 \times [0, 1]$ denote an annulus, and define $\tau\colon A \to A$ by $\tau(\lambda, t) = (e^{2\pi i t}\lambda, t)$. This is a twist diffeomorphism of $A$. Notice that the boundary of the annulus remains fixed (See Figure 13). More generally, choose an embedded curve $\alpha \in \mathcal{C}(N)$ contained in the interior of $N$. Let $A_\alpha$ be an annular neighborhood of $\alpha$ and $\tau_\alpha\colon N \to N$ be the diffeomorphism obtained by applying $\tau$ to $A_\alpha$ and the identity to the rest of $N$. The map $\tau_\alpha$ is called a *twist along* $\alpha$.

Since there is some choice as to the direction of the twist, a convention is

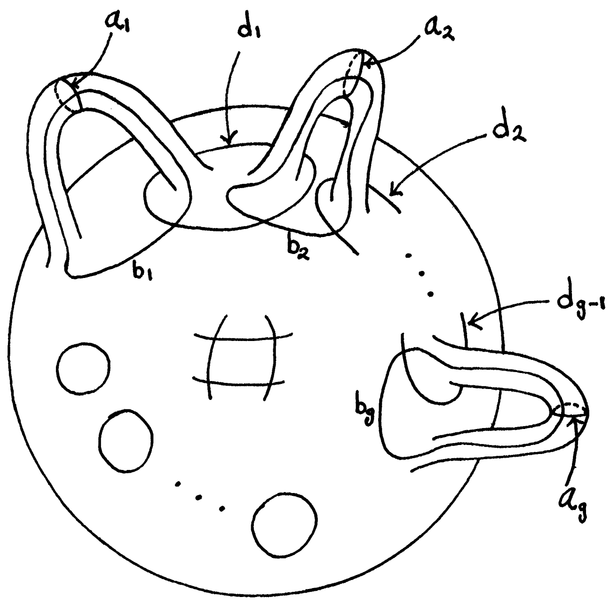


Figure 10

called for. The convention is that a segment transverse to $\alpha$ shall *turn to the right* along $\alpha$ when $\tau_\alpha$ is applied to it. Turning to the right is meant to apply to an observer on the surface whose head points in the positive normal direction as he or she approaches $\alpha$ along the transverse segment. (See Figure 14.) Note that $\tau_\alpha$ does not depend upon the orientation of $\alpha$.

The next lemma shows how the twist mappings act on $\mathcal{R}(N)$.

LEMMA 3.1. *Let $\alpha, \beta \in \mathcal{C}(N)$ be transversely intersecting curves on $N$, an*

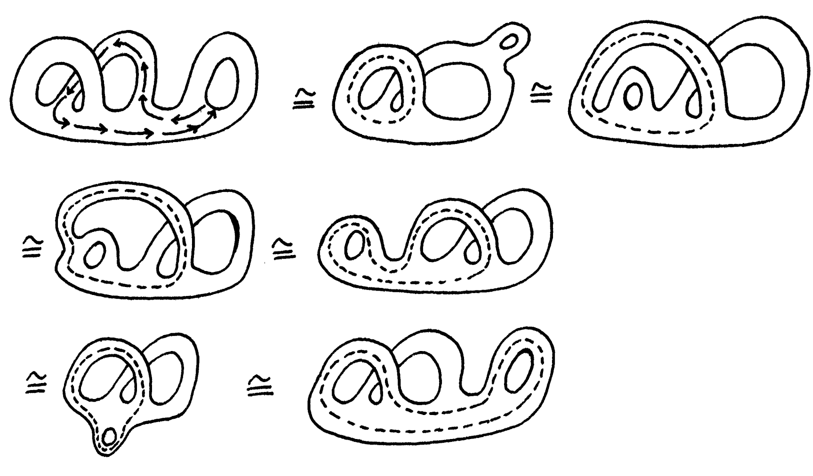

Figure 11

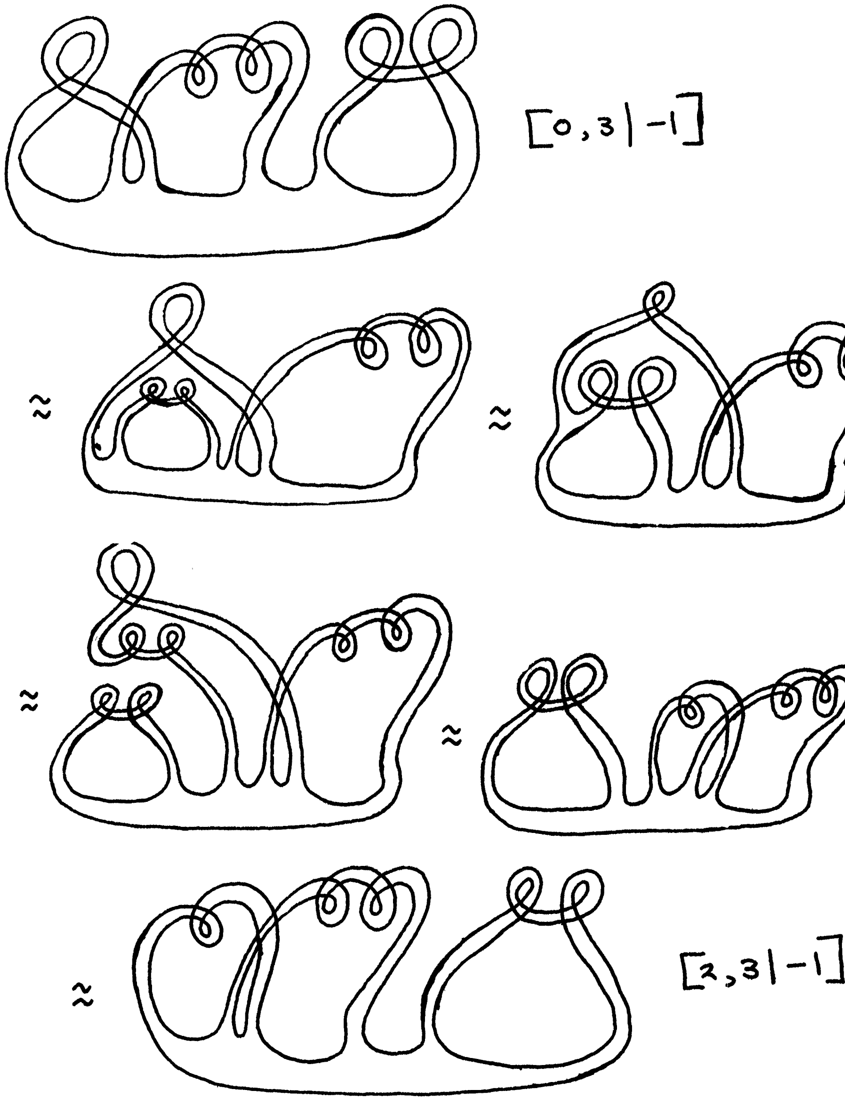


Figure 12

*oriented surface with boundary. Let* $f: N \to \mathbf{R}^2$ *be an orientation-preserving immersion, and let D denote* $D_f$, *and* $\alpha \cdot \beta \in \mathbf{Z}$ *denote the intersection number of* $\alpha$ *and* $\beta$ *on N. Then* $D(\tau_\alpha(\beta)) = (\alpha \cdot \beta)D(\alpha) + D(\beta)$.

The proof of this lemma will be omitted.

The fundamental theorem for $\mathcal{M}(N)$ is as follows (see [B]):

THEOREM 3.2. *Let N be an oriented surface with boundary. If N has genus*

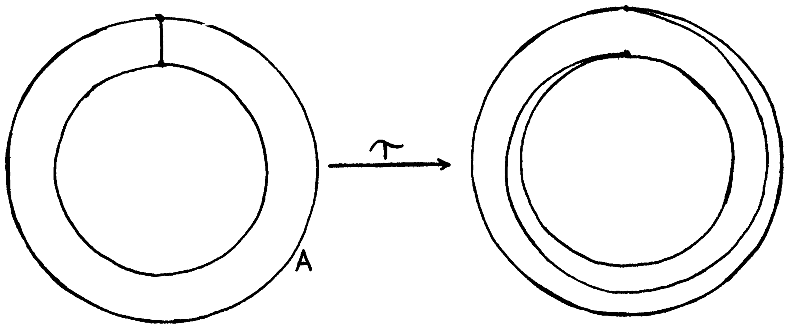


Figure 13

*g, then $\mathcal{M}(N)$ is generated by braiding diffeomorphisms and the* $3g-1$ *twists along the* $3g-1$ *curves illustrated in Figure* 10.

*Examples.* (1) Let $N=T$, a singly punctured torus with canonical band curves $a$ and $b$. Then $\mathcal{M}(T)$ is generated by $\tau_a$ and $\tau_b$ (the braiding diffeomorphisms are isotopic to the identity). In fact, $\mathcal{M}(T)\cong SL(2,\mathbf{Z})$. The handle sliding moves corresponding to $\tau_a$ and $\tau_b$ consist in sliding the foot of one band all the way around the other band as in Figure 1. From this, or from Lemma 3.1, it is easy to see that they induce image homotopies $[\alpha,\beta]\approx[\alpha\pm\beta,\beta]$ and $[\alpha,\beta]\approx[\alpha,\pm\alpha+\beta]$. Thus $\mathcal{M}(T)\cong SL(2,\mathbf{Z})$ acts on $\mathcal{R}(T)\simeq\mathbf{Z}\oplus\mathbf{Z}$ in standard fashion. By Euclid's algorithm we see that $[\alpha,\beta]\approx[d,0]$ where $d=\gcd(\alpha,\beta)$ (gcd is the non-negative greatest common divisor). Hence $\mathcal{I}(T)\simeq\mathbf{Z}^+$.

(2) Let $N=T\#A$ be a doubly punctured torus (viewed as the boundary connected sum (#) of $T$ and an annulus $A$). Then $\mathcal{R}(N)\simeq\{[\alpha,\beta\mid\gamma]\}$ as in Section 2. We can perform unimodular transformations on $[\alpha,\beta]$ as in Example 1. But we also have a relation

$$[\alpha,\beta\mid\gamma]\approx[\alpha-\gamma+1,\beta\mid\gamma]$$

coming from braiding as illustrated in Figures 11 and 12. Furthermore, $[\alpha,\beta\mid\gamma]\approx[\alpha,\beta\mid-\gamma]$ via boundary exchange braiding as in Section 2.

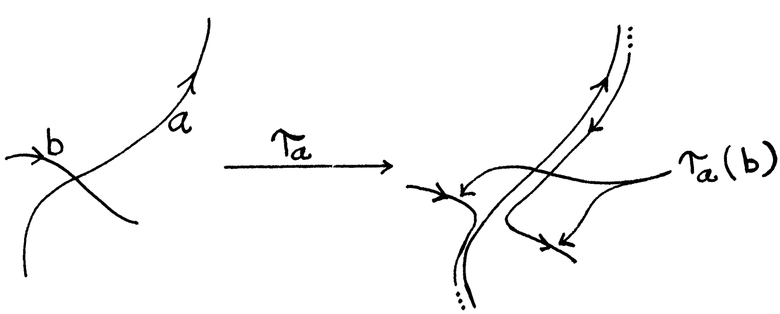


Figure 14

LEMMA 3.3. *Let* $d = \gcd(\alpha, \beta)$. *Then*

$$[\alpha, \beta \mid \gamma] \approx \begin{cases} [0, 0 \mid \gamma] & \textit{if} \quad d \quad \textit{is even}, \\ [1, 1 \mid \gamma] & \textit{if} \quad d \quad \textit{is odd}. \end{cases}$$

*If* $\gamma$ *is even, then* $[0, 0 \mid \gamma] \approx [1, 1 \mid \gamma]$.

*Proof.* Let $\Delta = 1 - \gamma$. Then $1 + \gamma = -\Delta + 2$ and

$$[\alpha, \beta \mid \gamma] \approx [\alpha + \Delta, \beta \mid \gamma] \approx [\alpha + \Delta, \beta \mid -\gamma]$$
$$\approx [\alpha + \Delta - \Delta + 2, \beta \mid -\gamma] \approx [\alpha + 2, \beta \mid \gamma].$$

Hence $[\alpha, \beta \mid \gamma] \approx [\alpha - 2, \beta \mid \gamma]$. But $[\alpha, \beta \mid \gamma] \approx [d, 0 \mid \gamma]$. Hence

$$[d, 0 \mid \gamma] \approx [d-2, 0 \mid \gamma] \approx \cdots \approx [0, 0 \mid \gamma] \quad \text{or} \quad [1, 0 \mid \gamma].$$

Since $[1, 0 \mid \gamma] \approx [1, 1 \mid \gamma]$, this proves the first part of the lemma. If $\gamma$ is even then

$$[1, 0 \mid \gamma] \approx [1 + 1 - \gamma, 0 \mid \gamma] = [e, 0 \mid \gamma]$$

where $e$ is even. Hence $[1, 0 \mid \gamma] \approx [0, 0 \mid \gamma]$, completing the proof of the lemma.

This lemma shows that the image homotopy class of an immersion of $T \# A$ is determined by the boundary invarient when it is even. For odd boundary invariant, the two possibilities $[0, 0 \mid \gamma]$ and $[1, 1 \mid \gamma]$ are in fact distinct and will be distinguished by the mod-2 quadratic form described in Section 4. The extra boundary component introduced by taking connected sum with $A$ acts as a catalyst, reducing the toral part of the immersion modulo two.

(3) Let $N = T \# T$, a once-punctured double torus. Then $\mathcal{M}(N)$ contains twists about the canonical curves $a_1, b_1, a_2, b_2$ and the curve $e$ shown in Figure 15. ($e$ corresponds to $d_1$ of Figure 10.) The canonical curves act as $SL(2, \mathbf{Z}) \oplus SL(2, \mathbf{Z})$ on $\mathcal{R}(N) \approx \{[\alpha, \beta \mid \gamma, \delta]\}$. The curve $e$ involves the two toral groups. Rather than describing $\tau_e$ directly, we give a related handle-sliding maneuver $\mathcal{T}: N \to N$. It is shown in Figure 16.

As in Figure 16, we perform $\mathcal{T}$ by first sliding the 1-2 group over band 3

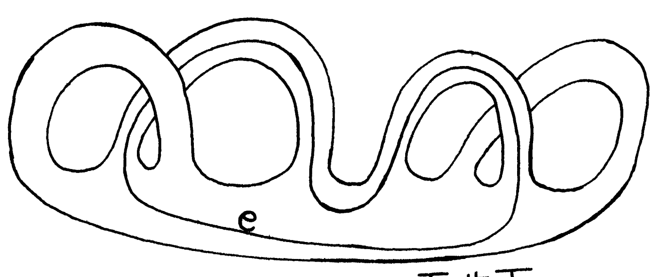


Figure 15

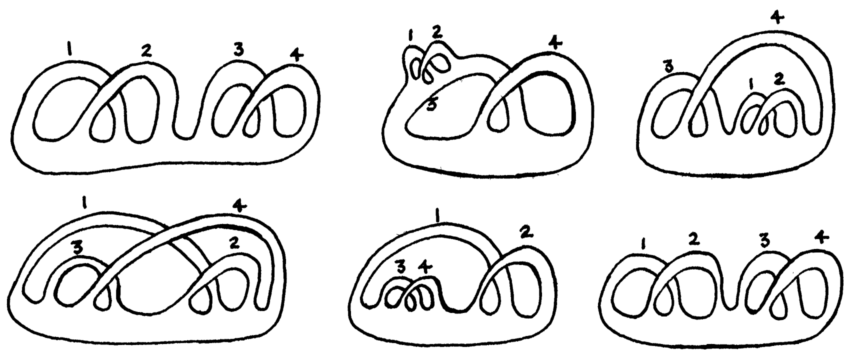


Figure 16

so that it sits under band 4. Then the left foot of band 1 is slid over band 3, and the right foot of band 4 is slid over band 2. Now the 3-4 group sits under band 1. It is slid out across band 2, completing the move.

An analysis (that we omit) shows that $\mathcal{T}$ corresponds to a diffeomorphism that is isotopic to $\tau_e^{-1} \circ \tau_{a_2} \circ \tau_{b_1}$. Thus the twist part of $\mathcal{M}(T \# T)$ is generated by handle-sliding moves.

The image homotopy of Figure 3 is based on $\mathcal{T}$. Steps 1 through 14 consist of $\mathcal{T}$ combined with the regular homotopy in steps 9–12. Note that $1 \to 14$ gives an image homotopy $[0, 0 \mid 0, 0] \approx [-1, 0 \mid 0, 1]$. In general, $\mathcal{T}$ has the effect

$$[\alpha, \beta \mid \gamma, \delta] \approx [\alpha + \gamma - 1, \beta \mid \gamma, -\beta + \delta + 1].$$

This image homotopy effects an interaction between the two torus groups that reduces $\mathcal{I}(T \# T)$ to only two elements!

Lemma 3.4. *For any choice of* $\alpha, \beta, \gamma, \delta$

$$[\alpha, \beta \mid \gamma, \delta] \approx [1, 1 \mid 1, 1] \quad \text{or} \quad [0, 0 \mid 1, 1].$$

*Proof.* Let $x = [\alpha, \beta \mid \gamma, \delta]$. We first show that

$$x \approx [0, 0 \mid 0, 1] \quad \text{or} \quad [0, 0 \mid 0, 2].$$

Let $d = \gcd(\alpha, \beta)$, $d' = \gcd(\gamma, \delta)$. Then

$$\begin{aligned} x \approx [d, 0 \mid d', 0] &\approx_{\mathcal{T}} [d + d' - 1, 0 \mid d', 1] \\ &\approx [d + d' - 1, 0 \mid 0, 1] \\ &\approx [d + d' - 1, 0 \mid -(d + d') + 2, 1] \\ &\approx_{\mathcal{T}} [0, 0 \mid -(d + d') + 2, 2] \\ &\approx [0, 0 \mid 0, 2] \quad \text{or} \quad [0, 0 \mid 0, 1]. \end{aligned}$$

The same calculation shows that

$$[1, 1 \mid 1, 1] \approx [0, 0 \mid 0, 2]$$

while

$$[0, 0 \mid 0, 1] \approx [0, 0 \mid 1, 1].$$

This proves the lemma.

Thus $\mathcal{I}(T \# T)$ has at most two elements. The mod-2 quadratic form will distinguish them.

These examples are basic for the image homotopy classification of surfaces. The next section provides the tools needed to complete the job.

## 4. The mod-2 quadratic form

Given an immersion $f: N \to \mathbf{R}^2$, there is an associated mod-2 quadratic form

$$q_f: H_1(N; \mathbf{Z}_2) \to \mathbf{Z}_2.$$

This form is constructed just as the quadratic form in [KB] for immersions into $S^2$. That is, if $[\alpha] \in H_1(N; \mathbf{Z}_2)$ is represented by an embedded curve $\alpha \in \mathscr{C}(N)$, then $q_f([\alpha]) \equiv D_f(\alpha) + 1 \pmod 2$.

If $\langle \ , \ \rangle: H_1(N; \mathbf{Z}_2) \times H_1(N; \mathbf{Z}_2) \to \mathbf{Z}_2$ denotes the mod-2 intersection pairing on the surface $N$, then $q_f(x+y) = q_f(x) + q_f(y) + \langle x, y\rangle$ for $x, y \in H_1(N; \mathbf{Z}_2)$. When two immersions $f$ and $g$ are image homotopic, their quadratic forms are isomorphic.

The reader is referred to [KB] for further information on mod-2 forms. Given a form $q$, we let $A(q)$ denote the Arf invariant of $q$ (when it is defined).

The quadratic form cleans up the examples of Section 3: Let $T^0 = [1, 1]$ and $T^1 = [0, 0]$. We have shown that $T^1 \# T^1 \approx T^0 \# T^0$, but $T^0 \# T^1 \not\approx T^1 \# T^1$ because $q(T^0 \# T^1)$ has Arf invariant 1, while $q(T^1 \# T^1)$ has Arf invariant 0. Hence by Lemma 3.4, $\mathcal{I}(T \# T) \simeq \mathbf{Z}_2$.

Similarly, if $\gamma$ is odd, then the quadratic form $q([\alpha, \beta \mid \gamma])$ has zero radical, and hence a well-defined Arf invariant that is equal to the Arf invariant of $q([\alpha, \beta])$. This shows that $[1, 1 \mid \gamma]$ and $[0, 0 \mid \gamma]$ are not image homotopic for odd $\gamma$ (see Lemma 3.3).

Definition 4.1. Let $f: N \to \mathbf{R}^2$ be an immersion of an oriented surface with boundary. Let $\beta(f)$ denote the number of boundary components $C$ of $N$ such that $D_f(C) \equiv 0 \pmod 2$.

Recall that we have also defined the boundary invariant $B(f) = \{D_f(C) \mid C$ is a boundary component of $N\}$.

The next theorem completes the image homotopy classification.

THEOREM 4.2. *Let $f, f': N \to \mathbf{R}^2$ be orientation-preserving immersions of an oriented surface with boundary into $\mathbf{R}^2$.*

(i) *If $N$ has genus 1 and connected boundary, then $d: \mathcal{I}(N) \simeq \mathbf{Z}^+$, where $d$ assigns to an immersion the greatest common divisor of the Whitney degrees of the two canonical curves on $N$.*

(ii) *If $N$ has genus $>1$ and connected boundary, then $\mathcal{I}(N) \simeq \mathbf{Z}_2$. Two immersions are image homotopic if and only if their quadratic forms have the same Arf invariant.*

(iii) *If $N$ has more than one boundary component then:*

(a) *If $\beta(f) = \beta(f') = 0$, then $\mathcal{A}(q_f)$ and $\mathcal{A}(q_{f'})$ are defined. In this case $f \approx f'$ if and only if $B(f) = B(f')$ and $\mathcal{A}(q_f) = \mathcal{A}(q_{f'})$.*

(b) *If $\beta(f) \neq 0$ then $f \approx f'$ if and only if $B(f) = B(f')$.*

*Proof.* Part (i) has already been shown in Example 1 of Section 3. To prove (ii) note that in Example 3 of Section 3 we showed that any immersion of $T \# T$ is image homotopic to $T^0 \# T^1$ or to $T^0 \# T^0$ where $T^0 = [1, 1]$ and $T^1 = [0, 0]$. In fact, if $T_d = [d, 0]$ then $T_d \# T_{d'} = [d, 0 \mid d', 0] \approx T^0 \# T^0$ exactly when $d + d'$ is even (see Lemma 3.4). If $N = T \# T \# \cdots \# T$ (g factors and $g \geq 2$) then the same arguments apply: Any two torus groups can be made adjacent by permutation sliding. Then apply Lemma 3.4 to such pairs. The result is that the immersion $X$ is image homotopic to $T^{\varepsilon_1} \# \cdots \# T^{\varepsilon_g}$ where $\varepsilon_i = 0$ or 1. But $T^1 \# T^1 \approx T^0 \# T^0$. Hence, using more permutations, $X \approx T^0 \# T^0 \# \cdots \# T^0 \# T^\varepsilon$ where $\varepsilon = 0$ or 1. But $\mathcal{A}(q_X) = \varepsilon$ and hence $\mathcal{I}(N) \simeq \mathbf{Z}_2$ as desired.

We have already shown part (iii) when $g = \text{genus}(N) = 1$ (Example 2 of Section 3). For $g > 1$, we proceed slightly differently: The immersion is represented by

$$X \approx [d_1, 0 \mid d_2, 0 \mid \cdots \mid d_g, 0 \mid \gamma_1, \ldots, \gamma_k].$$

By the same argument as in part (ii) ($g > 1$)

$$X \approx [1, 1 \mid 1, 1 \mid \cdots \mid 1, 1 \mid \varepsilon, 0 \mid \gamma_1, \ldots, \gamma_k] \quad \text{where} \quad \varepsilon = 0 \quad \text{or} \quad 1.$$

If $\beta(X) \neq 0$ then $\gamma_i \equiv 0 \pmod 2$ for some $i = 0, 1, \ldots, k$. By permutation, or boundary exchange we may assume that $\gamma_1$ is even. But Lemma 3.3 shows that

$$[1, 0 \mid \gamma] \approx [0, 0 \mid \gamma] \quad \text{for} \quad \gamma \text{ even}.$$

In our context ($g > 1$) we can still use $[\alpha, \beta \mid \gamma] \approx [\alpha + 1 - \gamma, \beta \mid \gamma]$. Hence

$$X \approx [1, 1 \mid 1, 1 \mid \cdots \mid 1, 1 \mid \varepsilon + 1 - \gamma_1, 0 \mid \gamma_1, \gamma_2, \ldots, \gamma_k].$$

Let $d = \varepsilon + 1 - \gamma_1$. If $\varepsilon = 0$ then $d$ is odd; if $\varepsilon = 1$ then $d$ is even. Since

$$[1, 1 \mid d, 0] \approx [1, 1 \mid \varepsilon', 0] \quad \text{where} \quad \varepsilon' = 0 \text{ or } 1 \text{ and } \varepsilon' \equiv d \pmod 2$$

we conclude that $X \approx [1, 1 \mid 1, 1 \mid \cdots \mid \varepsilon', 0 \mid \gamma_1, \ldots, \gamma_k]$. Hence

$$\beta(X) \neq 0 \Rightarrow X \approx [1, 1 \mid 1, 1 \mid \cdots \mid 1, 1 \mid \gamma_1, \ldots, \gamma_k].$$

Since these are distinguished precisely by the boundary invariant, we have shown part (b) of (iii).

If $\beta(X)=0$ then $q_X$ has no radical. Hence $q_X$ has a well-defined Arf invariant that is equal to the Arf invariant of $Y=[1,1\mid 1,1\mid\cdots\mid 1,1\mid \varepsilon,0]$. Since $\mathcal{A}(q_Y)=1-\varepsilon$, this proves part (a).

This completes the proof of the theorem.

*Remark.* There is an interesting relationship between the quadratic form of an immersion and the Seifert pairing used in knot theory. Suppose that $N\subset\mathbf{R}^3$ is embedded so that the projection $f=p\mid N\colon N\to\mathbf{R}^2$ is an immersion. The Seifert pairing (see [K])

$$\theta\colon H_1(N;\mathbf{Z})\times H_1(N;\mathbf{Z})\to\mathbf{Z}$$

is defined by the equation $\theta(x,y)=l(ix,y)$ where $i\colon N\to\mathbf{R}^3-N$ is obtained by translating away from $N$ in the positive normal direction, and $l(\ ,\ )$ denotes linking number in $\mathbf{R}^3$. Then $q_f(\bar{x})\equiv\theta(x,x)$ (mod 2) where $\bar{x}\in H_1(N;\mathbf{Z}_2)$ and $x\in H_1(N;\mathbf{Z})$ is a representative of $\bar{x}$. It follows from this that the Arf invariant of a knot or link (see [K]) is the Arf invariant of an immersion of some spanning surface into $\mathbf{R}^2$. The Seifert pairing can also be used to find obstructions to covering a given image homotopy by an ambient isotopy in $\mathbf{R}^3$ of a given embedding $N\subset\mathbf{R}^3$.

## References

[B] J. Birman, *Braids, links, and mapping class groups*, Ann. of Math. Studies no. 82, Princeton Univ. Press Princeton, N.J. 1975.

[K] L. Kauffman, *Link manifolds*, Mich. Math. J. vol. 31 (1974), pp. 33–44.

[KB] L. Kauffman and T. Banchoff, *Immersions and mod-2 quadratic forms*, Amer. Math. Monthly, vol. 84 (1977), pp. 168–185.

[L] W. B. R. Lickorish, *A finite set of generators for the homeotopy group of a 2-mainfold*, Proc. Cambridge Philos. Soc., vol. 60 (1964), pp. 769–778.

[W] H. Whitney, *On regular closed curves in the plane*, Compositio Math., vol. 4 (1937), pp. 276–284.

The University of Illinois at Chicago Circle
Chicago, Illinois