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GEOMETRY NEAR A C.R. SINGULARITY

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This paper is concerned with the local geometry of a real-analytic submanifold of $\mathbf{C}^n$ in the vicinity of a C.R. singular point. The concept of local generic embeddability near a C.R. singularity is introduced in Section 1. Unlike the C.R. situation, not all non-C.R. real-analytic submanifolds of $\mathbf{C}^n$ are locally generically embeddable. Hence a new concept, pseudo-generic embeddability, is developed. The section concludes with a description of generic embeddability in the context of the theory of analytic local rings. Unfortunately this description is in terms of some dimensions which are not generally computable by known methods. Therefore the emphasis of this paper is Sections 2 and 3 where a simple scheme is developed to study these concepts for two-dimensional real-analytic submanifolds of $\mathbf{C}^3$.

In Section 2 the problem of determining all generically embeddable real-analytic two-dimensional submanifolds of $\mathbf{C}^3$ is solved by "reducing" the given submanifold to a special form, Form VI, for which the solution is trivial, Theorem 2.6 and Corollary 2.7. It is interesting that this reduction process need not be biholomorphic. It must only not affect certain algebraic properties which are necessary and sufficient for generic embeddability. This is made precise in the detailed development which follows.

In Section 3 a further reduction in form Form IX, yields numerical invariants which determine a large class of pseudo-generically embeddable submanifolds which are not generically embeddable, Theorem 3.6 and Theorem 3.7. Moreover these invariants provide an explicit calculation of the order at the origin of the smallest complex subvariety into which the submanifold can be holomorphically embedded, Proposition 3.8.

1. Let M denote the germ at 0 of a k-dimensional real-analytic submanifold of $\mathbf{C}^n$. We will make no notational distinction between M the germ, and M a particular representative. We will use the same abuse of notation for holomorphic functions and mappings. Let Φ be the germ at 0 of a real-analytic parametrization of M and let $\tilde{\Phi}$ be the complexification of Φ. Assume $\Phi(0) = 0$.

For convenience we will say that $\tilde{\Phi}$ is *associated with* M, denoted $\tilde{\Phi} \sim M$, and *the rank of* $\tilde{\Phi}$, denoted $rk\tilde{\Phi}$, is defined as the maximum value of $rk[\partial\tilde{\phi}_i/\partial z_j(z)]$

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for z arbitrarily near 0. In [3] the author shows that $rk\Phi$ depends uniquely on M up to biholomorphic equivalence. Hence the following definitions are reasonable.

(i) M is *C.R. at* 0 provided $rk[\partial\tilde{\phi}_i/\partial z_j(\cdot)]$ is constant arbitrarily near 0.
(ii) M (not necessarily C.R.) is *generic in* $\mathbf{C}^n$ provided $rk\tilde{\Phi} = n$.

It is also shown in [3] that (i) agrees with the definition of C.R. usually found in the literature and (ii) means that M is generic in the usual sense off its C.R. singular set.

The problem of determining the local restrictions to M of ambient holomorphic functions is solved in [3] assuming the hypothesis "M is generic in $\mathbf{C}^n$". Hence the following question is natural.

Question. Suppose $rk\tilde{\Phi} = s < n$. Does there exist a biholomorphic mapping $H: \mathbf{C}^n \to \mathbf{C}^n$ such that $H(M) \subset \mathbf{C}^s \times \{0\} \subset \mathbf{C}^n$?

DEFINITION. If such H exists we say that M is *generically embeddable at* 0.

Results of Tomassini [5] yield that M C.R. at 0 implies M is generically embeddable at 0. However this is not necessarily the case in general, as demonstrated by the following example which provided the initial motivation for this study.

Example 1.1. $M \equiv \{(x + iy, (x + iy)y, (x + iy)ye^y) \mid (x, y) \in \mathbf{R}^2\}$.

As seen in [3], M is a local uniqueness set for holomorphic functions on $\mathbf{C}^3$ and hence is not generically embeddable.

Let $\mathbf{C}_n$ denote the ring of convergent power series centered at $0 \in \mathbf{C}^n$ with indeterminates $z_1, \ldots, z_n$. We will also think of $\mathbf{C}_n$ as the germ at 0 of holomorphic functions on $\mathbf{C}^n$. For any n-tuple $\Phi \equiv (\phi_1, \ldots, \phi_n)$ with $\phi_i \in \mathbf{C}_k$ and $\phi_i(0) = 0$ for all $i = 1, 2, \ldots, n$, let Φ_* denote the homomorphism from $\mathbf{C}_n$ to $\mathbf{C}_k$ defined by $\Phi_*(f) \equiv f \circ \Phi$ for any $f \in \mathbf{C}_n$. Clearly $f \in \mathbf{C}_n$ vanishes on M, denoted $f \in \mathcal{O}(M)$, if and only if $f \in \ker \tilde{\Phi}_*$ for $M \sim \tilde{\Phi}$.

The question of generic embeddability can now be stated somewhat more algebraically. The $\sim$ will henceforth be omitted as all maps are assumed to be holomorphic.

Question. Suppose $\Phi: \mathbf{C}^k \to \mathbf{C}^n$ is a holomorphic mapping with $\Phi(0) = 0$ and $rk\Phi = s < n$. Does there exist a biholomorphic mapping

$$H \equiv (H_1, \ldots, H_n): \mathbf{C}^n \to \mathbf{C}^n$$

such that $(H_{s+1}, \ldots, H_n)\mathbf{C}_n = \ker \Phi_*$? $(H_{s+1}, \ldots, H_n)\mathbf{C}_n$ denotes the ideal in $\mathbf{C}_n$ generated by $H_{s+1}, \ldots, H_n$.

Notice that generic embeddability in the C.R. case follows immediately from the Implicit Function Theorem.

The above question naturally leads to some standard concepts from algebraic geometry.

DEFINITIONS. Let $\mathscr{A}$ be any ideal in $\mathbf{C}_n$, $\mathscr{M}$ the maximal ideal of $\mathbf{C}_n$, and δ: $\mathscr{M} \to \mathscr{M}/\mathscr{M}^2$ the usual mapping. Define $jg\mathscr{A} \equiv \dim_{\mathbf{C}} \delta(\mathscr{A})$ and $cg\mathscr{A} \equiv \dim_{\mathbf{C}} \mathscr{A}/\mathscr{M}\mathscr{A}$ [1, pg. 100].

THEOREM 1.2 *Suppose $M \sim \Phi$. Then $jg(\ker \Phi_*) = n - s$ if and only if M is locally holomorphically embeddable in $\mathbf{C}^s$ but not in $\mathbf{C}^{s-1}$.*

Thus a purely algebraic statement of the general question can be given.

Question. Given a homomorphism $\Phi_*: \mathbf{C}_n \to \mathbf{C}_k$ under what conditions does

$$jg(\ker \Phi_*) = n - rk\Phi_*? \qquad rk\Phi_* \equiv rk(\Phi_*(w_1), \ldots, \Phi_*(w_n))$$

where $w_1, \ldots, w_n$ are the coordinates for $\mathbf{C}^n$.

In general $jg(\ker \Phi_*) \leq n - rk\Phi_*$ with Example 1.1 yielding strict inequality. An answer to this question can be viewed as a generalized implicit function theorem in the non-constant rank situation. Namely suppose $rk\Phi_* = r$. $jg(\ker \Phi_*) = n - r$ if and only if (after reindexing if necessary) for each $i = r + 1, \ldots, n$ there exists $G_i \in \mathbf{C}_r$ so that $\phi_i = G_i(\phi_1, \ldots, \phi_r)$.

Necessary and sufficient conditions for the existence of such G_i's are developed in [3, Theorem 3.5], thus answering the question. However the conditions given in [3] are so complicated as to make general application practically impossible. A tractable solution for two dimensional submanifolds of $\mathbf{C}^3$ is given in Section 2.

An interesting geometric phenomenon is exhibited by the following example.

Example 1.3. $\Phi = (z, zw, (zw)^2, zw^2, zwe^w)$.

Clearly $\ker \Phi_* = (z_3 - z_2^2, z_1 z_4 - z_2^2)\mathbf{C}_5$ and $cg(\ker \Phi_*) = 2$ while $jg(\ker \Phi_*) = 1$. Φ is associated, after a holomorphic coordinate change in $\mathbf{C}^2$, with the submanifold

$$M \equiv \{(x + iy, (x + iy)y, (x + iy)^2y^2, (x + iy)y^2, (x + iy)e^y \mid (x, y) \in \mathbf{R}^2\} \subset \mathbf{C}^5.$$

If follows that M is biholomorphically embeddable in $\mathbf{C}^4$ but not in $\mathbf{C}^3$. However, M is contained in a three dimensional complex subvariety of $\mathbf{C}^5$ but cannot be holomorphically embedded in any two dimensional complex subvariety of $\mathbf{C}^5$. We are lead to the following definition.

DEFINITION. Suppose $M \sim \Phi$. Then M is *pseudo-generically embeddable at* 0 provided $cg(\ker \Phi_*) = n - rk\Phi_*$.

Clearly generic embeddability implies pseudo-generic embeddability; however, the above example can be used to show there are no other a priori relations between these two concepts.

We can now summarize this section with the following theorem which consists of formal statements of well known results about analytic local rings [1, Chapter II] and their relation to the above discussion.

THEOREM 1.4. *Suppose* $\Phi\colon \mathbf{C}^k \to \mathbf{C}^n$ *is a holomorphic map associated with a submanifold* M *and* $rk\Phi = r$. *The following statements are equivalent:*

(i) $\mathbf{C}_n/\ker \Phi_*$ *is regular of dimension* r;
(ii) $cg(\ker \Phi_*) = jg(\ker \Phi_*) = n - r$;
(iii) $\Phi_*(\mathbf{C}_n) \approx \mathbf{C}_r$;
(iv) M *is generically embeddable at* 0.

As indicated above, the results from [3] yield a method for determining exactly when $jg(\ker \Phi_*) = n - rk\Phi_*$. However, in addition to being practically impossible to apply, the results of [3] yield no information about $cg(\ker \Phi_*)$ or about $jg(\ker \Phi_*)$ if $jg(\ker \Phi_*) < n - rk\Phi_*$. Thus in Section 2 we consider the special case of a real non-C.R. two dimensional submanifold of $\mathbf{C}^3$. In this case $0 \le jg(\ker \Phi_*) \le cg(\ker \Phi_*) \le 1$ and a given submanifold must be either generically embeddable, pseudo-generically embeddable, or a local uniqueness set. Thus examples like that arising from Example 1.3 can not exist in $\mathbf{C}^3$.

2. Henceforth M will denote the germ at 0 of a 2-dimensional real-analytic submanifold of $\mathbf{C}^3$. For submanifolds M_1 and M_2, respectively holomorphic mappings Φ_1 and Φ_2, we use the notation $M_1 \approx M_2$, respectively $\Phi_1 \approx \Phi_2$, to denote biholomorphic equivalence. We must guarantee that any conclusion we make about M is invariant, thus the following straightforward proposition will be useful.

PROPOSITION 2.1. *Suppose* $M_1 \sim \Phi_1$ *and* $M_2 \sim \Phi_2$; *then* $M_1 \approx M_2$ *if* $\Phi_1 \approx \Phi_2$.

PROPOSITION 2.2. *Suppose* $\Phi_i\colon \mathbf{C}^k \to \mathbf{C}^n$, $i = 1, 2$, *and* $H\colon \mathbf{C}^k \to \mathbf{C}^k$ *are holomorphic mappings with* $\Phi_i(0) = 0$ *and* $H(0) = 0$.

(i) *If* $H_*\colon \mathbf{C}_k \to \mathbf{C}_k$ *is injective then* $\ker \Phi_* = \ker (H_* \circ \Phi_*) = \ker (\Phi \circ H)_*$.
(ii) *If* $rkH = k$ *then* H_* *is injective.*
(iii) *If* H_* *is injective then* $\Phi_1 \approx \Phi_2$ *if and only if* $\Phi_1 \circ H \approx \Phi_2 \circ H$.

Assume without loss of generality that M (not C.R.) is parametrized by $(x + iy, P(x, y), Q(x, y))$ such that P and Q are complex valued convergent power series in the real indeterminates x and y which vanish at 0 to order 2. For $f \in \mathbf{C}_2$ let $\operatorname{ord}_0 f(z, w)$ denote the largest number d so that $f(z, w) \in \mathcal{M}^d$. M, parametrized as above, is associated with the holomorphic mapping $\Phi\colon \mathbf{C}^2 \to \mathbf{C}^2$ given by:

Form I. $\Phi = (z + iw, \phi_1, \phi_2)$ with $\operatorname{ord}_0 \phi_1(z, w) \ge 2$ and $\operatorname{ord}_0 \phi_2(z, w) \ge 2$.

We are interested in computing $jg(\ker \Phi_*)$ and $cg(\ker \Phi_*)$. The idea is to "reduce" Φ in Form I to a form for which $jg(\ker \Phi_*)$ and $cg(\ker \Phi_*)$ can easily be computed, with each "reduction" preserving $jg(\ker \Phi_*)$ and $cg(\ker \Phi_*)$. By *a reduction in form* we mean a reduction which has the following properties: following properties:

(i) If Ψ is a reduction of Φ then $jg(\ker \Psi_*) = jg(\ker \Phi_*)$ and $cg(\ker \Psi_*) = cg(\ker \Phi_*)$.
(ii) If Ψ_1 and Ψ_2 are reductions of Φ_1 and Φ_2 respectively with Φ_1 and Φ_2 of the same form then Ψ_1 and Ψ_2 are of the same form. Moreover $\Phi_1 \approx \Phi_2$ implies $\Psi_1 \approx \Psi_2$.

Holomorphic coordinate changes in $\mathbf{C}^2$ and $\mathbf{C}^3$ clearly yield reductions in form. Thus Form I can be replaced by the following:

Form II. $\Phi = (z, \phi_2, \phi_3)$ with

(i) $\operatorname{ord}_0 \phi_2(z, w) \geq 2$ and $\operatorname{ord}_0 \phi_3(z, w) \geq 2$,
(ii) $w \mid \phi_2$ and $w \mid \phi_3$.

Suppose m, respectively n, is the largest integer so that $w^m \mid \phi_2$, respectively $w^n \mid \phi_3$. If ϕ_2, respectively ϕ_3, is identically 0 we say m, respectively n, is ∞, otherwise m and n are both finite. Because we are interested in the case of non-C.R. M, either m or n must be finite. Clearly rearranging the component functions ϕ_2 and ϕ_3 is a reduction in form. Thus assume the following:

Form III. $\Phi = (z, w^m h, w^n g)$ with

(i) $w \nmid h$ and $w \nmid g$,
(ii) $1 \leq m \leq n$.

Form III yields our first numerical invariant by way of the following theorem.

THEOREM 2.3. *If $\Phi_1 = (z, w^{m_1}h_1, w^{n_1}g_1)$ and $\Phi_2 = (z, w^{m_2}h_2, w^{n_2}g_2)$ are both of Form III and $\Phi_1 \approx \Phi_2$ then $m_1 = m_2$.*

Proof. Suppose $m_1 < m_2$. Because $\Phi_1 \approx \Phi_2$ there exists $G \in \mathbf{C}_3$ such that $w^{m_1}h_1 = G(z, w^{m_2}h_2, w^{n_2}g_2)$. By assuming $m_1 < m_2$ and $w \nmid h_1$ we must have $w^{m_2} \nmid G(z, w^{m_2}h_2, w^{n_2}g_2)$, but $w^{m_2} \mid w^{n_2}g_2$ since Φ_2 is of Form III. Thus G must have a "pure" z_1 term which contradicts $w^{m_1} \mid G(z, w^{m_2}h_2, w^{m_2}g_2)$. Hence $m_1 \geq m_2$. The assumption $m_1 > m_2$ leads to a similar contradiction and the theorem is proved.

If $m = n$ in Form III we can rearrange component functions if necessary producing a further reduction in form.

Form IV. $\Phi = (z, w^m h, w^n g)$ with

(i) $w \nmid h, w \nmid g$,
(ii) $1 \leq m \leq n$,
(iii) if $m = n$ then $\operatorname{ord}_0 h(z, 0) \leq \operatorname{ord}_0 g(z, 0)$.

Notice, the integer n in Form IV is not an invariant for let $\Phi_1 = (z, w^2, w^4)$ and $\Phi_2 = (z, w^2, 0)$. Clearly $\Phi_1 \approx \Phi_2$ but $n_1 = 4$ while $n_2 = \infty$. However $\operatorname{ord}_0 h(z, 0)$ is invariant.

THEOREM 2.4. $\operatorname{ord}_0 h(z, 0)$ *in Form IV is invariant.*

Proof. Suppose $\Phi_1 = (z, w^m h_1, w^{n_1} g_1)$ and $\Phi_2 = (z, w^m h_2, w^{n_2} g_2)$ are of Form IV and $\Phi_1 \approx \Phi_2$. Thus there exist $G \in \mathbf{C}_3$ such that $w^m h_1 = G(z, w^m h_2, w^{n_2} g_2)$. Suppose $G = \sum a_\alpha z^\alpha$. (We are using the usual multi-index notation.) Clearly $a_{j,0,0} = 0$ for all $j = 0, 1, 2, \ldots, \infty$. Thus

$$h_1 = \left(\sum_{j=0}^{\infty} a_{j,1,0} z^j\right) h_2 + \left(\sum_{j=0}^{\infty} a_{j,0,1} z^j\right) w^{n_2 - m} g_2 + \mathcal{O}(w)$$

Either $n_2 > m$ or $\operatorname{ord}_0 h_2(z, 0) \leq \operatorname{ord}_0 g_2(z, 0)$. In either case

$$\operatorname{ord}_0 h_1(z, 0) \geq \operatorname{ord}_0 h_2(z, 0).$$

By similar arguments we obtain $\operatorname{ord}_0 h_1(z, 0) \leq \operatorname{ord}_0 h_2(z, 0)$.

Let k denote $\operatorname{ord}_0 h(z, 0)$ in Form IV and define $H_k\colon \mathbf{C}^2 \to \mathbf{C}^2$ by $H_k(z, w) \equiv (z, z^k w)$. H_k is not necessarily invertible. However $rkH_k = 2$, and thus Proposition 2.2 implies that a reduction in form can be obtained by replacing Φ in Form IV by $\Phi \circ H_k$.

Form V. $\Phi = (z, (wz^k)^m z^k U, (wz^k)^n g(z, z^k w))$ with

(i) $w \nmid g(z, z^k w)$,
(ii) $1 \leq m \leq n$,
(iii) if $m = n$ then $\operatorname{ord}_0 g(z, 0) \geq k$.
(iv) U is a unit (i.e. $U(0) \neq 0$).

Define $H(U)\colon \mathbf{C}^2 \to \mathbf{C}^2$ by

$$H(U)(z, w) \equiv (z, w\sqrt[3]{U(z, w)}). \tag{2.4.1}$$

$H(U)$ is biholomorphic. Thus replacing Φ in Form V by $\Phi \circ H^{-1}$ is a reduction in form yielding the following:

Form VI. $\Phi = (z, z^k w^m, w^n g(z, w))$ with

(i) $w \nmid g$,
(ii) $1 \leq m \leq n$,
(iii) if $m = n$ then $\operatorname{ord}_0 g(z, 0) \geq k$.

For notational convenience we have let k in Form VI denote the expression $k(m+1)$ in Form V.

Remark 2.5. The reductions to Form V are all reversible in the sense that $M_1 \approx M_2$ if $\Phi_1 \approx \Phi_2$. However Form VI is not. For example let

$$M_1 \equiv \{(x+iy, x^2-y^2, 0) \mid (x, y) \in \mathbf{R}^2\}$$

and

$$M_2 \equiv \{(x+iy, x^2, 0) \mid (x, y) \in \mathbf{R}^2\}.$$

The reduction to Form VI in both cases yields $\Phi = (z, (zw)z, 0)$ although $M_1 \not\approx M_2$. The reason for this is the dependence of $H(U)$ in (2.4.1) on U. Clearly going from Form V to Form VI for M_1 and M_2 loses any information contained in U.

We are now ready to answer the question of generic embeddability.

THEOREM 2.6. *Suppose g in Form VI has the power series representation $g = \sum a_{\alpha,\beta} z^\alpha w^\beta$. M is generically embeddable if and only if*

(2.6.1) $a_{\alpha,\beta} = 0$ *for all (α, β) such that either $m \nmid (\beta+n)$ or $m \mid (\beta+n)$ but $\alpha < (\beta+n)k/m$.*

Proof. Theorem 2.6 follows easily by inspection of Form VI. That is, M is generically embeddable if and only if there exists $G \in \mathbf{C}_2$ such that $w^n g(z, w) = G(z, w^m z^k)$.

If such G exists then Φ in Form VI can be replaced by $\Gamma \circ \Phi$ where $\Gamma: \mathbf{C}^3 \to \mathbf{C}^3$ is defined via $\Gamma(z_1, z_2, z_3) \equiv (z_1, z_2, z_3 - G(z_1, z_2))$. Thus Theorem 2.6 has an immediate corollary.

COROLLARY 2.7. *M is generically embeddable if and only if Φ, $M \sim \Phi$, can be reduced in form to $\Phi = (z, w^m z^k, 0)$.*

Remark 2.8. Because of Remark 2.5 we are not able to use Corollary 2.7 to classify up to biholomorphic equivalence all generically embeddable real two-submanifolds of $\mathbf{C}^3$. However we have classified all generically embeddable submanifolds of $\mathbf{C}^3$ *up to a reduction in form*. That is, M is generically embeddable if and only if M is *equivalent up to form* to $\{(x+iy, (x+iy)^k y^m, 0) \mid (x, y) \in \mathbf{R}^2\}$ for some k and m.

3. The previous section solves the problem of determining when a given non-C.R. real-analytic two-submanifold of $\mathbf{C}^3$ is generically embeddable. We know that two-submanifolds of $\mathbf{C}^3$ exist which are not generically embeddable but are pseudo-generically embeddable. In this section we continue the above "reduction in form" process to develop more numerical invariants which are relevant to the question of pseudo-generic embeddability.

The discussion leading to Corollary 2.7 provides our next reduction in form. Namely, we may "subtract off" all terms of $w^n g(z, w)$ in Form VI which satisfy (2.6.1). If every term of $w^n g(z, w)$ satisfies (2.6.1) then we have generic embeddability; thus in this section we assume some non-zero term in $w^n g(z, w)$ fails to satisfy (2.6.1). This yields the following reduction in form.

Form VII. $\Phi = (z, w^m z^k, w^n g(z, w))$ with

(i) $w \nmid g$,
(ii) $1 \leq m < n < \infty$,
(iii) $g = \sum a_{\alpha,\beta} z^\alpha w^\beta$ where $a_{\alpha,\beta} = 0$ for all (α, β) such that $m \mid (\beta + n)$ and $\alpha \geq (\beta + n)k/m$.

The integer n appearing in VII is not necessarily the same as that appearing in VI but is at least as large. In addition, the possibility for $m = n$ is ruled out by the following argument. If $m = n$ then, by Form VI, $\operatorname{ord}_0 g(z, 0) \geq k$. Moreover $w \nmid g$ implies $a_{\alpha,0} \neq 0$ for some $\alpha \geq k$ which contradicts Form VII (iii). We now have more numerical invariants.

THEOREM 3.1. *The integer n in Form VII is invariant.*

Proof. Suppose $\Phi_1 = (z, z^k w^m, w^{n_1} g_1)$ and $\Phi_2 = (z, z^k w^m, w^{n_2} g_2)$ are both of Form VII with $\Phi_1 \approx \Phi_2$. Assume $n_1 < n_2$. There exist $G \in \mathbf{C}_3$ with $w^{n_2} g_2 = G(z, z^k w^m, z^{n_1} g_1)$ and either $\partial G/\partial z_2(0) \neq 0$ or $\partial G/\partial z_3(0) \neq 0$. Thus

$$w^{n_2} g_2 = P(z) z^k w^m + Q(z) w^{n_1} g_1 + R(z, z^k w^m) + \mathcal{O}(w^{n_1 + 1}) \tag{3.1.1}$$

where $R(z_1, z_2) = \sum_{\beta \geq 2} a_{\alpha,\beta} z_1^\alpha z_2^\beta$ and either $P(z)$ or $Q(z)$ is a unit. However $m < n_1 < n_2$ implies $P(z) \equiv 0$, hence $Q(z)$ is a unit. Thus from (3.1.1) and $n_1 < n_2$ we have

$$Q(z) w^{n_1} g_1 + R(z, z^k w^m) \in \mathcal{O}(w^{n_1 + 1}). \tag{3.1.2}$$

But (3.1.2) implies the lowest degree non-zero pure z term of g_1 times w^{n_1} is a power series in z and $z^k w^m$ contradicting Form VII. If $n_1 > n_2$ a similar contradiction is obtained, hence $n_1 = n_2$.

THEOREM 3.2. $\operatorname{ord}_0 g(z, 0)$ *in Form VII is invariant.*

Proof. Suppose $\Phi_1 = (z, z^k w^m, w^n g_1)$ and $\Phi_2 = (z, z^k w^m, w^n g_2)$ are both in Form VII with $\Phi_1 \approx \Phi_2$. There exists $G \in \mathbf{C}_3$ with $w^n g_2 = G(z, z^k w^m, w^n g_1)$. Suppose $G = \sum a_\alpha z^\alpha$. Thus

$$w^n g_2 = \sum a_\alpha z^{\alpha_1 + k\alpha_2} w^{m\alpha_2 + n\alpha_3} g_1^{\alpha_3}. \tag{3.2.1}$$

Because no term of $w^n g_2$ is a power series in z and $z^k w^m$, it follows that $a_{\alpha_1, \alpha_2, 0} = 0$ for all α_1 and α_2. Thus (3.2.1) yields

$$g_2 = \sum_{\alpha_3 \geq 1} a_\alpha z^{\alpha_1 + k\alpha_2} w^{m\alpha_2 + (\alpha_3 - 1)n} g_1^{\alpha_3}.$$

Thus

$$g_2 = \left(\sum a_{\alpha_1,0,1} z^{\alpha_1}\right) g_1 + \mathcal{O}(w). \tag{3.2.2}$$

It follows from (3.2.2) that $\operatorname{ord}_0 g_2(z, 0) \geq \operatorname{ord}_0 g_1(z, 0)$. A similar argument yields $\operatorname{ord}_0 g_1(z, 0) \geq \operatorname{ord}_0 g_2(z, 0)$.

THEOREM 3.3. *Suppose g is from Form VII and μ is the largest integer so that $z^\mu | g$, then μ is invarient.*

Proof. Suppose $\Phi_1 = (z, z^k w^m, z^{\mu_1} w^n \tilde{g}_1)$ and $\Phi_2 = (z, z^k w^m, z^{\mu_2} w^n \tilde{g}_2)$ are both in Form VII with $z \nmid \tilde{g}_1$, $z \nmid \tilde{g}_2$, and $\Phi_1 \approx \Phi_2$. Let $G = \sum a_\alpha z^\alpha \in \mathbf{C}_3$ with the property that $z^{\mu_1} w^n \tilde{g}_1 \equiv G(z, z^k w^m, z^{\mu_2} w^n \tilde{g}_2)$. So

$$z^{\mu_1} w^n \tilde{g}_1 = \sum a_\alpha z^{\alpha_1 + k\alpha_2 + \mu_2 \alpha_3} w^{m\alpha_2 + n\alpha_3} \tilde{g}_2^{\alpha_3}. \tag{3.3.1}$$

As in the proof of Theorem 3.2 we have $a_{\alpha_1,\alpha_1,0} = 0$ for all α_1, α_2. Thus (3.3.1) yields

$$z^{\mu_1} \tilde{g}_1 = \sum_{\alpha_3 \geq 1} a_\alpha z^{\alpha_1 + k\alpha_2 + \mu_2 \alpha_3} w^{m\alpha_2 + (\alpha_3 - 1)n} \tilde{g}_2^{\alpha_3}. \tag{3.3.2}$$

Because z^{μ_1} divides the right side of (3.3.2) but does not divide g_2 it follows that $a_\alpha = 0$ for all α such that $\alpha_1 + k\alpha_2 + \mu_2 \alpha_3 < \mu_1$. Hence (3.3.2) yields

$$\tilde{g}_1 = \sum_{\alpha_3 \neq 0} a_\alpha z^{\alpha_1 + k\alpha_2 + \mu_2 \alpha_3 - \mu_1} w^{m\alpha_2 + (\alpha_3 - 1)n} \tilde{g}_2^{\alpha_3}. \tag{3.3.3}$$

Now $z \nmid \tilde{g}_1$; thus $a_\alpha \neq 0$ in (3.3.3) for some α with $\alpha_1 + k\alpha_2 + \mu_2 \alpha_3 - \mu_1 = 0$. That is there exist α_1, α_2 and $\alpha_3 \neq 0$ such that $\alpha_1 + k\alpha_2 + \mu_2 \alpha_3 = \mu_1$. Therefore $\mu_1 \geq \mu_2$. Similar arguments yield $\mu_2 \geq \mu_1$.

We now simply rewrite Form VII using the invariants from Theorems 3.1 and 3.3.

Form VIII. $\Phi = (z, z^k w^m, z^\mu w^n g)$ with

(i) $w \nmid g$ and $z \nmid g$,
(ii) $1 \leq m < n < \infty$,
(iii) $g = \sum a_{\alpha,\beta} z^\alpha w^\beta$ where $a_{\alpha,\beta} = 0$ for all (α, β) such that $m | (\beta + n)$ and $\alpha + \mu \geq (\beta + n)k/m$.

Theorems 3.2 and 3.3 yield an immediate corollary.

COROLLARY 3.4. $\operatorname{ord}_0 g(z, 0)$ *in Form VIII is invariant.*

Let $l = \operatorname{ord}_0 g(z, 0)$ for g from Form VIII. Define $H_l\colon \mathbf{C}^2 \to \mathbf{C}^2$ by $H_l(z, w) \equiv (z, z^l w)$. Thus H_l is uniquely determined by M and $H_{l*}\colon \mathbf{C}_2 \to \mathbf{C}_2$ is injective. As in Section 2 we can replace Φ in VIII by $\Phi \circ H_l$ to obtain:

Form IX. $\Phi = (z, z^p w^m, z^q w^n U)$ with

(i) $p \equiv k + ml$ and $q \equiv \mu + nl + l$,
(ii) $U(0) \neq 0$,
(iii) $1 \leq m < n < \infty$,
(iv) $U = \sum a_{\alpha,\beta} z^\alpha w^\beta$ where $a_{\alpha,\beta} = 0$ for all (α, β) with $m \mid (\beta + n)$ and $\alpha + q \geq (\beta + n)p/m$.

Remark 3.5. The above reduction to Form IX differs from that used in Section 2 to obtain Form V. The reason is to yield the smallest number l so that replacing Φ by $\Phi \circ H_l$ produces Form IX.

Clearly pseudo-generic embeddability of M will depend on the properties of U. In general this dependence appears quite complicated and difficult to study. However we can study a large class of submanifolds and obtain some interesting results. Let $\mathscr{P}$ denote the set of all two-submanifolds of $\mathbf{C}^3$ for which Φ in Form IX is equivalent to $\Phi \equiv (z, z^p w^m, z^q w^n)$.

THEOREM 3.6. *$M \in \mathscr{P}$ if and only if there exist $G \in \mathbf{C}_3$ such that $U = G(z, z^p w^m, z^q w^n)$. ($U$ is from Form IX.)*

Proof. ($\Leftarrow$) If such G exists then Γ: $\mathbf{C}^3 \to \mathbf{C}^3$ defined by

$$\Gamma(z_1, z_2, z_3) \equiv (z_1, z_2, z_3 G(z_1, z_2, z_3))$$

is the necessary equivalence.

($\Rightarrow$) Suppose $M \in \mathscr{P}$ and M is associated with $\Phi = (z, z^p w^m, z^q w^n U)$ of Form IX. Then there is $H = \sum a_\alpha z^\alpha \in \mathbf{C}_3$ so that

$$z^q w^n U = \sum a_\alpha z^{\alpha_1 + p\alpha_2 + q\alpha_3} w^{m\alpha_2 + n\alpha_3}. \tag{3.6.1}$$

By (iii) of Form IX we see $a_{\alpha_1,\alpha_2,0} = 0$ for all α_1, α_2; thus (3.6.1) yields

$$U = \sum_{\alpha_3 \geq 1} a_\alpha z^{\alpha_1 + p\alpha_2 + q(\alpha_3 - 1)} w^{m\alpha_2 + n(\alpha_3 - 1)}.$$

That is,

$$U = \sum_{\alpha_3 \geq 1} a_\alpha z^{\alpha_1} (z^p w^m)^{\alpha_2} (z^q w^n)^{\alpha_3 - 1}.$$

Notice that the existence of such G in Theorem 3.6 can also be explicitly determined from the coefficients of U. In particular any submanifold M associated to Φ of the form $\Phi = (z, z^p w^m, z^q w^n U(z))$ is in $\mathscr{P}$; let $G \equiv z_3 U(z_1)$. Moreover it follows from the above process that any submanifold belonging to $\mathscr{P}$ is equivalent up to a holomorphic equivalence of Form IX to

$$\{(x + iy, (x + iy)^p y^m, (x + iy)^q y^n) \mid (x, y) \in \mathbf{R}^2\}$$

for some m, n, p, q satisfying $1 \leq m < n$ and if $m \mid n$ then $q < (n/m)p$.

As indicated, members of $\mathscr{P}$ are not generically embeddable, however we will see that any submanifold contained in $\mathscr{P}$ can be holomorphically embedded in a two-dimensional complex subvariety of $\mathbf{C}^3$.

THEOREM 3.7. *If $M \in \mathscr{P}$ then M is pseudo-generically embeddable.*

Proof. Suppose $M \sim \Phi$ where $\Phi = (z, z^p w^m, z^q w^n)$ with $1 \leq m < n$ and $m \mid n$ implies $q < (n/m)p$. Let $r \equiv \gcd(m, n)$ and define $\tilde{m}$ and $\tilde{n}$ via $m = r\tilde{m}$ and $n = r\tilde{n}$. If $m \mid n$ then $r = m$, $\tilde{m} = 1$, and $\tilde{n} = n/m$. Let $k \equiv p\tilde{n} - q\tilde{m}$. Case (i): If $k > 0$ then $(z^p w^m)^{\tilde{n}} - (z^q w^n)^{\tilde{m}} z^k \equiv 0$. Thus $z_2^{\tilde{n}} - z_3^{\tilde{m}} z_1^k \in \ker \Phi_*$. Case (ii): If $k \leq 0$ then $z^{-k}(z^p w^m)^{\tilde{n}} - (z^q w^n)^{\tilde{m}} \equiv 0$. Thus $z_1^{-k} z_2^{\tilde{n}} - z_3^{\tilde{m}} \in \ker \Phi_*$. In either case, $cg(\ker \Phi_*) = 1 = 3 - rk\Phi_*$. Because a reduction in form preserves $cg(\ker \Phi_*)$ we have the desired result.

The proof of Theorem 3.7 along with a technical algebra lemma yields an interesting result.

PROPOSITION 3.8. *Suppose $M \in \mathscr{P}$ and $k = p\tilde{n} - q\tilde{m}$ as in the proof of Theorem 3.7. The smallest complex subvariety of $\mathbf{C}^3$ into which M can be holomorphically embedded is two-dimensional with singularity at 0 of order* $\min\{\tilde{n}, \tilde{m} + k\}$ *if $k > 0$ and of order* $\min\{\tilde{m}, \tilde{n} - k\}$ *if $k \leq 0$.*

LEMMA 3.9. *Suppose $f = x^n - y^m z^k$ and there are no common prime divisors of n, m, and k; then f is irreducible in $\mathbf{C}[x, y, z]$.*

(The author thanks Professor William Gustafson for producing the proof of this lemma.)

Proof. Let $\zeta = e^{2\pi i/n}$ and λ be a root of f in an extension field of $\mathbf{C}(y, z)$, the quotient field of $\mathbf{C}[y, z]$. Then $f = \prod_{t=0}^{n-1} (x - \zeta^t \lambda)$. If follows that f is irreducible in $\mathbf{C}(y, z)[x]$ and thus, by Gauss' Lemma, f is irreducible in $\mathbf{C}[x, y, z]$.

Proof of Proposition 3.8. In either Case (i), $f = z_2^{\tilde{n}} - z_3^{\tilde{m}} z_1^k$, or Case (ii), $f = z_1^{-k} z_2^{\tilde{n}} - z_3^{\tilde{m}}$, the lemma yields f irreducible in $\mathbf{C}[z_1, z_2, z_3]$. Hence as a Weierstrass polynomial f is irreducible in $\mathbf{C}_3$ [2, page 71]. Thus $\ker \Phi_* = (f)\mathbf{C}_3$ and the conclusion follows.

We conclude by recalling Example 1.1, namely

$$M = \{(x + iy), (x + iy)y, (x + iy)ye^y) \mid (x, y) \in \mathbf{R}^2\}.$$

The above reduction in form to Form IX yields

$$\Phi = (z, zw, zw^2 U) \quad \text{where } U = 1 + \frac{1}{2!}w + \frac{1}{3!}w^2 + \cdots.$$

As indicated in Section 1, $\ker \Phi_* = (0)$. The natural question at this point is: "What properties of U from Form IX cause $\ker \Phi_*$ to be the zero ideal?" We conjecture that U being a polynomial in w is sufficient to give pseudo-generic embeddability.

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