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CANONICAL GENERATORS OF FUCHSIAN GROUPS

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In [3], Purzitsky defined an element A in a Fuchsian group Γ to be a *canonical generator* if and only if A pairs the sides of some convex, locally finite fundamental domain of Γ. Every elliptic and every parabolic element in Γ is canonical and Purzitsky gives a necessary and sufficient condition for a hyperbolic element in Γ to be canonical (Theorem 1, [3]). Unfortunately, there are gaps in the proof of this theorem and Purzitsky has since then completed the proof (unpublished). The idea of a canonical generator has proved to be very significant and has been used by several writers (see, for example, [1] and [2]). The purpose of this note is to offer a simpler proof of a more general result.

Let Γ be a Fuchsian group containing a hyperbolic element A with axis $\mathscr{A}$ and let D be a fundamental domain for Γ with sides s_1 and s_2 paired by A. Note that D need not be convex and the sides of D are assumed only to be Jordan arcs. Let w be an interior point of s_1 not fixed by any elliptic element in Γ, let σ be a Jordan arc joining w to Aw in $D \cup \{w, Aw\}$ and define

$$\Sigma = \bigcup_{n=-\infty}^{+\infty} A^n(\sigma).$$

Then $V(\sigma)$ intersects σ if and only if V is I (the identity), A or A^{-1}; if $V(\Sigma)$ intersects Σ then $V = A^n$ for some integer n.

Now consider Γ as acting on the unit disc Δ. Both $\mathscr{A}$ and Σ are simple cross-cuts of Δ with endpoints x_A, y_A (the fixed points of A). Thus if $V(\mathscr{A})$ crosses $\mathscr{A}$ (that is if $V(\mathscr{A}) \cap \mathscr{A} \neq \phi$, $V(\mathscr{A}) \neq \mathscr{A}$) then $V(\Sigma)$ intersects Σ and $V = A^n$ for some n. We rewrite this as follows:

(i) if A pairs the sides s_1, s_2 and if $V \neq A^n$ for any n, then $V(\mathscr{A})$ and $\mathscr{A}$ do not cross.

Let us now relate this to Purzitsky's work. As in [3], we define

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$$

and, when x_A, y_A, Vx_A, Vy_A are all distinct, $K(A, V) = (Vx_A, y_A, x_A, Vy_A)$. We also define $K(A, V) = 1$ if V fixes x_A, y_A and $K(A, V) = 0$ if V interchanges x_A, y_A (see [3, p. 488, line 17]).

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The necessity in Theorem 1 [3], states that if A pairs the sides s_1, s_2 and if V is not a power of A, then $K(A, V) \notin (0, 1]$. In fact, the proof in [3, p. 488] shows only that $K(A, V) \notin (0,1)$ and in this part,

(ii) it is not clear that s_1, s_2 need be arcs of circles which contain the origin and

(iii) it is not clear that θ_1 and θ_2 exist.

(Both can fail for some polygons, though perhaps not for fundamental polygons).

Now $K(A, V) \in (0, 1)$ if and only if $V(\mathscr{A})$ crosses $\mathscr{A}$ (consider A to act on the upper half plane with $x_A = 0$ and $y_A = \infty$). Thus by (i), if A pairs the sides s_1, s_2 and if $V \neq A^n$ for any n, then $K(A, V) \notin (0, 1)$. This avoids justification of (ii) and (iii) and it allows the sides of D to be Jordan arcs.

It is still necessary to show that $K(A, V) \neq 1$. If $K(A, V) = 1$, then V fixes x_A and y_A. It is then easy to see that $V(\Sigma)$ must meet Σ. Thus $V = A^n$ for some n. This completes the proof of the "necessity" of Theorem 1.

References

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