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SINGULAR MEASURES AND TENSOR ALGEBRAS

BY

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Let X and Y be two compact (Hausdorff) spaces, and let

$$V = V(X, Y) = C(X) \otimes C(Y)$$

be the tensor algebra over X and Y [8]. We denote by $V^{\sim}$ the space of all $f \in C(X \times Y)$ for which there exists a sequence (f_n) in V such that $f_n \to f$ uniformly and $\sup_n \|f_n\|_V < \infty$. Then $V^{\sim}$ forms a Banach algebra with norm $\|f\|_{V^{\sim}} = \inf \sup_n \|f_n\|_V$, where the infimum is taken over all sequences (f_n) as above (cf. [9] and [10]). The algebra $V^{\sim}$ is often called the tilde algebra associated with V. Notice that the natural imbedding of V into $V^{\sim}$ is an isometric homomorphism (cf. Theorem 4.5 of [5]).

For infinite compact spaces X and Y, C. C. Graham [1] constructs a function $f \in V^{\sim}\backslash V$ such that $f^n \in V$ for all $n \geq 2$. In the present note, we shall prove that a natural analog of Theorem 2.4 of [7] holds for V. Let r be a natural number and let E be a subset of $\mathbf{Z}_+^r$. As in [7], we shall say that E is *dominative* if (a) it contains all the unit vectors $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ and (b) whenever $(m_j) \in \mathbf{Z}_+^r$, $(n_j) \in E$, and $m_j \leq n_j$ for all indices j, then $(m_j) \in E$.

THEOREM. *Let X and Y be two infinite compact spaces, and let E be a dominative subset of $\mathbf{Z}_+^r$. Then there exist functions $f_1, \ldots, f_r$ in $V^{\sim}$ such that*

(a) $f_1^{m_1} \cdots f_r^{m_r} \notin V$ *if* $(m_j) \in E\backslash\{0\}$,
(b) $f_1^{n_1} \cdots f_r^{n_r} \in V$ *if* $(n_j) \in \mathbf{Z}_+^r \backslash E$.

In order to prove this, let Γ be a locally compact abelian group with dual G. We denote by $A(\Gamma) = M_a(G)^{\wedge}$ the Fourier algebra of Γ(cf. [3] and [4]).

LEMMA 1. *Let Γ be an infinite locally compact abelian group, let F be a finite dominative set in $\mathbf{Z}_+^r$, and let $\eta > 0$. Then there exist $f_1, \ldots, f_r \in A(\Gamma)$ such that*

(i) $\|f_j\|_{A(\Gamma)} < 3$ *and* $\|f_j\|_\infty < \eta$ *for all indices j*,
(ii) $\|f_1^{m_1} \cdots f_r^{m_r}\|_{A(\Gamma)} > 1$ *if* $(m_j) \in F\backslash\{0\}$,
(iii) $\|f_1^{n_1} \cdots f_r^{n_r}\|_{A(\Gamma)} < \eta$ *if* $(n_j) \in \mathbf{Z}_+^r \backslash F$.

Proof. We may assume that $\eta < 1$. We first deal with the case where Γ is discrete or, equivalently, G is compact (and infinite). By Theorem 2.4 of [7], there exist probability measures $\mu_1, \ldots, \mu_r$ in $M(G)$ such that the measure

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$\mu_1^{n_1} * \cdots * \mu_r^{n_r}$ is singular (resp. absolutely continuous) if and only if $(n_j) \in F$ (resp. $(n_j) \in \mathbf{Z}_+^r \backslash F$). Choose a compact set K in Γ such that

$$\bigcup_{j=1}^{r} \{\gamma \in \Gamma : |\hat{\mu}_j(\gamma)| \geq \eta\} \subset K. \tag{1}$$

Then there exists a measure ν in $M_a(G)$ such that $0 \leq \hat{\nu} \leq 1$ on Γ, $\hat{\nu} = 1$ on K, $\|\nu\|_{M(G)} < 1 + \eta$, and

$$\|[\mu_1 * (\delta_0 - \nu)]^{n_1} * \cdots * [\mu_r * (\delta_0 - \nu)]^{n_r}\|_{M(G)} < \eta \tag{2}$$

for all $(n_j) \in \mathbf{Z}_+^r \backslash F$, where δ_0 denotes the unit point measure at $0 \in G$. The existence of such a ν is an easy consequence of Section 2.6 of [4] and our choice of $\mu_1, \ldots, \mu_r$. (Notice that F is a finite set and $\eta < 1$.) If $(m_j) \in F \backslash \{0\}$, then the measure

$$[\mu_1 * (\delta_0 - \nu)]^{m_1} * \cdots * [\mu_r * (\delta_0 - \nu)]^{m_r} \tag{3}$$

is *not* singular and its singular part is $\mu_1^{m_1} * \cdots * \mu_r^{m_r}$. It follows that the measure in (3) has norm strictly larger than 1. Therefore there exists a measure $\tau \in M_a(G)$ such that $\|\tau\|_{M(G)} \leq 1$ and

$$\|[\mu_1 * (\delta_0 - \nu) * \tau]^{m_1} * \cdots * [\mu_r * (\delta_0 - \nu) * \tau]^{m_r}\|_{M(G)} > 1$$

for all $(m_j) \in F \backslash \{0\}$ (cf. Theorem 1.9.1 of [4]). It is now easy to check that the functions $f_j = \hat{\mu}_j(1 - \hat{\nu})\hat{\tau}$, $1 \leq j \leq r$, have the required properties.

The general case can be proved by passing to the Bohr compactification of G. Since we only need the result for discrete Γ, we omit the details.

LEMMA 2. *Given a finite dominative set F in $\mathbf{Z}_+^r$ and $\eta > 0$, there exists a finite discrete space H and $g_1, \ldots, g_r \in V(H, H)$ such that*

(i) $\|g_j\|_V < 3$ *and* $\|g_j\|_\infty < \eta$,
(ii) $\|g_1^{m_1} \cdots g_r^{m_r}\|_V > 1$ *if* $(m_j) \in F \backslash \{0\}$,
(iii) $\|g_1^{n_1} \cdots g_r^{n_r}\|_V < \eta$ *if* $(n_j) \in \mathbf{Z}_+^r \backslash F$.

Proof. Let Γ be an infinite, discrete, abelian, torsion group, and let $f_1, \ldots, f_r \in A(\Gamma)$ be as in Lemma 1. There is no loss of generality in assuming that every f_j has finite support. (Notice that the measure τ in the proof of Lemma 1 can be chosen so that $\hat{\tau}$ has finite support.) Since Γ is a torsion group, we can find a finite subgroup H of Γ such that $f_j = 0$ outside of H for all $j = 1, \ldots, r$. Then the restrictions of the f_j to H satisfy conditions (i)–(iii) in Lemma 1 with H in place of Γ. We define $g_j \in V(H, H)$ by setting

$$g_j(x, y) = f_j(x + y) \quad \text{for } x, y \in H \quad (j = 1, 2, \ldots, r).$$

By the well-known (P, M)-mappings theorem (see [2] or [3; p. 588]), the functions g_j have the required properties.

Proof of the theorem. This is now routine. Let X, Y and $E \subset \mathbf{Z}^r_+$ be as in the hypotheses of the present theorem. For each natural number p, let H_p be any finite space for which there exist $g_{1p}, \ldots, g_{rp}$ in $V(H_p, H_p)$ satisfying the conclusions of Lemma 2 with $F = E \cap \{0, 1, \ldots, p\}^r$ and $\eta = 2^{-p}$. Let $N_p = \text{Card}\,(H_p)$. Since X is an infinite compact space, there exists a sequence (X_p) of compact subsets of X such that

$$X_p \cap \left(\text{the closure of } \bigcup_{k=p+1}^{\infty} X_k\right) = \emptyset \quad (p = 1, 2, \ldots)$$

and such that each X_p contains at least N_p interior points. Similarly there exists a sequence (Y_p) of compact subsets of Y which satisfies the same conditions as (X_p). It follows from our choice of H_p that there exist $f_{1p}, \ldots, f_{rp}$ in $V(X, Y)$ such that

(1) $\|f_{jp}\|_V < 3$ and $\|f_{jp}\|_\infty < 2^{-p}$,
(2) $\|f_{1p}^{m_1} \cdots f_{rp}^{m_r}\|_V > 1$ if $0 \neq (m_j) \in E \cap \{0, 1, \ldots, p\}^r$,
(3) $\|f_{1p}^{n_1} \cdots f_{rp}^{n_r}\|_V < 2^{-p}$ if $(n_j) \in \mathbf{Z}^r_+ \backslash E$,
(4) $\operatorname{supp} f_{jp} \subset X_p \times Y_p$ $(j = 1, \ldots, r)$.

For the proof of this fact, the reader is referred to the proof of Lemma 4.4 in [5].

Finally we define

$$f_j = \sum_{p=1}^{\infty} f_{jp} \quad (j = 1, \ldots, r). \tag{5}$$

Notice that the series in (5) converges uniformly by (1). Moreover, the functions f_j belong to $V^{\sim}$ by (1) and (4), since the sets $X_p \times Y_p$ are pairwise "bidisjoint" (cf. [1; Lemma B], [5; Lemma 2.2] and [9; Lemma 2]). Now let (n_j) be any nonzero element of $\mathbf{Z}^r_+$, and let $g = f_1^{n_1} \cdots f_r^{n_r} \in V^{\sim}$. By (4) and (5), we then have

$$g = \sum_{p=1}^{\infty} f_{1p}^{n_1} \cdots f_{rp}^{n_r}. \tag{6}$$

Therefore (3) guarantees that g is in V if $(n_j) \notin E$. In order to prove that g is *not* in V if $(n_j) \in E$, notice that g vanishes on

$$K = \left(X \backslash \bigcup_{p=1}^{\infty} X_p^0\right) \times \left(Y \backslash \bigcup_{p=1}^{\infty} Y_p^0\right), \tag{7}$$

where D^0 denotes the interior of D, and that K is a set of synthesis for the algebra V, as is easily seen. Therefore the required conclusion is an immediate consequence of Lemma B of [1] (see also [6; Proposition 2.2]). This completes the proof.

Remarks. The functions in the theorem and in Lemma 1 can be chosen to be nonnegative. Moreover our result holds for

$$V_0 = C_0(X) \hat{\otimes} C_0(Y),$$

where X and Y are two infinite locally compact spaces (cf. [10]).

REFERENCES

1. C. C. GRAHAM, *On a Banach algebra of Varopoulos*, J. Functional Analysis, vol. 4 (1969), pp. 317–328.
2. C. S. HERZ, *Remarques sur la note procedente de M. Varopoulos*, C. R. Acad. Sci. Paris, vol. 260 (1965), pp. 6001–6004.
3. E. HEWITT and K. A. ROSS, *Abstract harmonic analysis, Vol. II, Structure and Analysis for Compact Groups; Analysis on Locally Compact Abelian Groups*, Springer-Verlag, New York, 1970.
4. W. RUDIN, *Fourier analysis on groups*, Interscience Tracts in Pure and Appl. Math. no. 12, Interscience, New York, 1962.
5. S. SAEKI, *Homomorphisms of tensor algebras*, Tôhoku Math. J., vol. 23 (1971), pp. 173–199.
6. ———, *On restriction algebras of tensor algebras*, J. Math. Soc. Japan, vol. 25 (1973), pp. 506–522.
7. ———, *Singular measures having absolutely continuous convolution powers*, Illinois J. Math., vol 21 (1977), pp. 395–412.
8. N. TH. VAROPOULOS, *Tensor algebras and harmonic analysis*, Acta Math., vol. 119 (1967), pp. 51–112.
9. ———, *On a problem of A. Beurling*, J. Functional Analysis, vol. 2 (1968), pp. 24–30.
10. ———, *Tensor algebras over discrete spaces*, J. Functional Analysis, vol. 3 (1969), pp. 321–335.

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