ILLINOIS JOURNAL OF MATHEMATICS
Volume 25, Number 3, Fall 1981

ERGODIC MEASURES, ALMOST PERIODIC POINTS AND DISCRETE ORBITS

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Let K be a compact space, and ρ be a homeomorphism of K. A set $L \subset K$ is said to be *invariant* if $\rho L \subset L$, and is said to be *minimal* if it is closed, invariant and minimal with respect to these two properties.

A point $\omega \in K$ is said to be *almost periodic* if for each neighborhood V of ω, the set $\{i \in \mathbf{N};\ \rho^i\omega \in V\}$ is relatively dense in $\mathbf{N}$. Denote by A^ρ the set of all almost periodic points. It is known that $\omega \in A$ if and only if the closure of $\{\rho^i(\omega);\ i \geq 0\}$ is minimal.

A point $\omega \in K$ will be said to be *recurrent* if it is not almost periodic and if each neighborhood V of ω, the set $\{i \in \mathbf{N};\ \rho^i(\omega) \in V\}$ is infinite. Denote by R^ρ the set of recurrent points, and denote by D^ρ the complement of $R^\rho \cup A^\rho$, that is the set of points whose orbit is discrete. The sets A^ρ, R^ρ, D^ρ are invariant.

Denote by M^ρ the set of all ρ-invariant Radon probabilities on K. It is a convex w^*-compact set, and an invariant probability μ on K is said to be *extremal* if it is extremal in M^ρ. For $\mu \in M^\rho$ and X any subset of K, we denote by $\mu^*(X)$ and $\mu_*(X)$ the outer and inner measure of X. If μ is extremal and X invariant, then for all X we have $\mu^*(X), \mu_*(X) \in \{0, 1\}$.

Let us denote by τ the map $n \to n + 1$ from $\mathbf{N}$ to $\mathbf{N}$, and again by τ the restriction to $\beta\mathbf{N}\backslash\mathbf{N}$ of its canonical extension to the Stone-Čech compactification $\beta\mathbf{N}$ of $\mathbf{N}$.

In [1], very interesting results concerning A^τ, R^τ, D^τ are proved. Our aim is to investigate, from a slightly different point of view, for an extremal $\mu \in M^\rho$, what can be the inner and outer measure of A^ρ, R^ρ, D^ρ. If the support supp μ is minimal, it is contained in A^ρ. So we have to investigate only what happens if this support is not minimal. Let

$$E^\rho = \{\mu \in M^\rho\colon \mu \text{ is extremal, supp } \mu \text{ is not minimal}\}.$$

The following result shows that if $\mu \in E^\rho$, then A^ρ is small for μ.

THEOREM 1. *Let $\mu \in E^\rho$. Then $\mu^*(A^\rho) = 0$.*

Proof. Let F be the support of μ. If F is not minimal, F contains an invariant closed G such that $G \neq F$. Let U be an open set of K such that $U \cap F \neq \phi$, $\bar{U} \cap G = \phi$. For all n let $V_n = \bigcup_{i \leq n} \rho^i(U)$. Since $V_n \cap G = \phi$, V_n

Received October 15, 1979.

does not support μ; hence $\mu(V_n) < 1$. Let $V = \bigcup_n V_n$. Since $\mu(V) > 0$ and V is invariant, $\mu(V) = 1$. For $n \in \mathbf{N}$ let

$$B_n = \{\omega \in A^\rho; \quad \forall i \in \mathbf{N},\ \tau^i(\omega) \in V_n\}.$$

Then B_n is invariant, and since $B_n \subset V_n$, $\mu^*(B_n) < 1$, and hence $\mu^*(B_n) = 0$, which shows that $\mu^*(\bigcup_n B_n) = 0$. If $\omega \in A^\rho \cap V$, then

$$H_\omega = \overline{\{\rho^i(\omega);\ i \in \mathbf{N}\}}$$

is minimal. Since $H_\omega \cap V \neq \phi$, and V is invariant, $H_\omega \backslash V$ is invariant, and since H_ω is minimal, $H_\omega \backslash V = \phi$, i.e. $H_\omega \subset V$. By compacity, there exists n such that $H_\omega \subset V_n$. Hence $A^\rho \cap V \subset \bigcup B_n$, which shows $u^*(A^\rho) = 0$ since $\mu(V) = 1$.

Q.E.D.

THEOREM 2. *If* $\mu \in E^\rho$ *then* $\mu^*(R^\rho) = 1$.

Proof. First let us notice that for each compact L with $\mu(L) > 0$, we have $\mu(\bigcup_{i>p} \rho^{-1}(L)) = 1$ since the complement of $\bigcup_{i>p} \rho^{-i}(L)$ is invariant of measure less than 1. We have to show that if $L \subset K$ is a compact such that $\mu(L) > 0$, then $L \cap R \neq \emptyset$. Let us construct by induction a decreasing sequence T_n of compacts, with $T_0 = L$ and a sequence k_n of integers with $k_n \geq n$, satisfying

$$\text{(i)}\quad \mu(T_n) > 0, \qquad \text{(ii)}\quad \rho^{k_{n+1}}(T_{n+1}) \subset T_n.$$

If T_n and k_n are constructed, since $\mu(\bigcup_{i \geq n+1} \rho^{-i}(T_n)) = 1$, there exists k_{n+1} with $\mu(\rho^{-k_{n+1}}(T_n) \cap T_n) > 0$. Then $T_{n+1} = \rho^{-k_{n+1}}(T_n) \cap T_n$ satisfies (i) and (ii). It follows now from [1, Prop. 3.1] that $\bigcap_n T_n$, hence L, contains a recurrent point.

Q.E.D.

It follows from Theorem 1 and 2 that only two possibilities exist: $\mu_*(R^\rho) = 1$ (and hence $\mu^*(D^\rho) = 0$) or $\mu_*(R^\rho) = 0$ (and hence $\mu^*(D^\rho) = 1$). We are now going to give two examples where $K = \beta\mathbf{N}\backslash\mathbf{N}$ and $\rho = \tau$ to show that both possibilities can occur. (It is known that the first possibility can occur in a metric space, for example in the shift of $\{0, 1\}$, but of course the second cannot since R^ρ is then Borel and hence measurable.)

Example 3. There exists $\mu \in E^\tau$ such that $\mu_*(R^\tau) = 1$.

Proof. Let $T \subset \mathbf{N}$ be a set such that for all n and $F \subset [0, n]$ the set

$$\bigcap_{p \in F} \tau^{-p} T \cap \bigcap_{q \in [0,n] \backslash F} \tau^{-q}(\mathbf{N} \backslash T)$$

is infinite. For example, T has this property if, for each n and $F \subset [0, n]$, there exists $m \in \mathbf{N}$ with $T \cap [m, n + m] = \tau^m(F)$.

Let s be the shift of $\{0, 1\}^\mathbf{N} = Y$, given by $s((a_n)) = (a_{n+1})$, and $\tilde{T} \in \mathscr{C}(\beta\mathbf{N})$ be the extension to $\beta\mathbf{N}$ of the characteristic function of T. Let ϕ: $\beta\mathbf{N}\backslash\mathbf{N} \to Y$ given by $\phi(\omega) = (\tilde{T}(\tau^n(\omega))$. This map is continuous, and $\phi \circ \tau = s \circ \phi$. Moreover it is onto, since if

$$\bigcap_{p \in F} \tau^{-p} T \cap \bigcap_{q \in [0,n]F} \tau^{-q}(\mathbf{N} \backslash T) \in \omega,$$

then, for $p \leq n$, $\phi(\omega)(p) = 1$ if and only if $p \in F$.

Let H be an invariant closed set of $\beta\mathbf{N}\backslash\mathbf{N}$ such that $\phi(H) = Y$, and which is minimal with respect to these three properties. Let P be the set of invariant probabilities ν on H such that $\phi(\nu) = \lambda$, where λ is the Haar measure on Y. The set P is non-empty, since if η is any probability on H such that $\phi(\eta) = \lambda$, any w^*-cluster point of $n^{-1} \sum_{i=1}^{n} \tau^i(\eta)$ belongs to P. Let μ be an extreme point of P. Then μ is an extreme point of M^τ; since s is ergodic with respect to λ, and since the support of μ is closed, invariant and such that $\phi(\text{supp } \mu) = Y$, it is equal to H by the minimality of H with respect to these properties. Moreover, H is not minimal, since under s, Y is not minimal. This shows that $\mu \in E$.

Let $\omega \in D^\tau \cap H$. Then there exist a neighborhood V of ω such that for $n \geq 1$, $\tau^n \omega \notin V$. Let $G = H \backslash \bigcup_{i \geq 0} \tau^{-i} V$. Since G is invariant and $G \neq H$, we have $\phi(G) \neq Y$. Hence there exists a clopen set $B \subset Y$ such that $\phi^{-1}(B) \subset \bigcup_{i \geq 0} \tau^{-i} V$. Hence

$$s^l(\phi(\omega)) = \phi(\tau^l(\omega)) \notin B \quad \text{for } l \geq 0,$$

for, if not, we would have $\tau^l(\omega) \in \bigcup_{i \geq 0} \tau^{-i} V$, i.e. $\tau^l(\omega) \in \tau^{-i} V$ for $i \geq 0$, $\tau^{i+l}(\omega) \in V$, which is impossible. This shows that

$$\phi(D^\tau) \cap \bigcup_{B \neq \phi} \bigcup_{l \geq 0} s^{-l}(B) = \phi$$

where the intersection is taken over all nonempty clopen B sets of Y. But for each B, $\bigcup_{l \geq 0} s^{-l}(B)$ is invariant, and of positive measure, and hence of measure 1. This shows that $\lambda^*(\phi(D^\tau)) = 0$, hence $\mu^*(D^\tau) = 0$. Q.E.D.

This result points in the same direction as Section 3 of [1]: there are many recurrent points in $\beta\mathbf{N}$.

Example. There exists $\mu \in E^\tau$ such that $\mu_*(R^\tau) = 0$ (hence $\mu^*(D^\tau) = 1$).

Proof. This example is entirely based on the theory of [2], that we shall explain briefly. For $B \subset \mathbf{N}$, let $\tilde{B}$ be the corresponding clopen set of $\beta\mathbf{N}$. For a Radon measure ν on $\beta\mathbf{N}$, consider the real function $\tilde{\nu}$ on $P(\mathbf{N})$ given by $\tilde{\nu}(B) = \nu(\tilde{A})$. Let λ be the Haar measure of $\{0, 1\}^{\mathbf{N}}$, which can be identified with $P(\mathbf{N})$. We say that ν is *measurable* if the map $\tilde{\nu}$ is λ-measurable. It then turns out that $\tilde{\nu}(B) = \frac{1}{2}\tilde{\nu}(\mathbf{N}) = \frac{1}{2}\|\nu\|$ for λ-almost all B. It is shown in [2] that:

(a) There exists $\mu \in M^\tau$ which is measurable.
(b) If η is measurable, and $\nu \leq \eta$, then ν is measurable.

From the methods of [2, 1J] one can easily show that:

(c) If ν is measurable, then $\tilde{\nu}(B \cap \tau^{-1}(\mathbf{N}\backslash B) \cap \cdots \cap \tau^{-k}(\mathbf{N}\backslash B)) = 2^{-k-1}\|\nu\|$ for λ—almost all $B \in P(\mathbf{N})$.

Now, let us show that $\mu^*(D^\tau) = 1$, i.e. that for each compact $L \subset \beta\mathbf{N}\backslash\mathbf{N}$ with $\mu(L) > 0$ we have $L \cap D^\tau \neq \phi$. Let ν be the restriction of μ to L. By (b), ν is measurable. By (c), there exists $B \subset \mathbf{N}$ such that

$$\nu(B \cap \tau^{-1}(\mathbf{N}\backslash B) \cdots \cap \tau^{-k}(\mathbf{N}\backslash B)) > 0 \quad \text{for all } k.$$

It means that for all p, $\tilde{B} \cap \tau^{-1}(\tilde{B}^c) \cap \cdots \cap \tau^{-k}(\tilde{B}^c) \cap L \neq \phi$, where $\tilde{B}^c$ is the complement of $\tilde{B}$. Now let $\omega \in L \cap \tilde{B} \cap \bigcap_{i \geq 1} \tau^{-i}(\tilde{B}^c)$. For $i \geq 1$, $\tau^i(\omega) \notin \tilde{B}$, which shows that $\omega \in D^\tau$. Q.E.D.

Remark. For a homeomorphism ρ of K, one could also consider the points $\rho^{-i}(\omega)$ for $\omega \in K$. One can define in this way left recurrent points. A slight modification of the proof of Theorem 2 shows that for $\mu \in E^\rho$ the outer measure of the set of points which are both left and right recurrent is 1. It is also possible to extend Prop. 23 of [1] in the following way: for each invariant closed set K, then $K\backslash A^\rho \subset \overline{K \cap D'}$, where D' is the set of $\omega \in K$ such that the orbit $\{\tau^i(\omega); i \in \mathbf{Z}\}$ is discrete.

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