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THE FREE BOUNDARY FOR A FOURTH ORDER VARIATIONAL INEQUALITY[1]

BY

LUIS A. CAFFARELLI, AVNER FRIEDMAN AND ALESSANDRO TORELLI

Abstract

Consider the variational inequality

$$\min_{v \in K} \left\{ \int_\Omega |\Delta v|^2 - 2 \int_\Omega fv \right\} = \int_\Omega |\Delta u|^2 - 2 \int_\Omega fu, \quad u \in K, \tag{0.1}$$

where Ω is a bounded domain in R^2 and

$$K = \{v \in H_0^2(\Omega), \alpha \le \Delta v \le \beta\} \quad (\alpha < 0 < \beta). \tag{0.2}$$

This problem was studied by Brezis and Stampacchia [3] who proved that the solution u belongs to $W_{\text{loc}}^{3,p}(\Omega)$ if $f \in L^p$ $(p > 2)$. In this paper we study the free boundary for this problem. Particular attention will be given to the case $-\alpha = \beta \to 0$. It will be shown, for a special choice of f and Ω, that $u/\beta \to w$ where w is the solution of a variational inequality for the Laplace operator with obstacle $\frac{1}{2} d^2$ and d is the distance function to $\partial\Omega$.

1. Introduction

The problem (0.1) (for Ω in R^2) has the physical interpretation of a horizontal plate whose "linearized" mean curvature is restricted to lie between two levels, α and β. The plate is clamped at the boundary and is pressured by a vertical force of magnitude f.

Throughout this paper it is assumed that Ω is a bounded domain whose boundary is piecewise $C^{2+\delta}$, for some $\delta > 0$, that is, $\partial\Omega$ consists of a finite number of disjoints $C^{2+\delta}$ arcs S_i $(1 \le i \le m)$ with endpoints V_i, V_{i+1} where $V_{m+1} = V_1$. It is also assumed that there exists a function F such that

$$F \in L^2(\Omega), \quad F = 0 \text{ on } \partial\Omega, \quad \Delta F = f; \tag{1.1}$$

the last two conditions are taken in the usual distribution sense. Thus f belongs to $H^{-2}(\Omega)$.

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The variational inequality (0.1), with K defined by (0.2), can also be written in the form

$$\int_\Omega \Delta u \cdot \Delta(v-u)\,dx \geq \int_\Omega f(v-u)\,dx, \quad v \in K;\ u \in K. \tag{1.2}$$

Since the bilinear form on the left hand side is coercive, there exists a unique solution u.

The following result is due to Brezis and Stampacchia [3].

THEOREM 1.1. *The solution u satisfies*

$$\Delta u = \tau(F + z) \tag{1.3}$$

where $\tau(t)$ *is the truncation*

$$\tau(t) = \begin{cases} \alpha & \text{if } t < \alpha, \\ t & \text{if } \alpha \leq t \leq \beta, \\ \beta & \text{if } t > \beta, \end{cases} \tag{1.4}$$

and z *is some function such that*

$$\Delta z = 0 \text{ in } \Omega, \quad z \in L^1(\Omega). \tag{1.5}$$

Thus if, in particular, $f \in H^{-1,p}(\Omega)$ $(p > 2)$ *then*

$$\Delta u \in W^{1,\infty}_{\mathrm{loc}}(\Omega), \quad u \in W^{3,p}_{\mathrm{loc}}(\Omega). \tag{1.6}$$

Actually Theorem 1.1 is proved only in case $\partial\Omega$ is sufficiently smooth. However the $L^1(\Omega)$ estimate on z is independent of the smoothness of $\partial\Omega$. Approximating Ω from inside by domains Ω_m with smooth boundary and applying (1.3) to each solution u_m of (0.1), (0.2) in Ω_m, we obtain the assertion (1.3) for u in Ω. (We use here the easily verified fact that $u_m \to u$ as $m \to \infty$.)

Theorem 1.1, and in fact all the results of Sections 1–3, are valid (with the same proofs) for n-dimensional domains Ω $(n \geq 2)$. However the proofs of the main results of this paper (Sections 4–6) definitely require that $n = 2$.

A generalization of Theorem 1.1 to more generalized operators and convex sets K is given by Torelli [11].

In Section 2 we derive some properties of the harmonic function z and study the coincidence sets

$$I_\beta = \{x \in \Omega;\ F(x) + z(x) \geq \beta\}, \quad I_\alpha = \{x \in \Omega;\ F(x) + z(x) \leq \alpha\}, \tag{1.7}$$

i.e., the sets where $\Delta u = \beta$ and $\Delta u = \alpha$ respectively.

In Section 3 we take $\alpha = -\beta$, and denote the corresponding solution by u^β. We make a preliminary study of the behavior of

$$I_{\pm\beta} \quad \text{and} \quad \frac{1}{\beta}u^\beta, \quad \text{as } \beta \to 0. \tag{1.8}$$

In Section 4 we study the second order variational inequality

$$(1.9)\qquad \int_\Omega \nabla w \cdot \nabla(v-w)\,dx \ge \int_\Omega (v-w)\,dx \quad \text{for all } v \in K_0;\ w \in K_0,$$

where

$$(1.10)\qquad K_0 = \{v \in H_0^1(\Omega);\ v(x) \le \tfrac{1}{2}\, d^2(x)\},$$

$$(1.11)\qquad d(x) = \text{dist}\,(x, \partial\Omega)$$

in the special case where Ω is a square. We find that the coincidence set I consists of four convex regions, each containing one of the sides of $\partial\Omega$; write $\Lambda = \Omega \backslash \bar{I}$ for the non-coincidence set.

In Section 5 we study the following special case of (1.8):

(1.12) Ω is a square with center $0 = (0, 0)$ and f is the Dirac measure supported at 0.

We prove that, as $\beta \to 0$,

$$(1.13)\qquad \frac{1}{\beta}u^\beta \to w, \quad I_\beta \to I, \quad I_{-\beta} \to \Lambda.$$

This statement is the main result of the paper; it is valid, with minor changes, also in case Ω is a rectangle. It encourages one to ask the intriguing question: for which pairs Ω, f do the limits in (1.8) exist and how can they be identified in terms of simpler free boundary problems. In Section 6 we answer this question in another special case, where Ω is an equilateral triangle and f is the Dirac function supported at its center. Some "negative" results on this question are given in Section 7.

2. General properties of u and z

We assume the following throughout this paper, in addition to (1.1):

(2.1) $F(x)$ is continuous in $\bar{\Omega}$ except for a finite number of points $\xi_i \in \Omega$ where $F(\xi_i) = +\infty$ or $F(\xi_i) = -\infty$.

This means that either $F(x) \to +\infty$ or $F(x) \to -\infty$ as $x \to \xi_i$.

The condition (2.1) is satisfied if $f \in H^{-1,p}(\Omega)$ where $p > 2$; it is also satisfied in the case (of special interest to us later on) where f is the Dirac function; here $f \in H^{-1,p}(\Omega)$ for any $p < 2$ but not for $p \ge 2$.

The condition (2.1) together with (1.3) imply that

(2.2) Δu is continuous in Ω.

DEFINITION. The set I_β (defined in (1.7)) is called the *upper coincidence set* and the set I_α is called the *lower coincidence set*. The set

$$\Omega_0 = \Omega\backslash(I_\alpha \cup I_\beta)$$

is called the *non-coincidence set*.

Since (2.1) holds, the sets I_β, I_α are closed with respect to Ω and the non-coincidence set Ω_0 is open. Further,

(2.3) Ω_0 is nonempty.

Indeed, if Ω_0 is empty then sgn (Δu) is constant in Ω. Since $u = 0$ on $\partial\Omega$, sgn u is also constant in Ω and the strong maximum principle gives $\partial u/\partial \nu \neq 0$ along the smooth part of $\partial\Omega$. This contradicts the fact that $u \in H_0^2(\Omega)$.

THEOREM 2.1. *The function z is uniquely determined.*

Proof. Suppose z_1, z_2 are two z functions. Then

$$\Delta u = F + z_1 = F + z_2 \quad \text{in } \Omega_0. \tag{2.4}$$

It follows that the harmonic function $z_1 - z_2$ vanishes in the nonempty open set Ω_0. Hence $z_1 - z_2 \equiv 0$ in Ω.

We shall assume from now on that

(2.5) Ω is star-shaped with respect to the origin 0.

Let

$$Z = \{v \in L^1(\Omega);\ \Delta v = 0\}. \tag{2.6}$$

LEMMA 2.2. *We have*

$$\int_\Omega v\tau(F + z)\, dx = 0, \quad v \in Z. \tag{2.7}$$

Proof. Since $\tau(F + z) = \Delta u$, (2.7) follows by integration by parts provided $v \in C^2(\bar{\Omega})$. For general v in Z notice, by (2.5), that the function

$$v_m(x) = v\left(\frac{m}{m+1}x\right) \quad (m > 1)$$

is harmonic and in $C^2(\bar{\Omega})$. Writing (2.7) for each v_m and taking $m \to \infty$, the assertion follows.

THEOREM 2.3. *If*

$$\int_\Omega \tau(F + \beta)\, dx \leq 0 \tag{2.8}$$

then

(2.9) $\bar{I}_\beta$ *intersects* $\partial\Omega$.

Proof. Indeed otherwise there exists an Ω-neighborhood N of $\partial\Omega$ such that $F + z < \beta$ in N. Hence $z < \beta$ in another (smaller) Ω-neighborhood N_0 of $\partial\Omega$. The maximum principle then implies that $z < \beta$ in Ω. Hence

$$\tau(F + z) \leq \tau(F + \beta) \text{ in } \Omega, \qquad \tau(F + z) < \tau(F + \beta) \text{ near } \partial\Omega.$$

Integrating over Ω and using Lemma 2.2, we get

$$\int_\Omega \tau(F + \beta)\, dx > \int_\Omega \tau(F + z)\, dx = 0,$$

contradicting (2.8).

Analogously to Theorem 2.3 we have:

(2.10) If $\int_\Omega \tau(F + \alpha)\, dx \geq 0$ then $\bar{I}_\alpha$ intersects $\partial\Omega$.

THEOREM 2.4. *Let $w \in Z$, $w > 0$ in Ω, and suppose γ is a constant such that $z \not\equiv \gamma$ and*

$$\int_\Omega w\tau(F + \gamma)\, dx = 0. \tag{2.11}$$

Then there exist points x^0, y^0 on $\partial\Omega$ such that

$$\overline{\lim_{x \to x^0}}\, z(x) > \gamma, \tag{2.12}$$

$$\underline{\lim_{x \to y^0}}\, z(x) < \gamma. \tag{2.13}$$

Proof. It is enough to prove (2.12). If the assertion is not true then

$$\overline{\lim_{x \to x^0}}\, z(x) \leq \gamma \quad \text{for any } x^0 \in \partial\Omega.$$

The strong maximum principle then gives $z < \gamma$ in Ω. Hence $\tau(F + z) \leq \tau(F + \gamma)$ with strict inequality on the non-coincidence set Ω_0. Multiplying this inequality by w and integrating over Ω, we get, after using Lemma 2.2 with $v = w$,

$$0 < \int_\Omega w\tau(F + \gamma)\, dx,$$

which contradicts (2.11).

THEOREM 2.5. *If $f \leq 0$ in Ω then*

$$\int_\Omega (\beta - z)\, dx \geq 0. \tag{2.14}$$

Thus the set $\bar{I}_\beta \cap \partial\Omega$ cannot be "too large."

Proof. By monotonicity of τ,

$$(\tau(F+\beta) - \tau(F+z))(\beta - z) \geq 0 \quad \text{in } \Omega.$$

Integrating over Ω and using Lemma 2.2, we get

$$\int \tau(F+\beta)(\beta - z)\, dx \geq 0.$$

Since $f \leq 0$, $F \geq 0$ and, consequently, $\tau(F+\beta) = \beta$; (2.14) thereby follows.

Similarly:

(2.15) If $f \geq 0$ in Ω then $\int_\Omega (\alpha - z)\, dx \leq 0$.

THEOREM 2.6. *Let x^0 be a point of $\partial\Omega \cap \bar{I}_\beta$ $(\partial\Omega \cap \bar{I}_\alpha)$ such that $\partial\Omega$ is not analytic in any neighborhood of x^0. Then any $\bar{\Omega}$-neighborhood of x^0 must intersect $\Omega_0 \cup I_\alpha$ $(\Omega_0 \cup I_\beta)$.*

Proof. Suppose the assertion is not true. Then, for definiteness, we may assume that in an Ω-neighborhood N of x^0, $\tau(F+z) = \beta$ and

$$x^0 \in \operatorname{Int}(\partial N \cap \partial\Omega).$$

Thus

$$\Delta u = \beta \quad \text{in } N, \tag{2.16}$$

$$u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial N \cap \partial\Omega \tag{2.17}$$

(assuming that x^0 is not a vertex). Using the hodograph mapping as in Kinderlehrer-Nirenberg [8] it follows that $\partial\Omega$ must be analytic in a neighborhood of x^0; a contradiction. Finally, x^0 cannot be a vertex; indeed, (2.16) and (2.17) (away from x^0) imply that $u > 0$ in some Ω-neighborhood of x^0, so that, by Caffarelli [4], $\partial\Omega$ must be C^1 in a neighborhood of x^0.

3. Asymptotic behavior as $-\alpha = \beta \to 0$

We now take $\alpha = -\beta$ and write $u = u^\beta$, $\tau = \tau^\beta$, $z = z^\beta$, $K = K^\beta$. We also set

$$U^\beta = \Delta u^\beta. \tag{3.1}$$

Thus

$$U^\beta = \tau^\beta(F + z^\beta). \tag{3.2}$$

LEMMA 3.1. *Let $C = 1 + 2\int_\Omega |F|\, dx$. Then*

$$|z^\beta|_{L^1(\Omega)} \leq C \quad \textit{if } 0 < \beta < 1. \tag{3.3}$$

Proof. By [3; Lemma 3.2], U^β solves the variational inequality

$$\text{(3.4)}\quad \int_\Omega U^\beta(V - U^\beta)\,dx \geq \int_\Omega (F + z^\beta)(V - U^\beta)\,dx, \quad V \in K_0^\beta,\ U^\beta \in K_0^\beta$$

where

$$\text{(3.5)}\quad K_0^\beta = \{V \in L^2(\Omega),\ -\beta \leq V \leq \beta\}.$$

Recalling, from Lemma 2.2, that U^β is orthogonal to z^β, we get, from (3.4),

$$\int_\Omega (F + z^\beta - U^\beta)V \leq \int_\Omega (F - U^\beta)U^\beta \leq \int_\Omega FU^\beta$$

Consequently,

$$|F + z^\beta - U^\beta|_{L^1(\Omega)} \leq \frac{1}{\beta}\int_\Omega |FU^\beta| \leq \int_\Omega |F|,$$

and (3.3) follows.

Set

$$H_\beta(t) = \begin{cases} -1 & \text{if } t < -\beta, \\ t/\beta & \text{if } -\beta \leq t \leq \beta, \\ 1 & \text{if } t > \beta, \end{cases}$$

i.e., $H_\beta(t) = \tau^1(t/\beta)$. Let

$$H(t) = \begin{cases} -1 & \text{if } t < 0, \\ [-1, 1] & \text{if } t = 0, \\ 1 & \text{if } t > 1, \end{cases}$$

be the Heaviside graph. Finally let

$$\text{(3.6)}\quad \tilde{u}^\beta = u^\beta/\beta, \quad \tilde{U}^\beta = U^\beta/\beta.$$

Thus $\tilde{U}^\beta = \Delta\tilde{u}^\beta$, $-1 \leq \tilde{U}^\beta \leq 1$, and

$$\text{(3.7)}\quad \tilde{U}^\beta = H_\beta\left(\frac{F + z^\beta}{\beta}\right).$$

Lemma 3.1 implies that from any sequence $\{\beta^*\}$ converging to zero we can extract a subsequence $\{\beta'\}$ such that

(3.8) $z^{\beta'} \to z^0$ uniformly on compact subsets of Ω,
(3.9) $\tilde{U}^{\beta'} \to \tilde{U}^0$ in the weak star topology of $L^\infty(\Omega)$,
(3.10) $\tilde{u}^{\beta'} \to \tilde{u}^0$ weakly in $W^{2,p}(\Omega_1)$, $p < \infty$,

for any subdomain Ω_1 of Ω whose boundary does not contain the vertices of $\partial\Omega$; in special cases like Ω a rectangle or Ω an equilateral triangle, we can take $\Omega_1 = \Omega$.

From (3.8) we deduce that

(3.11) z^0 is harmonic in Ω, $z^0 \in L^1(\Omega)$.

From (3.9) and Lemma 2.2 we obtain

$$\int_\Omega v\tilde{U}^0\,dx = 0, \quad v \in Z. \tag{3.12}$$

Taking $\beta = \beta' \to 0$ in (3.7) we obtain

$$\tilde{U}^0 \in H(F + z^0). \tag{3.13}$$

We wish to study the functions $\tilde{u}^0$, $\tilde{U}^0$ and the sets

$$I_+ = \{x \in \Omega;\ (F + z^0)(x) > 0\}, \tag{3.14}$$

$$I_- = \{x \in \Omega;\ (F + z^0)(x) < 0\}, \tag{3.15}$$

$$\Gamma_0 = \{x \in \Omega;\ (F + z^0)(x) = 0\}. \tag{3.16}$$

DEFINITION. I_+ is called the *upper set*, I_- is called the *lower set* and Γ_0 is called the *free boundary*.

Notice that these sets, as well as $\tilde{u}^0$, $\tilde{U}^0$, may depend in general on the sequence $\{\beta'\}$.

Now take another sequence $\{\beta''\}$ for which $z^{\beta''} \to z^*$, $\tilde{u}^{\beta''} \to \tilde{u}^*$, $\tilde{U}^{\beta''} \to \tilde{U}^*$ in the sense of (3.8)–(3.10), and define I_+^*, I_-^*, Γ_* analogously to I_+, I_-, Γ_0.

THEOREM 3.2. *The following relations hold:*

$$I_+ \subset I_+^* \cup \Gamma_*, \quad I_+^* \subset I_+ \cup \Gamma_0 \tag{3.17}$$

$$I_- \subset I_-^* \cup \Gamma_*, \quad I_-^* \subset I_- \cup \Gamma_0. \tag{3.18}$$

Proof. It is enough to prove the first part of (3.17). Since $H(t)$ is a monotone graph,

$$[H(F + z^0) - H(F + z^*)][(F + z^0) - (F + z^*)] \geq 0. \tag{3.19}$$

On the other hand, from (3.12), (3.13) and its counterpart for $\tilde{U}^*$ we get

$$\int_\Omega [H(F + z^0) - H(F + z^*)][z^0 - z^*]\,dx = 0.$$

Comparing with (3.19) we conclude that

$$[H(F + z^0) - H(F + z^*)][(F + z^0) - (F + z^*)] = 0 \quad \text{in } \Omega\backslash(\Gamma_0 \cup \Gamma_*).$$

Thus, if $(F + z^0)(x^0) > 0$ then we cannot have $(F + z^*)(x^0) < 0$. This proves the assertion.

COROLLARY 3.3. *If $y^0 \in \Gamma_0$ and* sgn $(F + z^0)$ *changes in any neighborhood of y^0, then $y^0 \in \Gamma_*$.*

Indeed, if $y^0 \notin \Gamma_*$ then $(F + z^*)(y^0) \neq 0$; suppose for definiteness that

$$(F + z^*)(y^0) > 0.$$

Then there exists a neighborhood N of y^0 in which $F + z^* > 0$, i.e., $N \subset I_+^*$. Since, by assumption, $N \cap I_- \neq \emptyset$, we get $I_- \cap I_+^* \neq \emptyset$, which contradicts the first relation in (3.18).

THEOREM 3.4. *Let x^0 be a point of $\partial\Omega$ such that $\partial\Omega$ is not analytic in any neighborhood of x^0. Then $x^0 \in \bar{\Gamma}_0$.*

The proof is similar to the proof of Theorem 2.6.

From now on we assume, in addition to (2.1), that

(3.20) $F(x)$ is analytic for all $x \in \Omega$, $x \neq \xi_i$.

Then $F + z^0$ is also analytic if $x \neq \xi_i$ and therefore Γ_0 consists of piecewise smooth curves (with branch points, in general).

THEOREM 3.5. meas $I_+ =$ meas I_-.

Proof. Take $v = 1$ in (3.12) and note that

$$\tilde{U}^0 = 1 \text{ on } I_+, \quad \tilde{U}^0 = -1 \text{ on } I_-, \quad \text{and} \quad \text{meas } \Gamma_0 = 0. \tag{3.21}$$

THEOREM 3.6. *Under the assumptions of Theorem* 3.2,

$$\text{int } \bar{I}_+ = \text{int } \overline{I_+^*}, \tag{3.22}$$

$$\text{int } \bar{I}_- = \text{int } \overline{I_-^*}. \tag{3.23}$$

This follows from Theorem 3.2 and the fact that Γ_0, Γ_* consist of piecewise smooth curves.

COROLLARY 3.7. *If F is harmonic for all $x \neq \xi_i$, then*

$$I_+ = I_+^*, \quad I_- = I_-^*, \quad \Gamma_0 = \Gamma_*. \tag{3.24}$$

Proof. If $y^0 \in \Gamma_0$ then the harmonic function $F + z^0$ must change sign in any neighborhood of y^0. Applying Corollary 3.3 we deduce that $y^0 \in \Gamma_*$. Similarly, if $y^0 \in \Gamma_*$ then $y^0 \in \Gamma_0$. Thus $\Gamma_0 = \Gamma_*$. The rest follows by Theorem 3.6.

In Section 5 we shall determine the limits in (3.8)–(3.10) and the sets (3.14)–(3.16) in the special case of (1.12). Some preliminary results needed in that section are given in Section 4.

4. A second order variational inequality

In this section and in Section 5 we always assume that Ω is a square:

$$\Omega = \{x = (x_1, x_2);\ -1 < x_1 < 1,\ -1 < x_2 < 1\}. \tag{4.1}$$

Let

$$d(x) = \text{dist } (x, \partial\Omega). \tag{4.2}$$

Consider the variational inequality

$$\int_\Omega \nabla w \cdot \nabla(v-w)\,dx \geq \int_\Omega (v-w)\,dx, \quad v \in \hat{K}, \quad w \in \hat{K} \tag{4.3}$$

where

$$\hat{K} = \{v \in H_0^1(\Omega);\ v(x) \leq \tfrac{1}{2} d^2(x)\}. \tag{4.4}$$

We recall that the variational inequality with constraint $d(x)$ (instead of $\frac{1}{2} d^2(x)$) arises in the elastic-plastic torsion problem for a bar. Some of the methods used for that problem [6] will be useful also here.

Taking $v = w^+$ in (4.3) we find that $w \geq 0$.

LEMMA 4.1. $w \in C^{1,1}(\bar{\Omega})$.

Proof. Notice that

$$\tfrac{1}{2} d^2(x) = \inf_{1 \leq i \leq 4} l_i^2(x)$$

where the $l_i(x)$ are linear functions ($l_i(x)$ is distance from the ith side of $\partial\Omega$). Consequently, for any direction ξ,

$$\frac{\partial^2}{\partial \xi^2}\left(\tfrac{1}{2} d^2(x)\right) \leq c$$

(in the distribution sense). The method of Brezis-Kinderlehrer [2] then gives $w \in C_{\mathrm{loc}}^{1,1}(\Omega)$. The $C^{1,1}$ of w up to the boundary follows by first extending w into a neighborhood of any vertex (by reflections) and then using [2].

We introduce the coincidence set

$$I = \{x \in \Omega;\ w(x) = \tfrac{1}{2} d^2(x)\}, \tag{4.5}$$

the non-coincidence set

$$\Lambda = \{x \in \Omega;\ w(x) < \tfrac{1}{2} d^2(x)\} \tag{4.6}$$

and the free boundary

$$\Gamma = \partial\Lambda \cap \Omega. \tag{4.7}$$

DEFINITION. A point $x^0 \in \Omega$ is said to belong to the *ridge* R of Ω if for any neighborhood N_0 of x^0 the function $d^2(x)$ is not in $C^{1,1}(N_0)$.

The method of [6] shows that $R \subset \Lambda$; both the definition of the ridge and the last relation are valid for general domains Ω.

LEMMA 4.2.

$$w_{x_i}(\operatorname{sgn} x_i) \leq 0 \textit{ in } \Omega \quad (i = 1, 2). \tag{4.8}$$

Proof. It is enough to prove that $w_{x_2} \geq 0$ in $\Omega_- = \Omega \cap \{x_2 < 0\}$. On $I \cap \Omega_-$ we have $w_{x_2} = (\frac{1}{2} d^2(x))_{x_2} > 0$. Next, by symmetry, $w_{x_2} = 0$ on $x_2 = 0$, whereas on the remaining part of $\partial\Omega_-$ $w_{x_2} \geq 0$; thus by the maximum principle, in $\Lambda \cap \Omega_-$, $w_{x_2} > 0$ in $\Lambda \cap \Omega_-$, and the proof is complete.

Set $0 = (0, 0)$, $A = (-1, -1)$, $B = (1, -1)$, $C = (-1, 1)$, $D = (1, 1)$, and introduce the triangle T with vertices 0, A, B.

LEMMA 4.3.

$$\frac{\partial}{\partial x_2}(w - \tfrac{1}{2} d^2) \leq 0 \quad \textit{in } T. \tag{4.9}$$

Proof. We shall show that

$$z \equiv w_{x_2} - (x_2 + 1) \leq 0 \quad \text{in } \Omega_-. \tag{4.10}$$

Notice that $z = 0$ on $I \cap T$. On the remaining part of $I \cap \Omega_-$, $w_{x_2} = 0$ so that $z \leq 0$. Also $z(x_1, 0) = w_{x_2}(x_1, 0) - 1 = -1 < 0$. Using the maximum principle we deduce that $z < 0$ in $A \cap \Omega_-$, and (4.10) follows.

LEMMA 4.4. *For any neighborhood N of any vertex of Ω,*

$$N \cap \Lambda \neq \emptyset, \quad N \cap I \neq \emptyset. \tag{4.11}$$

Proof. If $N \cap \Lambda = \emptyset$ then $w = \frac{1}{2} d^2$ in $N \cap \Omega$, contradicting Lemma 4.1. Suppose next that $N \cap I = \emptyset$. Then $\Delta w = -1$ in $N \cap \Omega$, and $w = w_\nu = 0$ on $N \cap \partial\Omega$. Reflecting w across $x_1 = -1$ we conclude, by unique continuation, that $w(x) = -\frac{1}{2}(x_2 + 1)^2$ which is impossible (since $w \geq 0$).

From Lemmas 4.2, 4.3 it follows that the coincidence set in T consists of a set

$$\{(x_1, x_2);\ -1 < x_2 < \phi(x_1),\ -a < x_1 < a\}$$

where $\phi(x_1)$ is monotone increasing if $-a < x_1 < 0$ and $\phi(-x_1) = \phi(x_1)$. Lemma 4.4 implies that $a = 1$. By a general result of Lewy and Stampacchia [9] it follows that the free boundary has analytic parametrization. Since $\bar{w}_{x_2} < 0$ in $\Lambda \cap T$, the method of Alt [1] shows that $\phi(x_1)$ is Lipschitz; hence $\phi(x_1)$ is analytic.

The coincidence set in the other three triangles OAC, OCD, ODB has the same form as in T. Thus I consists of the four shaded regions in Fig. 1, and the free boundary Γ is analytic.

THEOREM 4.5. *Each of the four components of the coincidence set is convex.*

Thus, the function $x_2 = \phi(x_1)$ representing the free boundary in T is concave.

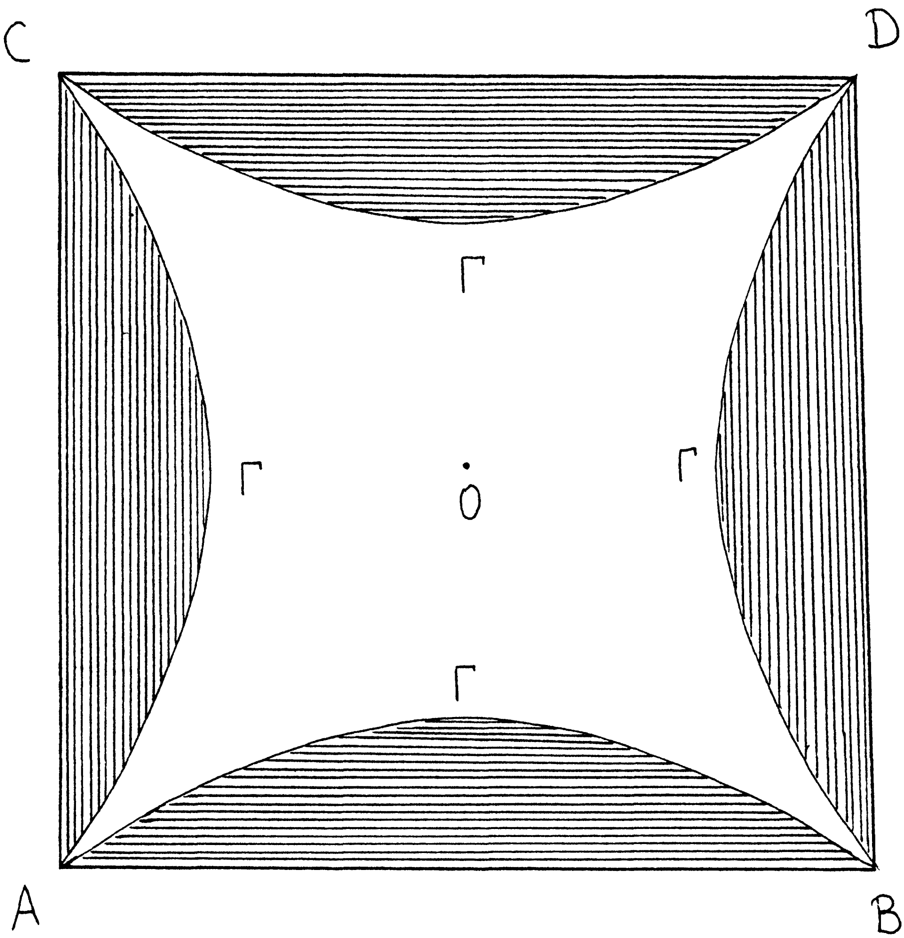

FIG. 1

Proof. If the assertion is not true then $\phi'(x_1)$ has a local maximum at some point $\bar{x}_1 \in (-1, 0)$. Then the function

$$w_{x_1x_1}(x_1, \phi(x_1)) = \frac{1}{1 + (\phi'(x_1))^2} - 1$$

has a local minimum at $\bar{x}_1$. Consider the "inflection domain" G with vertex $(\bar{x}_1, \phi(\bar{x}_1))$, i.e., a maximal connected component in Λ such that ∂G contains $(\bar{x}_1, \phi(\bar{x}_1))$ and

$$w_{x_1} < \mu \text{ in } G; \quad \mu = w_{x_1}(\bar{x}_1, \phi(\bar{x}_1)).$$

The construction of G is given in Caffarelli and Friedman [5].

G cannot lie entirely in T since on one hand $w_{x_1} = \mu$ on $\partial G \cap \Lambda$ and, on the other hand, $w_{x_1x_1} \equiv (w - \frac{1}{2} d^2)_{x_1x_1}$ (in T) cannot take a local maximum or a local minimum at any point of the free boundary $\Gamma \cap T$, by a result of Friedman and Jensen [7].

It follows that ∂G must intersect some of the other components of Γ (not in T). Using symmetry we can easily deduce that ∂G must in fact intersect $\Gamma \cap T_1$ where T_1 is the triangle OAC. But then G intersects the diagonal $\overline{AD}$ in some segment l; at least one endpoint ζ^0 of l lies in Λ.

We have

$$w_{x_1} = w_{x_2} \quad \text{on } \overline{AD}, \tag{4.12}$$

since $w(\tau x) = w(x)$ where τ is the reflection with respect to the diagonal $\overline{AD}$.

Differentiating (4.12) along l we find that $w_{x_1 x_1} = w_{x_2 x_2}$. Since $\Delta w = -1$ on l, we get $w_{x_1 w_1} = \text{const} = -\frac{1}{2}$ on l. But this is impossible, since $w_{x_1 x_1} < \mu$ in the interior of l and $w_{x_1 x_1} = \mu$ at the endpoint ζ^0 of l.

5. The limit problem in case (1.12)

We now specialize to the case (1.12), that is, Ω is the square (4.1) and f is the Dirac measure supported at 0. Thus

$$-F = \frac{1}{2\pi} \log \frac{1}{r} + h \text{ is the Green's function for } \Omega \text{ with pole at } 0; \tag{5.1}$$

h is harmonic in Ω,

$$h = -\frac{1}{2\pi} \log \frac{1}{r} \quad \text{on } \partial\Omega,$$

$r = (x^2 + y^2)^{1/2}$. Notice that F satisfies all the assumptions required in the previous sections, namely, (1.1), (2.1) and (3.20).

We shall need later on a version of the Phragmen-Lindelof theorem, which we now proceed to describe.

Let D be a domain in R^2 bounded by disjoint arcs $\gamma_0, \gamma_1, \gamma_2$ such that γ_1, γ_2 initiate at the origin 0, γ_0 lies on $r = \lambda$, for some $\lambda = 0$, D lies in the sector $0 < \theta < \pi/2, 0 < r < \lambda$.

Let ζ be a harmonic function in D such that

$$\begin{cases} \zeta \text{ is continuous in } \bar{D}\backslash\{0\}, \\ \zeta = 0 \text{ on } \gamma_1 \cup \gamma_2, \end{cases} \tag{5.2}$$

$$\zeta \in L^1(D). \tag{5.3}$$

LEMMA 5.1. *Under the foregoing assumptions,*

$$\lim_{x \in D, x \to 0} \zeta(x) = 0. \tag{5.4}$$

Proof. Introduce the region

$$T_\varepsilon = \{(r, \theta); 0 < \theta < \pi/2, \varepsilon < r < \lambda\} \quad (\varepsilon > 0)$$

and the functions w_ε satisfying:

$$\begin{aligned} &\Delta w_\varepsilon = 0 \quad \text{in } T_\varepsilon, \\ &w_\varepsilon(r, 0) = w_\varepsilon(r, \pi/2) = 0 \quad && \text{if } \varepsilon < r < \lambda, \\ &w_\varepsilon(\varepsilon, \theta) = |\zeta(\varepsilon, \theta)| && \text{if } (\varepsilon, \theta) \in D, \\ &w_\varepsilon(\varepsilon, \theta) = 0 && \text{if } (\varepsilon, \theta) \notin D, \\ &w_\varepsilon(\lambda, \theta) = C^* && \text{if } 0 < \theta < \pi/2, \end{aligned} \tag{5.5}$$

where $C^* = \sup_{(\lambda,\theta) \in \gamma_0} |\zeta(\lambda, \theta)|$. Then $w_\varepsilon \geq 0$ on $\gamma_1 \cup \gamma_2$ and, therefore, by the maximum principle,

$$w_\varepsilon \geq |\zeta| \quad \text{on } D_\varepsilon \equiv D \cap \{r > \varepsilon\}. \tag{5.6}$$

By repeated antireflections we can extend w_ε into the ring $\widetilde{T}_\varepsilon$: $\varepsilon < r < \lambda$. The extended function, say $\tilde{w}_\varepsilon$, is harmonic in $\widetilde{T}_\varepsilon$. We can write

$$\tilde{w}_\varepsilon = V_\varepsilon + W_\varepsilon \tag{5.7}$$

where V_ε, W_ε are both harmonic in $\widetilde{T}_\varepsilon$ and

$$V_\varepsilon = \begin{cases} \tilde{w}_\varepsilon & \text{on } r = \lambda, \\ 0 & \text{on } r = \varepsilon, \end{cases} \qquad W_\varepsilon = \begin{cases} 0 & \text{on } r = \lambda, \\ \tilde{w}_\varepsilon & \text{on } r = \varepsilon. \end{cases}$$

It is clear that

$$|V_\varepsilon| \leq C^*. \tag{5.8}$$

Introduce Green's function in the exterior of the disc $r < \varepsilon$:

$$G(r, \theta; \rho, \phi) = \frac{1}{4\pi} \log \frac{\varepsilon^4 - 2\varepsilon^2 r\rho \cos(\theta - \phi) + r^2\rho^2}{\varepsilon^2[\rho^2 + r^2 - 2r\rho \cos(\theta - \phi)]}.$$

By the maximum principle,

$$|W_\varepsilon(r, \theta)| \leq \int_0^{2\pi} \varepsilon |\tilde{w}_\varepsilon(\varepsilon, \phi)| \frac{\partial G}{\partial \rho}(r, \theta; \varepsilon, \phi)\, d\phi.$$

It is easy to compute that $G_\rho(2\varepsilon, \theta; \varepsilon, \phi) = O(1/\varepsilon)$ as $\varepsilon \to 0$. Hence

$$|W_\varepsilon(2\varepsilon, \theta)| \leq \frac{C}{\varepsilon} \int_0^{2\pi} \varepsilon |\tilde{w}_\varepsilon(\varepsilon, \phi)|\, d\phi \leq \frac{4C}{\varepsilon} \int \varepsilon |\zeta(\varepsilon, \phi)|\, d\phi \tag{5.9}$$

where the last integration is over the set $\delta_\varepsilon = \{\phi; (\varepsilon, \phi) \in D\}$. Since $\zeta \in L^1(D)$, the function $\varepsilon \to \int_{\delta_\varepsilon} \varepsilon |\zeta(\varepsilon, \phi)|\, d\phi$ belongs to L^1. Hence

$$\int_{\delta_{\varepsilon_n}} \varepsilon_n |\zeta(\varepsilon_n, \phi)|\, d\phi < \frac{1}{\varepsilon_n \log(1/\varepsilon_n)}$$

for a sequence $\varepsilon_n \downarrow 0$. Using this in (5.9) and recalling (5.6)–(5.8), we see that

$$\sup_{(2\varepsilon_n, \theta) \in D} |\zeta(2\varepsilon_n, \theta)| = O(1/\varepsilon_n^2) \quad (\varepsilon_n \downarrow 0).$$

This enables us to apply the usual Phragmen-Lindelof theorem [10] in order to conclude (5.4).

In order to identify the sets I_+, I_-, Γ_0 and the functions $\tilde{U}^0$, $\tilde{u}^0$, we return to the results of Section 4 and introduce Green's function G for the non-coincidence domain Λ, with a pole at 0. Thus

$$(5.10)\quad G(x) = \frac{1}{2\pi}\log\frac{1}{r} + \xi \text{ in } \Lambda,\ \xi \text{ is harmonic in } \Lambda, \text{ and } G = 0 \text{ on } \partial\Lambda.$$

Set $\Lambda_1 = \Lambda \cap \{x_2 < 0\}$ and $I_1 = I \cap T$ where T is the triangle OAB. The function $\psi(x) = \frac{1}{2}x_1^2 + w - x_2$ is harmonic in Λ_1. Therefore $\psi_{x_1} + i\psi_{x_2}$ is antianalytic in Λ_1, and the mapping

$$(5.11)\qquad \sigma\colon (x_1, x_2) \to (\psi_{x_1}, \psi_{x_2}) = (x_1 + w_{x_1}, w_{x_2} - 1)$$

is conformal; it is a special case of the mapping introduced by Lewy and Stampacchia [9].

We claim that

(5.12) σ maps Λ_1 onto I_1 in a 1-1 way.

Indeed, on the common boundary of Λ_1 and I_1 (it belongs to Γ) we have $w_{x_1} = 0$, $w_{x_2} = x_2 + 1$, and thus

$$(5.13)\qquad \sigma x = x \quad \text{on } \partial\Lambda_1 \cap \partial I_1.$$

Next, on $\partial\Lambda_1 \cap \{x_2 = 0\}$, $w_{x_2} = 0$ and thus $\sigma(x_1, 0) \subset \{x_2 = -1\}$ and on the remaining part of $\partial\Lambda_1$ (it lies on Γ) $w_{x_2} = (\frac{1}{2}d^2)_{x_2} = 0$, and again $\sigma x \subset \{x_2 = -1\}$.

Using these facts about σ and applying the argument principle, we conclude that σ maps Λ_1 onto I_1 in a 1-1 way. Notice that σ is the identity mapping on $\partial\Lambda_1 \cap \partial I_1$.

Define

$$(5.14)\qquad G_1(x) = \begin{cases} G(x) & \text{if } x \in \Lambda_1 \cup (\partial\Lambda_1 \cap \partial I_1), \\ -G(\sigma^{-1}x) & \text{if } x \in I_1. \end{cases}$$

This function is harmonic in $\Lambda_1 \cup I_1 \cup (\partial\Lambda_1 \cap \partial I_1)$; it has logarithmic singularities at the boundary points $(0, 0)$, $(0, -1)$.

In the same way we can extend G as a harmonic function into the remaining parts of I. Denote this extension by $\bar{G}$. This function has the following properties:

$$(5.15)\quad \begin{cases} \Delta\bar{G} = 0 \text{ in } \Omega\backslash\{0\}, \\ \bar{G} \text{ has logarithmic singularity at } 0 \text{ and at the points } (\pm 1, 0), (0, \pm 1), \\ \bar{G} \in L^1(\Omega), \\ \bar{G} = 0 \text{ on the free boundary } \Gamma \\ \bar{G} > 0 \text{ in } \Lambda, \\ \bar{G} < 0 \text{ in } I, \end{cases}$$

and

(5.16) $\Delta w \in H(-\bar{G})$ where H is the Heaviside graph.

LEMMA 5.2. *The function z^0 is given by*

$$z^0 = -\bar{G} - F. \tag{5.17}$$

Proof. For any harmonic function v in $C^2(\bar{\Omega})$,

$$\int_\Omega v \, \Delta w \, dx = 0, \tag{5.18}$$

by integration by parts. By approximation (cf. the proof of Lemma 2.2) we find that (5.18) holds for any $v \in Z$. Defining a function η by $-\bar{G} = F + \eta$ (η is harmonic in Ω, $\eta \in L^1(\Omega)$) and recalling (5.16), we obtain from (5.18)

$$\int_\Omega vH(F + \eta) \, dx = 0, \quad v \in Z. \tag{5.19}$$

By the monotonicity of H we have

$$[H(F + \eta) - H(F + z^0)][(F + \eta) - (F + z^0)] \geq 0.$$

Using this fact, (5.19) and Lemma 2.2, we can proceed as in Theorem 3.2 (with z^* replaced by η) and conclude that sgn $(F + \eta) =$ sgn $(F + z^0)$. Since $F + \eta = \bar{G} = 0$ on Γ, it follows that $F + z^0 = 0$ on Γ, and thus $\eta = z^0$ on Γ.

Applying Lemma 5.1 in Λ to the harmonic function $\eta - z^0$, we deduce that $\eta(x) - z^0(x) \to 0$ if x tends to a vertex of $\partial\Omega$. Hence, by the maximum principle, $\eta - z^0 \equiv 0$ in Λ; therefore also in Ω, and (5.17) is proved.

Remark. Lemma 5.2 implies that any possible limit function z^0 is uniquely determined. Hence the entire one-parameter family z^β is convergent to z^0 uniformly on compact subsets of Ω).

COROLLARY 5.3.

$$\tilde{U}^0 = \Delta w \quad in \ \Omega, \tag{5.20}$$

and hence

$$I_- = \Lambda, I_+ = I, \Gamma_0 = \Gamma. \tag{5.21}$$

Indeed, (5.20) follows from Lemma 5.2 and from (3.13), (5.16).

We can now give additional information on the free boundary Γ.

THEOREM 5.4. (a) *The two arcs of Γ initiating at each vertex of $\partial\Omega$ have tangents (at the vertex) which divide the angle of $\partial\Omega$ into three angles of equal size $\pi/6$.*

(b) *The area of Λ is equal to the area of I.*

Proof. Extend $\bar{G}$ by reflection into a neighborhood N of the vertex. Then the two arcs of Γ in N are two of the (say n) arcs (initiating at the vertex) on which $\bar{G} = 0$. Their tangents at the vertex divide 2π into n equal angles of size $2\pi/n$. This gives (a). The assertion (b) is a consequence of Theorem 3.5 and (5.21).

The final result of this section is the following:

THEOREM 5.5. *As* $\beta \to 0$,

$$\tilde{u}^{\beta} \to w \quad \text{in } W^{2,p}(\Omega) \quad (2 < p < \infty). \tag{5.22}$$

Proof. Since $\tilde{u}^{\beta}$ and w belong to $H_0^2(\Omega)$, it suffices to show that

$$\tilde{U}^{\beta} \to \tilde{U}^{0} \quad \text{in } L^p(\Omega), \tag{5.23}$$

that is,

$$\int_{\Omega} |H_{\beta}(F + z^{\beta}) - H(F + z^{0})|^p \, dx \to 0.$$

But this follows from the Lebesgue bounded convergence theorem.

Remark. Theorem 5.5 is valid also in case f is constant, say $f \equiv 1$. To prove it we only need to exhibit a function $\bar{G}$ in $L^1(\Omega)$ such that $\Delta\bar{G} = 1$ in Ω, $\bar{G} < 0$ in Λ, $\bar{G} > 0$ in I. Define

$$\alpha(x, y) = D_y[w - (1/2)(1 + y)^2].$$

Then $\Delta\alpha = 0$ in Λ_1, $\alpha < 0$ in Λ_1, $\alpha = 0$ on $\partial\Lambda_1 \cap \partial I_1$. Denote by $\tilde{\alpha}$ its harmonic continuation by means of the antireflection (5.11). Then $\tilde{\alpha} > 0$ in I_1. Let

$$A(x, y) = \int_{\phi(x)}^{y} \tilde{\alpha}(x, t)\, dt.$$

Notice that $\Delta A = -2$ in Λ_1 and

$$\frac{\partial}{\partial y}(\Delta A) = \Delta\left(\frac{\partial}{\partial y} A\right) = \Delta\tilde{\alpha} = 0 \quad \text{in } \Lambda_1 \cup I_1 \cup (\partial\Lambda_1 \cap \partial I_1);$$

consequently $\Delta A = -2$ also in I_1. Also $A = 0$ on $\partial\Lambda_1 \cap \partial I_1$, $A < 0$ in I_1. Define G by $\Delta G = 1$ in Λ, $G = 0$ on $\partial\Lambda$ and let $F = G + \frac{1}{2}A$. Then $\Delta F = 0$ in Λ_1, $F < 0$ in Λ_1, $F = 0$ on $\partial\Lambda_1 \cap \partial I_1$. Denote by $\tilde{F}$ the continuation of F by means of the antireflection (5.11); $\Delta\tilde{F} = 0$ and $\tilde{F} > 0$ in I_1. Then $\bar{G} = \tilde{F} - \frac{1}{2}A$ satisfies all the required properties in $\Lambda_1 \cup I_1 \cup (\partial\Lambda_1 \cap \partial I_1)$; the extension of $\bar{G}$ to the remaining I_j is similar.

Remark 2. All the results of Sections 4 and 5 (except for Theorem 4.5) extend with minor changes to the case where Ω is a rectangle (and f is the Dirac measure supported at the center). One can further show (using "inflection domains") that each of the four sections of the free boundary is a graph, and for a graph $x_2 = \phi(x_1)$ $(-a < x_1 < a)$, $\phi'(x_1)$ has at most one inflection point in the interval $-a < x_1 < 0$.

6. The case of an equilateral triangle

The results of Sections 4 and 5 can be extended to the case where

(6.1) Ω is an equilateral triangle and f is the Dirac measure supported at the center of Ω.

Take $A = (-1, 0)$, $B = (1, 0)$, $C = (0, \sqrt{3})$ to be the vertices of Ω. Then $D = (0, 1/\sqrt{3})$ is the center. As before, denote by w the solution of the variational inequality (4.3), (4.4).

The ridge of Ω consists of the line segments $\overline{AD}$, $\overline{BD}$, $\overline{CD}$.

The proof of Theorem 4.1 also gives, in this case,

$$w \in C^{1,1}(\bar{\Omega}); \tag{6.2}$$

near a vertex we employ several antireflections in order to extend w into a whole neighborhood of the vertex.

Next,

$$(\operatorname{sgn} x_1) w_{x_1} \leq 0; \tag{6.3}$$

the proof is by the same method as in Lemma 4.2. Also,

$$w_{x_2} \geq 0 \quad \text{in the triangle } ADB. \tag{6.4}$$

In proving (6.4) we use the fact that (since $w(\tau x) = w(x)$, τ the reflection with respect to the line containing A, D)

$$w_{x_1} = \sqrt{3} w_{x_2} \quad \text{on } \overline{AD}, \tag{6.5}$$

and therefore, in view of (6.3), $w_{x_2} \geq 0$ on $\overline{AD}$.

Introduce the function $\bar{w}(x) = w(x) - \frac{1}{2}x_2^2$. Then

$$\bar{w}_{x_2} \leq 0 \quad \text{in the triangle } T = ABD. \tag{6.6}$$

Indeed, on $\overline{AD}$ we have, by (6.5), $-\bar{w}_{x_1} + \sqrt{3}\bar{w}_{x_2} = \sqrt{3}x_2$. Applying the tangential derivative (to $\overline{AD}$) $\sqrt{3}\, \partial/\partial x_1 + \partial/\partial x_2$ to both sides and using the equation $\Delta \bar{w} + 2 = 0$, we discover the relation

$$\left(\frac{\partial}{\partial x_1} + \sqrt{3} \frac{\partial}{\partial x_2} \bar{w}_{x_2}\right) = -\sqrt{3} \quad \text{on } \overline{AD},$$

that is, $\partial \bar{w}_{x_2}/\partial \nu_1 < 0$ where $\partial/\partial \nu_1$ is some exterior derivative (to T) at the boundary points of $\overline{AD}$. Similarly, $\partial \bar{w}_{x_2}/\partial \nu_2 < 0$ on $\overline{BD}$ with another exterior derivative $\partial/\partial \nu_2$. The rest of the proof of (6.6) now follows by applying the maximum principle to $\bar{w}$ in $T \cap \Lambda$ (Λ is defined by (4.6)).

The proof of Lemma 4.4 also extends to the present case with obvious changes. We can now conclude that the coincidence set I consists of the three shaded regions in Fig. 2. In T, the free boundary Γ has the form $x_2 = \phi(x_1)$ $(-1 < x_1 < 1)$ where $\phi(x_1)$ is monotone increasing if $-1 < x_1 < 0$, $\phi(-x) = \phi(x_1)$, and ϕ is analytic.

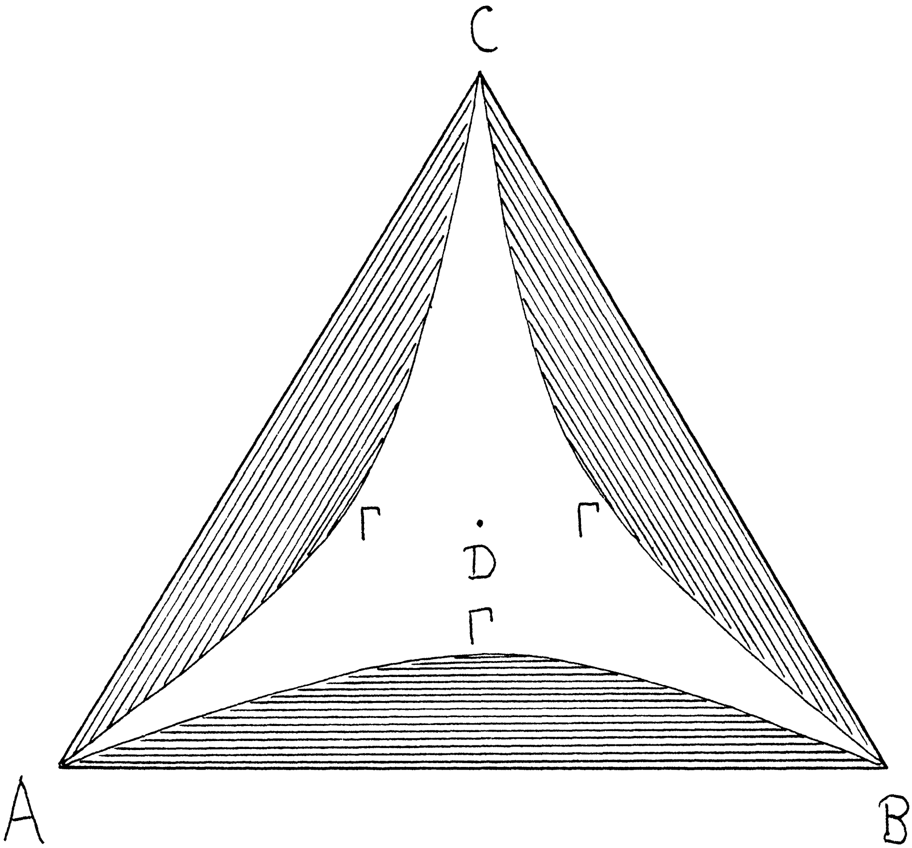

FIG. 2

We next introduce Green's function G in Λ with pole at D. We wish to extend it into Ω by a conformal mapping. It will be enough to carry out the extension into $I_1 \equiv I \cap T$. To do it, denote by E the intersection of Γ with the ray $\overrightarrow{BD}$ and by F the intersection of Γ with the ray $\overrightarrow{AD}$.

Consider the subset Λ_1 of Λ bounded by the three arcs of Γ from A to E, from B to F and from A to B, and by the two line segments $\overline{DE}$, $\overline{DF}$. We now introduce the conformal mapping $x \to \sigma x = (x_1 + w_{x_1}, w_{x_2})$. Clearly

$$\sigma x = x \quad \text{on } \partial\Lambda_1 \cap \partial I_1. \tag{6.7}$$

On $\overline{DE}$ (cf. (6.5)), $w_{x_1} + \sqrt{3} w_{x_2} = 0$ and, since $w_{x_1} \geq 0$, we get $w_{x_2} \leq 0$. Similarly $w_{x_2} \leq 0$ on $\overline{DF}$. On the arc of Γ from A to E we have $w_{x_2} = (\frac{1}{2} d^2(x))_{x_2} \leq 0$. The same holds on the part of Γ between B and F. Thus, altogether, $w_{x_2} \leq 0$ on $\partial\Lambda_1 \backslash \partial I_1$, i.e., $\sigma x \subset \{x_2 \leq 0\}$ if $x \in \partial\Lambda_1 \backslash \partial I_1$. Recalling (6.7) and using the argument principle, it follows that σ maps Λ_1 in a 1-1 way onto a domain containing I_1.

We can now repeat the remaining analysis of Section 5 and obtain the corresponding result for the present case of a triangle. Thus, setting $\bar{G}(x) = -G(\sigma^{-1}x)$ if $x \in I_1$, etc., we can state:

THEOREM 6.1. *The assertions* (5.17), (5.20), (5.21) *and* (5.22) *hold.*

Notice that two arcs of Γ initiating at the same vertex divide the angle at the vertex into three equal angles of size $\pi/9$.

7. Miscellaneous remarks

Consider the case where Ω is the square $ABCD$ as in Sections 5, 6 and let

$$E_1 = \{x;\ -1 < x_2 \le \psi_1(x_1),\ -1 < x_1 < 1\}$$

where $\psi_1(x_1)$ is any function such that $\psi_1(x_1) > -1$ and E_1 does not intersect the ridge R of Ω. Define E_2, E_3, E_4 in a similar way, using (arbitrary) functions ψ_2, ψ_3, ψ_4, and set $I_* = \bigcup_{i=1}^{4} E_i$, $\Lambda_* = \Omega\backslash\bar{I}_*$. For example, the sets I, Λ are a special case of I_*, Λ_*.

Consider the variational inequality (0.1), (0.2) with a general function f and with Ω the above square. We ask the following question: can the relations

$$I_- = \Lambda_*, \quad I_+ = I_* \tag{7.1}$$

hold for some f?

LEMMA 7.1. *If* (7.1) *holds then* $\tilde{u}^0 \equiv w$ *where* w *is the solution of* (4.3), (4.4); *consequently,* $\Lambda_* = \Lambda$ *and* $I_* = I$.

Proof. Suppose (7.1) holds. Then $\tilde{u}_y^0$ is harmonic in Λ_*. By uniqueness for the Cauchy problem, $\tilde{u}^0 = d^2/2$ in I_*; hence $\tilde{u}_{x_2}^0 \le x_2 + 1$ on $\partial\Lambda_* \cap \{x_2 < 0\}$. Also $\tilde{u}_{x_2}^0 = 0$ on $x_2 = -1$. Applying the maximum principle we get $\tilde{u}_{x_2}^0 < x_2 + 1$ in $\Lambda_* \cap \{x_2 < 0\}$. It follows that $\tilde{u}^0 \le \frac{1}{2} d^2$ in Λ_*. Finally since $\Delta\tilde{u}^0 = -1$ in Λ_*, $\tilde{u}^0$ is a solution of the same variational inequality as w; hence $\tilde{u}^0 \equiv w$.

Suppose now that

$$f(x) = \sum_{i=1}^{m} a_i\, \delta(x - \xi_i) \quad (a_i > 0,\ m \ge 1) \tag{7.2}$$

where $\delta(x)$ is the Dirac measure supported at $(0, 0)$ and

$$\xi_{i_0} \ne (0, 0) \quad \text{for at least one } i_0. \tag{7.3}$$

THEOREM 7.2. *Let* Ω *be a square with center* $(0, 0)$ *and let* f *be given by* (7.2), (7.3). *Then* (7.1) *cannot hold.*

Proof. Suppose (7.1) holds. Then, by Lemma 7.1, $\tilde{u}^0 \equiv w$, $\Lambda_* = \Lambda$, $I_* = I$. It follows that $\Delta w = \Delta\tilde{u}^0 = \tilde{U}^0$, so that $\Delta w \in H(F + z^0)$, by (3.13). We conclude that $F + z^0$ vanishes on the four arcs of Γ. Notice that the points ξ_i must all belong to I_-, hence to Λ. Suppose now, for simplicity, that ξ_{i_0} lies in Λ_1 (defined following (5.10)). Then the Lewy-Stampacchia type extension of

$F + z^0$ given by means of σ (cf. (5.11)), which we shall denote by ζ, has logarithmic singularity at the point $\sigma\xi_{i_0}$ of I_1. By unique continuation, $F + z^0$ must coincide with ζ on I_1. Consequently z^0 must also have a logarithmic singularity at $\sigma\xi_{i_0}$, a contradiction.

Remark 1. Lemma 7.1 and Theorem 7.2 extend to the case where Ω is an equilateral triangle. The proofs are similar.

Remark 2. Consider the problem (0.1), (0.2) where $\alpha = -\beta$, β is fixed and f depends on a parameter ε: $F_\varepsilon = g/\varepsilon$ ($\varepsilon \downarrow 0$). Denote the corresponding solution by u_ε and define $\bar{u}_\varepsilon = \varepsilon u_\varepsilon$. Then $\bar{u}_\varepsilon$ solves the variational inequality (0.1), (0.2) with $f = g$ and with β replaced by $\beta\varepsilon$. Thus the problem for f_ε can be reduced to the problem studied in this paper.

Remark 3. Consider the problem (0.1), (0.2) with $\alpha = -\beta$ when Ω depends on a parameter ε: $\Omega_\varepsilon = \{x/\varepsilon, x \in D\}$ ($\varepsilon \downarrow 0$). Denote the solution by u_ε and define

$$\tilde{u}_\varepsilon(x) = \varepsilon^4 u_\varepsilon(x/\varepsilon), \quad \tilde{f}_\varepsilon(x) = f(x/\varepsilon).$$

Then for $\tilde{u}_\varepsilon$ we get a variational inequality in D with f replaced by $\tilde{f}_\varepsilon$ and with β replaced by $\beta\varepsilon^2$. This problem is similar to the one studied in this paper and some of the results are applicable here.

REFERENCES

1. H. W. ALT, *The fluid flow through porous media. Regularity of the free surface*, Manuscripta Math., vol. 21 (1977), pp. 255–272.
2. H. BREZIS and D. KINDERLEHRER, *The smoothness of solutions to nonlinear variational inequalities*, Indiana Univ. Math. J., vol. 23 (1974), pp. 831–844.
3. H. BREZIS and G. STAMPACCHIA, *Remarks on some fourth order variational inequalities*, Ann. Scu. Norm. Sup. Pisa, vol. 4 (4) (1977), pp. 363–371.
4. L. A. CAFFARELLI, *The regularity of free boundaries in higher dimensions*, Acta Math., vol. 139 (1977), pp. 155–184.
5. L. A. CAFFARELLI and A. FRIEDMAN, *The dam problem with two layers*, Archive Rat. Mech. Anal., vol. 168 (1978), pp. 125–154.
6. ———, *The free boundary for elastic-plastic torsion problems*, Trans. Amer. Math. Soc., vol. 252 (1979), pp. 65–97.
7. A. FRIEDMAN and R. JENSEN, *Convexity of the free boundary in the Stefan problem and in the dam problem*, Archive Rat. Mech. Anal., vol. 67 (1978), pp. 1–24.
8. D. KINDERLEHRER and L. NIRENBERG, *Regularity in free boundary problems*, Ann. Scu. Norm. Sup. Pisa, vol. 4 (4) (1977), pp. 373–391.
9. H. LEWY and G. STAMPACCHIA, *On the regularity of solution of a variational inequality*, Comm. Pure Appl. Math., vol. 22 (1969), pp. 153–188.
10. M. H. PROTTER and H. F. WEINBERGER, *Maximum principles in differential equations*, Prentice-Hall, Englewood Cliffs, New Jersey, 1967.
11. A. TORELLI, *Some regularity results for a family of variational inequalities*, Ann. Scu. Norm. Sup. Pisa,

UNIVERSITY OF MINNESOTA
MINNEAPOLIS, MINNESOTA
NORTHWESTERN UNIVERSITY
EVANSTON, ILLINOIS
UNIVERSITY OF PAVIA
PAVIA, ITALY