ILLINOIS JOURNAL OF MATHEMATICS
Volume 28, Number 3, Fall 1984

BANACH'S CLOSED RANGE THEOREM AND FREDHOLM ALTERNATIVE THEOREM IN NON-ARCHIMEDEAN BANACH SPACES

BY

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1. Let K be a field with a non-trivial non-Archimedean valuation $|\ \ |$. Let E be a Banach space over K with norm $\|\ \ \|$. The unit ball

$$V = \{\lambda \in K : |\lambda| \leqq 1\}$$

is the valuation ring of K. Let E be a module over this ring. A nonempty subset A of E is called absolutely convex if it is a V-module of E; that is, if $a, b \in A$ and $\lambda, \mu \in V$, then $\lambda a + \mu b \in A$. A coset of an absolutely convex subset is said to be convex. A subset A of E is said to be compactoid if for every $\varepsilon > 0$ there exists a finite set $X \subset E$ such that $A \subset \{x \in E : \|x\| \leqq \varepsilon\} + \overline{C_0}X$, where $\overline{C_0}X$ denotes the closed convex hull of X (A. van Rooij [5], p. 134). The problem which we consider in this section is the following.

Let A and B be closed convex subsets of E. Under what circumstances is the subset $A + B$ closed? It is well known that if A is compact, then $A + B$ is closed. Further, A. van Rooij [5] has shown that if K is spherically complete and A is compactoid, $A + B$ is closed. By applying the results in this section to continuous linear operators, we can obtain Banach's closed range theorem and the Fredholm alternative theorem in non-Archimedean Banach space. In L. Narici, E. Beckenstein and G. Bachman [3, p. 91], the Fredholm alternative theorem is mentioned for the completely continuous operator. In Section 3, we shall extend it to compact operators as defined by A. van Rooij [6, p. 142]. The existence of the nonzero completely continuous linear operator implies that K is locally compact. However, even if K is not locally compact, there exists a nonzero compact linear operator of E to F, when E and F are Banach spaces [6, p. 182].

First we show the following result.

LEMMA 1. *Let A and B be subsets of E. If A is open and convex, then $A + B$ is closed. In particular, every open convex subset of E is closed.*

Received December 14, 1981.

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Proof. We may assume that A is absolutely convex. If $x \notin A + B$, then the subset $x + A$ is a neighborhood of x and $(x + A) \cap (A + B) = \emptyset$.

Let X and Y be subsets of E and let $0 < t \leqq 1$. We say that X is t-orthogonal to Y if for each $x \in X$ and each $y \in Y$,

$$\|x + y\| \geqq t \max(\|x\|, \|y\|).$$

The following statement shows that sums of t-orthogonal closed convex sets are closed.

PROPOSITION 2. *Let A and B be closed convex subsets of E. If A is t-orthogonal to B, then $A + B$ is closed.*

Let A and B be closed absolutely convex subsets of E and let $B_1(0)$ denote the unit ball $\{x \in E : \|x\| \leqq 1\}$. We make the following definitions:

$$p(A, B) = \sup\{|\alpha| : \alpha \in V, \alpha B_1(0) \cap \overline{(A + B)} \subset \overline{(A \cap B_1(0)) + B}\},$$

$$q(A, B) = \sup\{|\alpha| : \alpha \in V, \alpha B_1(0) \cap \overline{(A + B)} \subset (A \cap B_1(0)) + B\},$$

$$r(A, B) = \sup\{|\alpha| : \alpha \in V, \alpha B_1(0) \cap (A + B) \subset (A \cap B_1(0)) + B\}.$$

These quantities have been defined by R. Mennicken and B. Sagraloff [3] for closed linear subspaces of Banach spaces over the real number field. By modifying their proof we have the following lemma.

LEMMA 3. *If A and B are closed linear subspaces of E, then we obtain the equalities $p(A, B) = q(A, B) = r(A, B)$.*

Proof. To prove $p(A, B) \leqq q(A, B)$, take $\alpha \in V \setminus \{0\}$ such that

$$\alpha B_1(0) \cap \overline{(A + B)} \subset \overline{(A \cap B_1(0)) + B}.$$

For any $\beta \in K$ such that $|\beta| < 1$, $\beta \neq 0$, a subset $\alpha\beta B_1(0) \cap \overline{(A + B)}$ is a neighborhood of 0 in the normed space $\overline{A + B}$. Then we have the inclusion

$$\alpha B_1(0) \cap \overline{(A + B)} \subset (A \cap B_1(0)) + B + (\alpha\beta B_1(0) \cap \overline{(A + B)}).$$

Let $y_0 \in \alpha B_1(0) \cap \overline{(A + B)}$. Then we can choose

$$x_0 \in (A \cap B_1(0)) + B \quad \text{and} \quad y_1 \in \alpha\beta B_1(0) \cap \overline{(A + B)}$$

such that $y_0 = x_0 + y_1$. By induction we have two sequences $x_0, x_1, x_2, \ldots$ and $y_0, y_1, y_2, \ldots$ such that, for each i,

$$y_i = \beta^i x_i + y_{i+1}, \quad x_i \in (A \cap B_1(0)) + B$$

and

$$y_{i+1} \in \alpha\beta^{i+1} B_1(0) \cap \overline{(A + B)}.$$

Since y_i tends to 0, $y_0 = \Sigma_{i=0}^{\infty} \beta^i x_i$. Choose $u_i \in A \cap B_1(0)$ and $v_i \in B$ such that $x_i = u_i + v_i$. Since $\|x_i\| = |\beta^i|^{-1} \|y_i - y_{i+1}\| \leqq |\beta^i|^{-1} |\beta^i| = 1$ and $\|u_i\| \leqq 1$, it follows that $\|v_i\| \leqq 1$. Hence there exist $\Sigma_{i=0}^{\infty} \beta^i u_i$ and $\Sigma_{i=0}^{\infty} \beta^i v_i$. Let $u = \Sigma_{i=0}^{\infty} \beta^i u_i$ and $v = \Sigma_{i=0}^{\infty} \beta^i v_i$. Then we have $y_0 = u + v \in (A \cap B_1(0)) + B$ and it follows that

$$\alpha B_1(0) \cap \overline{(A + B)} \subset (A \cap B_1(0)) + B.$$

Therefore $p(A, B) \leqq q(A, B)$.

Next, by an elementary argument, we have

$$\overline{\rho B_1(0) \cap (A + B)} = \rho B_1(0) \cap \overline{(A + B)} \quad (\rho \in V)$$

from which we conclude that $r(A, B) \leqq p(A, B)$. The inequality $q(A, B) \leqq r(A, B)$ is trivial.

In particular, letting $\alpha = 1$ in the above argument we obtain:

PROPOSITION 4. *Let A and B be closed linear subspaces of E. Then the following conditions are equivalent.*

(1) $B_1(0) \cap \overline{(A + B)} \subset \overline{(A \cap B_1(0)) + B}$.
(2) $B_1(0) \cap \overline{(A + B)} \subset (A \cap B_1(0)) + B$.
(3) $B_1(0) \cap (A + B) \subset (A \cap B_1(0)) + B$.

Further it is easy to see the following lemma.

LEMMA 5. *Let A be an absolutely closed convex subset and let B be a closed linear subspace of E. For any $\alpha \in K$, $\alpha \neq 0$, the following conditions are equivalent.*

(1) $\alpha B_1(0) \cap (A + B) \subset (B_1(0) \cap A) + B$.
(2) $\alpha(B_1(0) + B) \cap A \subset B_1(0) + (A \cap B)$.

If A and B are closed absolutely convex subsets of E and A is t-orthogonal to B, then by Proposition 2 we have the same equality as in Lemma 3. Further we obtain the following theorem.

THEOREM 6. *Let A and B be closed absolutely convex subsets of E. If A is t-orthogonal to B, then $r(A, B) \geqq t$.*

Proof. Take $\alpha \in V$ such that $|\alpha| \leqq t$. Let $x + y \in \alpha B_1(0) \cap (A + B)$, $x \in A$ and $y \in B$. Then $|\alpha| \geqq \|x + y\| \geqq t \max(\|x\|, \|y\|)$. Hence $\|x\| \leqq 1$. So

$$x + y \in (A \cap B_1(0)) + B.$$

It follows that

$$\alpha B_1(0) \cap (A + B) \subset (A \cap B_1(0)) + B.$$

Therefore $t \leqq r(A, B)$.

We have the following theorem by a proof analogous to that in [4, p. 462].

THEOREM 7. *Let A and B be closed linear subspaces. Then $r(A, B) > 0$ if and only if $A + B$ is closed.*

Proof. If $r(A, B) > 0$, then by Lemma 3 we may choose $\beta \in K$ such that

$$q(A, B) \geqq |\beta| > 0 \quad \text{and} \quad \beta B_1(0) \cap \overline{(A + B)} \subset A + B.$$

Hence it follows that

$$\overline{(A + B)} = \bigcup_{\tau \in K} \tau(\beta B_1(0) \cap \overline{(A + B)}) \subset A + B.$$

Conversely, if $A + B$ is closed, then $A + B$ is a Banach space and it follows that

$$A + B = \bigcup_{\alpha \in K} \overline{\alpha\{(B_1(0) \cap A) + B\}}.$$

Hence there exists an $\alpha_0 \in K$ such that $\overline{\alpha_0\{(B_1(0) \cap A) + B\}}$ has an interior point x_0. Therefore we may take $\beta \in K$, $|\beta| > 0$ such that

$$x_0 + \beta\{B_1(0) \cap (A + B)\} \subset \overline{\alpha_0\{(B_1(0) \cap A) + B\}}.$$

It follows that

$$\beta B_1(0) \cap (A + B) = \beta\{B_1(0) \cap (A + B)\} \subset \overline{\alpha_0\{(B_1(0) \cap A) + B\}}.$$

Therefore $p(A, B) > 0$.

THEOREM 8. *Let A and B be closed linear subspaces such that $A \cap B = \{0\}$. Then the following conditions are equivalent.*

(1) *$A + B$ is closed.*
(2) *$r(A, B) > 0$.*
(3) *There exists t, $0 < t \leqq 1$ such that A is t-orthogonal to B.*

Proof. By Theorems 6 and 7 we have (3) $\Rightarrow$ (2) and (2) $\Leftrightarrow$ (1). We now show that (1) $\Rightarrow$ (3). Since $A \cap B = \{0\}$ and $A + B$ is closed, the closed linear subspaces are complementary to each other in the Banach space $A + B$. Therefore by Theorem [6, p. 63], there exists a positive number t such that

$$t \max(\|x\|, \|y\|) \leqq \|x + y\| \quad \text{for any } x \in A \text{ and } y \in B.$$

2. From now on we suppose that K is spherically complete and A, B are closed linear subspaces of E.

LEMMA 9. *Let π be a fixed element in K such that $0 < |\pi| < 1$. Then we have:*

1°. $A^\perp + (B_1(0))^\circ \subset (A \cap (B_1(0)))^\circ \subset A^\perp + \pi^{-1}(B_1(0))^\circ$.
2°. $\pi(B_1(0))^\circ \subset B_1'(0) \subset (B_1(0))^\circ$,

where $B_1'(0)$ denotes the subset $\{x' \in E' : \|x'\| \leqq 1\}$ of E'.
In particular, if $|K|$ is dense, then we have:

3°. $B_1'(0) = (B_1(0))^\circ$.

Proof. 1°. The inclusion

$$A^\perp + (B_1(0))^\circ \subset (A \cap (B_1(0)))^\circ$$

is clear. Let $x_0' \in (A \cap B_1(0))^\circ$. For any $x \in A$ there exists an integer n such that $|\pi|^{n+1} < \|x\| \leqq |\pi|^n$. Therefore $|x_0'(\pi^{-n}x)| \leqq 1$ and $|x_0'(x)| < |\pi|^{-1} \|x\|$. Let $\bar{x}_0'$ be the restriction of x_0' to A. Then, by Ingleton's version of Hahn-Banach theorem, we can define an extension x' of $\bar{x}_0'$ to E such that x' satisfies the inequality

$$|x'(x)| \leqq |\pi|^{-1} \|x\| \quad \text{for each } x \in E.$$

Hence we obtain $\pi x' \in (B_1(0))^\circ$ and $x' \in \pi^{-1}(B_1(0))^\circ$, so

$$x_0' = x' + (x_0' - x') \in \pi^{-1}(B_1(0))^\circ + A^\perp.$$

2°. It is clear that $B_1'(0) \subset (B_1(0))^\circ$. Let $x' \in (B_1(0))^\circ$. Then for any $\varepsilon > 0$ there exists $x_0 \in E$ such that

$$\|x'\| - \varepsilon < \frac{|x'(x_0)|}{\|x_0\|}.$$

Further there exists an integer $n \geqq 0$ such that $|\pi|^{n+1} < \|x_0\| \leqq |\pi|^n$. Hence

$$(\|x'\| - \varepsilon)\, \|(\pi^{-1})^n x_0\| < |x'((\pi^{-1})^n x_0)| \leqq 1.$$

Then

$$(\|x'\| - \varepsilon)\, \|(\pi^{-1})^{n+1} x_0\| < |\pi^{-1}|,$$

so

$$(\|x'\| - \varepsilon) < |\pi^{-1}|.$$

Since ε is arbitrary, it follows that $\|x'\| \leqq |\pi^{-1}|$ and $\pi x' \in B_1'(0)$. This means that $\pi(B_1(0))^\circ \subset B_1'(0)$.

3°. If $|K|$ is dense, then the reverse inclusion $B_1'(0) \supset (B_1(0))^\circ$ is shown using $\|x'\| = \sup\{|x'(x)|;\ 0 < \|x\| \leqq 1\}$.

THEOREM 10. *The subset $A + B$ is closed in E if and only if $A^\perp + B^\perp$ is closed in E'.*

Proof. If $A^\perp + B^\perp$ is closed in E', then by Theorem 7 we can take $\alpha \in K$ such that $r(B^\perp, A^\perp) > |\alpha| > 0$. It follows that

$$\alpha B_1'(0) \cap (B^\perp + A^\perp) \subset (B_1'(0) \cap B^\perp) + A^\perp.$$

We suppose that K is discrete. By Lemma 5 and Lemma 9.2° it follows that

$$\alpha\pi((B_1(0)^\circ + A^\perp) \cap B^\perp \subset (B_1(0))^\circ + (A^\perp \cap B^\perp).$$

Hence

$$\begin{aligned}\alpha\pi^2((B_1(0) \cap A) + B)^\circ &= \alpha\pi^2((B_1(0) \cap A)^\circ \cap B^\perp)\\ &\subset \alpha\pi^2((A^\perp + \pi^{-1}(B_1(0))^\circ) \cap B^\perp) \quad \text{(by Lemma 9.1°)}\\ &= \alpha\pi(A^\perp + B_1(0)^\circ) \cap B^\perp\\ &\subset ((B_1(0))^\circ + (A^\perp \cap B^\perp))\\ &\subset (B_1(0) \cap (A + B))^\circ.\end{aligned}$$

Hence we have

$$(B_1(0) \cap (A + B))^{\circ\circ} \subset (\alpha\pi^2)^{-1}\overline{((B_1(0) \cap A) + B)^{\circ\circ}}.$$

Since K is discrete, by J. van Tiel [7, p. 280] we have

$$(B_1(0) \cap (A + B))^{\circ\circ} \subset (\alpha\pi^2)^{-1}\overline{((B_1(0) \cap A) + B)}$$

and

$$(B_1(0) \cap (A + B))^{\circ\circ} = \overline{B_1(0) \cap (A + B)}.$$

Then

$$(\alpha\pi^2)\overline{(B_1(0) \cap (A + B))} \subset \overline{(B_1(0) \cap A) + B}.$$

Thus we conclude that

$$(\alpha\pi^2)B_1(0) \cap \overline{(A + B)} \subset \overline{(B_1(0) \cap A) + B}.$$

Therefore $0 < |\alpha\pi^2| \leqq p(A, B)$.

If $|K|$ is dense, then by using Lemma 9.3° and arguing as in the case where K is discrete, we have

$$(B_1(0) \cap (A + B))^{\circ\circ} \subset (\alpha\pi^2)^{-1}\overline{((B_1(0) \cap A) + B)^{\circ\circ}}.$$

By J. van Tiel [7, p. 281],

$$\overline{((B_1(0) \cap A) + B)^{\circ\circ}} \subset \pi^{-1}\overline{((B_1(0) \cap A) + B)}.$$

Hence

$$\begin{aligned}B_1(0) \cap (A + B) &\subset (B_1(0) \cap (A + B))^{\circ\circ}\\ &\subset (\pi^2\alpha)^{-1}\overline{((B_1(0) \cap A) + B)^{\circ\circ}}\\ &\subset (\pi^{-3}\alpha^{-1})\overline{(B_1(0) \cap A) + B)}.\end{aligned}$$

Then it follows that

$$\pi^3\alpha B_1(0) \cap \overline{(A + B)} \subset \overline{(B_1(0) \cap A) + B}.$$

This means that $0 < |\pi^3\alpha| \leqq p(A, B)$. Thus by Theorem 7, $A + B$ is closed. Since $(A^{\perp})^{\perp} = A$, the converse is trivial.

From this proof we can induce the inequality

$$|\pi|^2 p(B^{\perp}, A^{\perp}) \leqq p(A, B) \leqq |\pi|^{-3} p(B^{\perp}, A^{\perp}).$$

Moreover the following result can be also proved by Lemmas 5 and 9.

THEOREM 11. *The following are equivalent.*

(1) *$A + B$ is closed.*
(2) *$A^{\perp} + B^{\perp} = (A \cap B)^{\perp}$.*
(3) *$A + B = (A^{\perp} \cap B^{\perp})^{\perp}$.*

Proof. First we prove (1) $\Leftrightarrow$ (2). If $A + B$ is closed, then by Theorem 7 we have $r(A, B) > 0$. Hence there is an $\alpha \in K$, $\alpha \neq 0$, such that

$$\alpha B_1(0) \cap (A + B) \subset (B_1(0) \cap A) + B.$$

Then

$$\begin{aligned}((B_1(0))^{\circ} \cap B^{\perp}) + A^{\perp} &= (B_1(0) + B)^{\circ} + A^{\perp} \\ &\supset \pi((B_1(0) + B) \cap A)^{\circ} \\ &\supset \pi(\alpha^{-1}(B_1(0) + (A \cap B)))^{\circ} \quad \text{(by Lemma 5)} \\ &= \pi(\alpha^{-1}B_1(0) + (A \cap B))^{\circ} \\ &= \pi((\alpha^{-1}B_1(0))^{\circ} \cap (A \cap B)^{\perp}) \\ &= (\pi\alpha(B_1(0))^{\circ}) \cap (A \cap B)^{\perp}.\end{aligned}$$

Thus

$$\begin{aligned}A^{\perp} + B^{\perp} &= \bigcup_{\beta \in K} \beta((B_1(0))^{\circ} \cap B^{\perp}) + A^{\perp} \\ &= \bigcup_{\beta \in K} \beta(((B_1(0))^{\circ} \cap B^{\perp}) + A^{\perp}) \\ &\supset \bigcup_{\beta \in K} \beta(\pi\alpha(B_1(0))^{\circ} \cap (A \cap B)^{\perp}) \\ &= \left(\bigcup_{\beta \in K} \beta\pi\alpha(B_1(0))^{\circ}\right) \cap (A \cap B)^{\perp} \\ &= (A \cap B)^{\perp}.\end{aligned}$$

The reverse inclusion $A^{\perp} + B^{\perp} \subset (A \cap B)^{\perp}$ is trivial.

Conversely, if $A^{\perp} + B^{\perp} = (A \cap B)^{\perp}$, then $A^{\perp} + B^{\perp}$ is closed. Hence by Theorem 10 $A + B$ is closed.

Using Theorem 10 and the relation (1) $\Leftrightarrow$ (2) in this theorem, we can prove (1) $\Leftrightarrow$ (3).

3. In this section we apply the preceding results to the space of linear maps. Throughout this section let E and F be Banach spaces and $L(E, F)$

be the continuous linear maps taking E into F. The product space $E \times F$ can be normed by $\|(x, y)\| = \max(\|x\|, \|y\|)$ for $x \in E$ and $y \in F$. Then $E \times F$ is a Banach space. The dual space $(E \times F)'$ is the product space $E' \times F'$ and it is a Banach space with the norm

$$\|(x', y')\| = \max(\|x'\|, \|y'\|) \quad \text{for } x' \in E' \text{ and } y' \in F'.$$

Let T be a linear closed map (in the sense that it has a closed graph) between E and F with a dense domain $D(T)$ in E. We set $A = G(T)$, where $G(T)$ denotes the graph of T. Then A is a closed linear subspace of $E \times F$. Let T' be the conjugate of T. Then T' is closed. The set $R(T)$ is the range set of T.

THEOREM 12 (Banach's closed range theorem). *The following conditions are equivalent.*

(1) *$R(T)$ is closed.*
(2) *$R(T')$ is closed.*

Proof. Let $B = E \times \{0\}$. We have $A^{\perp} = G(-T')$ and $B^{\perp} = \{0\} \times F'$. Then $A^{\perp}$ and $B^{\perp}$ are closed in $E' \times F'$. Since

$$A + B = E \times R(T) \quad \text{and} \quad A^{\perp} + B^{\perp} = R(T') \times F',$$

the theorem is seen to follow from Theorem 10.

If T is a continuous linear map and $D(T) = E$, then T is closed. So the next corollary follows from the previous results.

COROLLARY 13. *Let $T \in L(E, F)$, $A = G(T)$, $B = E \times \{0\}$, $A' = G(T')$ and $B' = F' \times \{0\}$. Then we obtain the following diagram.*

(i) *T' is open*	(i)′ *T is open.*
$\Uparrow$	$\Uparrow$
(ii) *T' is onto.*	(ii)′ *T is onto.*
$\Updownarrow$	$\Updownarrow$
(iii) *There exists a real number t, $0 < t \leqq 1$ such that $\|T(x)\| \geqq t\,\|x\|$ $(x \in E)$.*	(iii)′ *There exists a real number s, $0 < s \leqq 1$ such that $\|T'(y')\| \geqq s\,\|y'\|$ $(y' \in F')$.*
$\Updownarrow$	$\Updownarrow$
(iv) *There exists a real number t, $0 < t \leqq 1$ such that A and B are t-orthogonal.*	(iv)′ *There exists a real number s, $0 < s \leqq 1$ such that A' and B' are s-orthogonal.*
$\Downarrow$	$\Downarrow$
(v) *$r(A, B) > 0$.*	(v)′ *$r(A', B') > 0$.*

$$\Updownarrow \qquad\qquad\qquad\qquad\qquad \Updownarrow$$

(vi) *$A + B$ is closed.* $\iff$ (vi)′ *$A' + B'$ is closed.*

$$\Updownarrow \qquad\qquad\qquad\qquad\qquad \Updownarrow$$

(vii) *$R(T)$ is closed.* (vii)′ $R(T')$ is closed.

$$\Updownarrow \qquad\qquad\qquad\qquad\qquad \Updownarrow$$

(viii) *T is open.* (viii)′ *T' is open.*

In particular, if T is injective, then (i)–(viii) *are equivalent and if T' is injective, then* (i)′–(viii)′ *are equivalent.*

Proof. The equivalences (ii) $\Leftrightarrow$ (iii) and (ii)′ $\Leftrightarrow$ (iii)′ are proved by R. Ellis [2]. The implications (ii) $\Rightarrow$ (i) and (ii)′ $\Rightarrow$ (i)′ are instances of the open mapping theorem. The eqivalences (v) $\Leftrightarrow$ (viii) and (v)′ $\Leftrightarrow$ (viii)′ can be easily shown [4, p. 464]. The implications (iv) $\Rightarrow$ (v) and (iv)′ $\Rightarrow$ (v)′ are proved by Theorem 6. We prove (v) $\Leftrightarrow$ (vi) and (v)′ $\Leftrightarrow$ (vi)′ using Theorem 7. Since $A + B = E \times R(T)$ and $A' + B' = F' \times R(T')$, (vi) $\Leftrightarrow$ (vii) and (vi)′ $\Leftrightarrow$ (vii)′ are trivial. Theorem 10 proves that (vi) $\Leftrightarrow$ (vi)′. We now show that (iii) $\Rightarrow$ (iv). For all $(x, T(x)) \in A$ and for all $(y, 0) \in B$, we have

$$\|(x, T(x)) + (y, 0)\| = \max(\|x + y\|, \|T(x)\|)$$

and

$$\max(\|(x, T(x))\|, \|(y, 0)\|) = \max(\|x\|, \|y\|, \|T(x)\|).$$

If $\|x\| \neq \|y\|$, then $\|x + y\| = \max(\|x\|, \|y\|)$. Hence it is trivial that for each t, $0 < t \leqq 1$,

$$\|(x, T(x)) + (y, 0)\| \geqq t \max(\|(x, T(x))\|, \|(y, 0)\|).$$

If $\|x\| = \|y\|$, then by (iii) there exists a real number t, $0 < t \leqq 1$, such that $\|T(x)\| \geqq t \max(\|x\|, \|y\|, \|T(x)\|)$ and it follows that

$$\|(x, T(x)) + (y, 0)\| \geqq t \max(\|(x, T(x))\|, \|(y, 0)\|).$$

Thus (iii) $\Rightarrow$ (iv).

Conversely, if there exists a real number t, $0 < t \leqq 1$ such that A and B are t-orthogonal, then we have

$$\begin{aligned}\|(x, T(x)) + (-x, 0)\| &\geqq t \max(\|(x, T(x))\|, \|(-x, 0)\|)\\ &= t \max(\|x\|, \|T(x)\|) \quad (x \in E).\end{aligned}$$

Hence it follows that for all $x \in E$ $\|T(x)\| \geqq t\, \|x\|$.

The proof of (iii)′ $\Leftrightarrow$ (iv)′ is similar. In particular, if T is injective, then $A \cap B = \{0\}$. Hence by Theorem 8 we can prove (iv) $\Leftrightarrow$ (v). Thus it follows that (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (vii) $\Leftrightarrow$ (viii) $\Leftrightarrow$ (viii)$'$ $\equiv$ (i). The proof of the case where T' is injective is also similar.

DEFINITION. A linear map $T : E \to F$ is called completely continuous if for any bounded sequence of vectors $\{x_n\}$ in E, then the sequence $\{T(x_n)\}$ contains a convergent subsequence.

However, if there exists a nontrivial completely continuous linear map, then K must be locally compact. A. van Rooij [6] has extended this concept as follows.

DEFINITION. A linear map $T : E \to F$ is called compact if the subset $T(B_1(0))$ is compactoid in F.

If T is compact, then T is continuous. And if K is locally compact, then T is completely continuous if and only if T is compact [6, p. 142]. The following conclusions are extensions of results for the completely continuous operator to results for the compact operator. Let $T \in L(E, E)$. Then it is obvious that for $\lambda \in K$ such that $\|T\| < |\lambda|$, $\lambda - T$ is injective and $R(\lambda - T)$ is closed, where the operator λ is defined by $\lambda(x) = \lambda x$ for $x \in E$. Let $N(T)$ be the null space of T and $R(T)$ be the range space of T. The linear span of subset X of E is indicated by $[X]$.

THEOREM 14. *Let $T \in L(E, E)$ be compact and $\lambda \in K$, $\lambda \neq 0$. If $\lambda - T$ is surjective, then $\lambda - T$ is injective.*

Proof. Let $S = \lambda - T$ and suppose $x_1 \neq 0$ satisfies the equation $S(x_1) = 0$. Since S is surjective, there exists x_2 such that $S(x_2) = x_1$ and $S^2(x_2) = 0$. By induction we can construct a sequence $\{x_n\}$ such that $x_n \neq 0$, $S(x_n) = x_{n-1}$ and $S^n(x_n) = 0$. Thus we may also conclude that $N(S^{n-1}) \subset N(S^n)$ and the inclusion is proper. By the Riesz theorem [3, p. 72] there exist $y_n \in N(S^n)$ $(n = 2, 3, \ldots)$ and a constant real number $a > 1$ such that $\|y_n\| \leqq a$ and $d(y_n, N(S^{n-1})) \geqq \frac{1}{2}$. Since

$$N(S^1) \subset N(S^2) \subset \cdots \subset N(S^{n-1}) \subset N(S^n) \ldots,$$

$y_n \notin N(S^{n-1})$ and $y_n \in N(S^n)$, the elements $y_2, y_3, \ldots, y_n, \ldots$ of E are linearly independent and $T(y_2), T(y_3), \ldots, T(y_n), \ldots$ are also linearly independent. For if $T(y_2), T(y_3), \ldots, T(y_n), \ldots$ are not linearly independent, then there exists an m such that

$$T(y_m) = \alpha_2 T(y_2) + \cdots + \alpha_{m-1} T(y_{m-1}) \quad (\alpha_i \in K, i = 2, \ldots, m-1).$$

Hence $T(y_m) = \alpha_2(\lambda y_2 - S(y_2)) + \cdots + \alpha_{m-1}(\lambda y_{m-1} - S(y_{m-1}))$. Since $y_2, y_3, \ldots, y_{m-1} \in N(S^{m-1})$, it follows that $S(y_2), S(y_3), \ldots, S(y_{m-1}) \in N(S^{m-2})$. Therefore we have $T(y_m) \in N(S^{m-1})$. This means that $y_m \in N(S^{m-1})$. This contradicts $y_m \notin N(S^{m-1})$. Thus $T(y_2), T(y_3), \ldots, T(y_n), \ldots$ can constitute the base of the closed linear span of a countable set $\{T(y_n); n = 2, 3, \ldots\}$. Therefore there exists a positive number t such that $T(y_2)$, $T(y_3)$, ... is a t-orthogonal sequence [6, p. 62]. Since $T(B_a(0))$ is compactoid, where $B_a(0)$ denotes the subset $\{x \in E : \|x\| \leqq a\}$, $T(y_n)$ tends to 0 [6, p. 139]. For $n > m$ it follows that

$$\|T(y_n) - T(y_m)\| \geqq |\lambda|/2 \quad [3, \text{p. } 87].$$

This is a contradiction. Thus $x_1 = 0$ and $\lambda - T$ is injective.

DEFINITION. The subset X of E is called locally compactoid if every bounded subset of X is compactoid.

If X is absolutely convex, then X is locally compactoid if and only if $X \cap B_1(0)$ is compactoid.

THEOREM 15. *Let $T \in L(E, E)$ be compact and let $\lambda \in K$, $\lambda \neq 0$. Then $N(\lambda - T)$ is a locally compactoid and finite-dimensional linear subspace of E.*

Proof. Let $x_1, x_2, \ldots, x_n, \ldots$ be a t-orthogonal sequence of elements of

$$N(\lambda - T) \cap B_1(0).$$

Then the sequence $T(x_1), T(x_2), \ldots, T(x_n), \ldots$ is a t-orthogonal sequence of elements of $T(B_1(0))$. Then $T(x_1), T(x_2), \ldots, T(x_n), \ldots$ tends to 0 [6, p. 139]. Hence $x_1, x_2, \ldots, x_n, \ldots$ tends to 0. Hence by A. van Rooij [6, p. 139], $N(\lambda - T) \cap B_1(0)$ is compactoid and so $N(\lambda - T)$ is locally compactoid. Further, since $N(\lambda - T)$ is a closed linear subspace, $N(\lambda - T)$ is finite-dimensional [5, p. 18].

Remark. The above theorem holds even if K is not spherically complete. However K is now spherically complete, so $N(\lambda - T)$ is c-compact and spherically complete [5, p. 26].

THEOREM 16. *Let $T \in L(E, E)$ be compact and let $\lambda \in K$, $\lambda \neq 0$. Then $R(\lambda - T)$ is closed.*

Proof. If E is finite-dimensional, then it is trivial. Hence we may assume that E is infinite-dimensional. Suppose that $R(\lambda - T)$ is not closed. Then there exists a sequence $x_1, x_2, \ldots, x_n, \ldots$ of elements of E such that $(\lambda - T)(x_n)$ tends to y where $y \neq R(\lambda - T)$. Hence we may assume $x_n \notin N(\lambda - T)$ for any n. Since K is spherically complete and $N(\lambda - T)$ is finite-dimensional, $N(\lambda - T)$ has an orthocomplement M_0 [6, p. 135]. Hence there exist $y_1 \in M_0$ and $z_1 \in N(\lambda - T)$ such that

$$x_1 = y_1 + z_1.$$

We have $y_1 \notin N(\lambda - T)$, $\|x_1\| = \max(\|y_1\|, \|z_1\|)$ and $(\lambda - T)(x_1) = (\lambda - T)(y_1)$. Let $N_1 = [y_1] + N(\lambda - T)$. For all $n \geqq 2$ if it should happen that $x_n \in N_1$, then we can conclude that $y \in R(\lambda - T)$. This is a contradiction. Hence there exists k such that for $n \geqq k$, $x_n \notin N_1$. We may assume $n = 2$. Since N_1 has an orthocomplement M_1, there exist $y_2 \in M_1$, $z_2 \in N(\lambda - T)$ and $\alpha_1 \in K$ such that

$$x_2 = y_2 + \alpha_1 y_1 + z_2.$$

The elements x_1 and x_2 of E are linearly independent. Further we set

$$N_2 = [y_1, y_2] + N(\lambda - T).$$

For all $n \geqq 3$, if $x_n \in N_2$, then there exist $\beta_n, \gamma_n \in K$ and $z_n \in N(\lambda - T)$ such that

$$x_n = \beta_n y_1 + \gamma_n y_2 + z_n.$$

From the orthogonality of y_2 and N_1, and the orthogonality of y_1 and $N(\lambda - T)$ where $(\lambda - T)(x_n) = (\lambda - T)(\beta_n y_1 + \gamma_n y_2)$ tends to y, the sequences $\beta_3, \beta_4, \beta_5, \ldots$ and $\gamma_3, \gamma_4, \gamma_5, \ldots$ are Cauchy sequences. Let $\lim \beta_n = \beta$ and $\lim \gamma_n = \gamma$. Then

$$y = (\lambda - T)(\beta y_1 + \gamma y_2).$$

Hence it follows that $y \in R(\lambda - T)$. This is a contradiction. Therefore we may assume $x_3 \notin N_2$. By induction, there exist $y_1, y_2, \ldots, y_n, \ldots$ such that y_n is orthogonal to $N_n = [y_1, y_2, \ldots, y_{n-1}] + N(\lambda - T)$ and $x_n \in N_n$ and $x_{n+1} \notin N_n$. We set $d_n = d(x_n, N(\lambda - T))$. Since $N(\lambda - T)$ is a closed subspace, the distance d_n is positive. Take $\pi \in K$, $0 < |\pi| < 1$. Then we can choose $w_n \in N(\lambda - T)$ such that for each n $(n = 1, 2, 3, \ldots)$,

$$d_n \leqq \|x_n - w_n\| < d_n |\pi|^{-1} < d_n |\pi|^{-2}.$$

The vectors $x_1 - w_1, x_2 - w_2, \ldots, x_n - w_n, \ldots$ are linearly independent. Suppose that the set $\{T(x_i - w_i); i = 1, 2, \ldots\}$ doesn't contain infinitely many linearly independent vectors. Then there would exist a number N such that $T(x_1 - w_1), T(x_2 - w_2), \ldots, T(x_N - w_N)$ are linearly independent vectors, and, for any $n > N$,

$$T(x_n - w_n) \in [T(x_1 - w_1), T(x_2 - w_2), \ldots, T(x_N - w_N)].$$

Hence we can take $\alpha_{n_i} \in K$ $(i = 1, 2, \ldots, N)$ such that

$$(\lambda - T)(x_n - w_n) = \lambda(x_n - w_n) + \alpha_{n_1} T(x_1 - w_1) + \cdots + \alpha_{n_N} T(x_N - w_N).$$

Since $x_1 - w_1, x_2 - w_2, \ldots$ are linearly independent vectors, $\{x_n - w_n\}$ contains a subsequence $\{x_{n_i} - w_{n_i}\}$ $(i = 1, 2, \ldots)$ such that $T(x_1 - w_1)$, $T(x_2 - w_2), \ldots, T(x_N - w_N), x_{n_1} - w_{n_1}, x_{n_2} - w_{n_2}, \ldots, x_{n_k} - w_{n_k}, \ldots$ are linearly independent vectors. Therefore there exists a number $t > 0$

such that

$$T(x_1 - w_1), T(x_2 - w_2), \ldots, T(x_N - w_N), x_{n1} - w_{n1}, x_{n2} - w_{n2}, \ldots$$

is a t-orthogonal sequence [6, p. 62]. Since $(\lambda - T)(x_{n_i} - w_{n_i})$ tends to y, for any $\varepsilon > 0$ there exists a number M such that $n_i, n_k > M$ implies

$$\|(\lambda - T)(x_{n_i} - w_{n_i}) - (\lambda - T)(x_{n_k} - w_{n_k})\| < \varepsilon.$$

While

$$\begin{aligned}\|(\lambda - T)(x_{n_i} - w_{n_i}) &- (\lambda - T)(x_{n_k} - w_{n_k})\| \\ &\geqq t \max(\|\lambda(x_{n_i} - w_{n_i})\|, \|\lambda(x_{n_k} - w_{n_k})\|, \ldots, \\ &\quad \|(\alpha_{n_{i1}} - \alpha_{n_{k1}})T(x_1 - w_1)\|, \ldots, \|(\alpha_{n_{iN}} - \alpha_{n_{kN}})T(x_N - w_N)\|) \\ &\geqq t\,\|\lambda(x_{n_i} - w_{n_i})\|.\end{aligned}$$

Hence

$$\|x_{n_i} - w_{n_i}\| < t^{-1}\,|\lambda|^{-1}\varepsilon.$$

Therefore $x_{n_i} - w_{n_i}$ tends to 0 and $(\lambda - T)(x_{n_i} - w_{n_i})$ tends to 0. This contradicts the assumption $y \neq 0$. Thus the set $\{T(x_i - w_i); i = 1, 2, \ldots\}$ contains infinitely many linearly independent vectors. So we may assume that the vectors $T(x_1 - w_1), T(x_2 - w_2), \ldots$ are linearly independent. Hence there exists a number $s > 0$ such that the sequence

$$T(x_1 - w_1), T(x_2 - w_2), \ldots$$

is s-orthogonal. Suppose there is a number a such that for each n, $\|x_n - w_n\| \leqq a$. It follows that $T(x_n - w_n) \in T(B_a(0))$. Because $T(B_a(0))$ is compactoid, $T(x_n - w_n)$ tends to 0 [6, p. 139]. Since

$$x_n - w_n = \lambda^{-1}((\lambda - T)(x_n - w_n) + T(x_n - w_n)),$$

the sequence $x_1 - w_1, x_2 - w_2, \ldots$ tends to $\lambda^{-1}y$. It follows that

$$y = (\lambda - T)(\lambda^{-1}y).$$

This contradicts $y \notin R(\lambda - T)$. Thus $\lim\|x_n - w_n\| = \infty$. Now choose m_n such that for each n,

$$|\pi|^{m_n} \leqq d_n < |\pi|^{m_n - 1}.$$

It follows that

$$d_n \leqq \|x_n - w_n\| < |\pi|^{m_n - 2} \leqq d_n\,|\pi|^{-2}.$$

Let $v_n = (\pi^{-1})^{m_n - 2}(x_n - w_n)$. Then the vectors $v_1, v_2, \ldots$ are linearly independent and $v_n \in B_1(0)$. Therefore the sequence $T(v_1), T(v_2), \ldots$ tends to 0. Since $(\pi^{-1})^{m_n - 2}$ tends to 0,

$$(\lambda - T)(v_n) = (\pi^{-1})^{m_n - 2}(\lambda - T)(x_n)$$

and

$$v_n = \lambda^{-1}((\lambda - T)(v_n) + T(v_n)),$$

the sequence $v_1, v_2, \ldots$ tends to 0, while we have the inequality

$$d_n \leqq \|x_n - w_n\| = |\pi|^{m_n-2} \|v_n\| \leqq d_n |\pi|^{-2} \|v_n\|.$$

It follows that $|\pi|^2 \leqq \|v_n\|$. This contradicts the fact $v_1, v_2, \ldots$ tends to 0. Hence our assumption that $R(\lambda - T)$ is not closed is false. The proof is completed.

THEOREM 17. *If $T \in L(E, F)$ is compact, then the conjugate T' of T is a compact linear map taking the Banach space F' into the Banach space E'.*

Proof. Since T is compact, for every $\varepsilon > 0$ there exists a continuous linear map S taking E into F such that $S(E)$ is finite-dimensional and $\|T - S\| \leqq \varepsilon$ [6, p. 142]. Because K is spherically complete, it follows that

$$\|T' - S'\| \leqq \varepsilon.$$

We now show that $S'(F')$ is finite-dimensional. Since $S(E)$ is finite-dimensional, there exist linearly independent vectors $e_i \in F$ $(i = 1, 2, \ldots, p)$ such that for each $x \in E$,

$$S(x) = a_1e_1 + a_2e_2 + \cdots + a_pe_p \quad (a_i \in K;\ i = 1, 2, \ldots p).$$

We define the elements f_i $(i = 1, 2, \ldots, p)$ of $(S(E))'$ as follows. For each $x \in E$,

$$f_i(S(x)) = a_i \quad (i = 1, 2, \ldots, p).$$

Since K is spherically complete, each f_i has an extension $\hat{f}_i$ to F. We can easily show that $S'(\hat{f}_1), S'(\hat{f}_2), \ldots, S'(\hat{f}_p)$ generate the linear subspace $S'(F')$ of E'. Hence $S'(F')$ is finite-dimensional. Then T' is compact.

THEOREM 18. *Let $T \in L(E, E)$ be compact and let $\lambda \in K$, $\lambda \neq 0$. If $\lambda - T$ is injective, then $\lambda - T$ is surjective.*

Proof. Since $\lambda - T$ is injective and $R(\lambda - T)$ is closed, by Corollary 13, $\lambda - T'$ is surjective. Hence by Theorems 14 and 16, $\lambda - T'$ is injective and $R(\lambda - T')$ is closed. Therefore $\lambda - T$ is surjective.

Further we can obtain the following theorems.

THEOREM 19. *Let $T \in L(E, E)$ be compact and let $\lambda \in K$, $\lambda \neq 0$. If $\lambda - T$ is injective, then there exists a real number $t > 0$ such that for all $x \in E$,*

$$\|(\lambda - T)(x)\| \geqq t\|x\|.$$

Proof. By Corollary 13 and Theorem 16, it is trivial.

THEOREM 20. *Let $T \in L(E, E)$ be compact and let $\lambda \in K$, $\lambda \neq 0$. Then there exists a constant real number c such that for all $x \in E$,*

$$d(x) \leqq c\|(\lambda - T)(x)\| \quad \textit{where } d(x) = d(x, N(\lambda - T)).$$

Proof. Since $N(\lambda - T)$ is finite-dimensional, $N(\lambda - T)$ is closed. Then the quotient space $E/N(\lambda - T)$ is a Banach space with the quotient norm $\| \ \|_1$. Let p be the quotient map

$$E \to E/N(\lambda - T).$$

For $x \in E$, let $\bar{x} = p(x)$. Then we can define the continuous linear operator H taking the Banach space $E/N(\lambda - T)$ into E by

$$H(\bar{x}) = (\lambda - T)(x).$$

Hence $R(H) = R(\lambda - T)$. Since $R(H)$ is closed and H is injective, by Corollary 13 there exists a real number t, $0 < t \leqq 1$, such that

$$\|H(\bar{x})\| \geqq t\|\bar{x}\|_1 \quad \text{for all } \bar{x} \in E/N(\lambda - T).$$

Then it follows that $\|(\lambda - T)(x)\| \geqq td(x)$. Set $c = 1/t$. Then we can conclude the proof.

COROLLARY 21. *Let $T \in L(E, E)$ be compact and let $\lambda \in K$, $\lambda \neq 0$. Then if $y \in R(\lambda - T)$, there exists an $x \in E$ such that $(\lambda - T)(x) = y$ and $\|x\| \leqq c\|y\|$, where c is the constant of Theorem 20.*

Proof. Since $y \in R(\lambda - T)$, there exists an $x_0 \in E$ such that

$$y = (\lambda - T)(x_0).$$

Let D be the orthocomplement to $N(\lambda - T)$. Then we may choose $y_0 \in D$ and $z_0 \in N(\lambda - T)$ such that $x_0 = y_0 + z_0$. It follows that $y = (\lambda - T)(y_0)$ and $d(x_0) = \|y_0\|$. Hence by Theorem 20 we have

$$\|y_0\| \leqq c\|(\lambda - T)(y_0)\|.$$

Set

$$R(\lambda - T)^a = \{x' \in E' : x'(x) = 0, x \in R(\lambda - T)\},$$
$${}^aR(\lambda - T') = \{x \in E : x'(x) = 0, x' \in R(\lambda - T')\}$$

and

$${}^aN(\lambda - T') = \{x \in E : x'(x) = 0, x' \in N(\lambda - T')\}.$$

Then we can prove the following theorem.

THEOREM 22. *Let $T \in L(E, E)$ be compact and let $\lambda \in K$, $\lambda \neq 0$. Then the following equalities hold.*

(1) $R(\lambda - T)^a = N(\lambda - T')$.
(2) ${}^aR(\lambda - T') = N(\lambda - T)$.
(3) $R(\lambda - T) = {}^aN(\lambda - T')$.
(4) $R(\lambda - T') = N(\lambda - T)^a$.

Proof. We have

$$\overline{R(\lambda - T)^a} = N(\lambda - T'), \overline{{}^aR(\lambda - T')} = N(\lambda - T),$$
$$\overline{R(\lambda - T)} = {}^aN(\lambda - T') \text{ and } \overline{R(\lambda - T')} \subset N(\lambda - T)^a \quad [1, \text{p. } 285].$$

By Theorem 16, $R(\lambda - T)$ and $R(\lambda - T')$ are closed. Then the equalities (1), (2), (3) and the inclusion $R(\lambda - T') \subset N(\lambda - T)^a$ hold. Therefore we must show that $R(\lambda - T') \supset N(\lambda - T)^a$. Using Corollary 21 and Ingleton's version of the Hahn-Banach theorem, we can show this inclusion in the same way as the Theorem A.7 in [1, p. 398].

The following corollary is the same statement as (3) and (4) of Theorem 22.

COROLLARY 23. *Let $T \in L(E, E)$ be compact and let $\lambda \in K$, $\lambda \neq 0$.*

(1) *The equation $(\lambda - T)(x) = y$ is solvable if and only if $y \in {}^aN(\lambda - T')$.*

(2) *Given a y' in E' there exists an x' in E' such that $y' = (\lambda - T')(x')$ if and only if $y' \in N(\lambda - T)^a$.*

LEMMA 24. (1) *If $x_1, x_2, \ldots, x_m$ are linearly independent vectors of E, then there exist elements $g_1, g_2, \ldots, g_m$ of E' such that $g_i(x_j) = \delta_{ij}$ $(i, j = 1, 2, \ldots, m)$.*

(2) *If $f_1, f_2, \ldots, f_n$ are linearly independent elements of E', then there exist vectors $y_1, y_2, \ldots, y_n$ of E such that $f_i(y_j) = \delta_{ij}$ $(i, j = 1, 2, \ldots, n)$.*

Proof. (1) Let $M = [x_1, x_2, \ldots, x_m]$. We define a functional h_i $(i = 1, 2, \ldots, m)$ on M as follows:

$$h_i : M \to K, \quad h_i(\alpha_1 x_1 + \cdots + \alpha_i x_i + \cdots + \alpha_m x_m) = \alpha_i.$$

Clearly, h_i is a continuous linear functional and $h_i(x_j) = \delta_{ij}$ $(i, j = 1, 2, \ldots, m)$. Since K is spherically complete, h_i can be extended to a continuous linear functional g_i defined on all of E. These functional $g_1, g_2, \ldots, g_m$ are desired elements of E'.

(2) As in [1, p. 399], we can prove that

$${}^a[f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_n] \subset {}^a[f_i] \quad (i = 1, 2, \ldots, n)$$

implies that f_i is a linear combination of $f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_n$. From which the statement follows.

THEOREM 25. *Let $T \in L(E, E)$ be compact and $\lambda \in K$, $\lambda \neq 0$. Then*

$$\dim N(\lambda - T) = \dim N(\lambda - T').$$

Proof. Using Theorems 17, 18, 22, Lemma 23, and the fact that a continuous finite-dimensional linear map is compact [6, p. 142], we can prove this theorem in the same way as the theorem for a Banach space over the real field [1, p. 400].

Acknowledgment. I would like to express my hearty gratitude to the editor, Professor M. M. Day, and to the referee for their help and kind suggestions.

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