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ON CONTINUITY OF THE VARIATION AND THE FOURIER TRANSFORM

BY

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Abstract

Let S be a commutative semitopological semigroup with identity and involution, Γ a compact subset in the topology of pointwise convergence of the set of semicharacters on S. Let f be a function which admits a (necessarily unique) integral representation of the form

$$f(s) = \int_{\Gamma} \rho(s)\, d\mu_f(\rho) \quad (\rho \in \Gamma, s \in S)$$

with respect to a complex regular Borel measure μ_f on Γ. The function $|f|(\cdot)$ defined by $|f|(s) = \int_{\Gamma} \rho(s) d|\mu_f|$ is called the *variation* of f. It is shown that the variation $|f|$ is bounded and continuous if and only if f is also bounded and continuous. This, coupled with the author's previous characterization of functions of bounded variation, gives a new description of the Fourier transforms of bounded measures on locally compact commutative groups.

1. Introduction and motivation

Consider the following three classical questions.

(1.1) When is a real sequence the difference of two completely monotonic sequences?

(1.2) When is a complex function on a locally compact commutative group G the Fourier transform of a regular complex measure on the character group?

(1.3) When is a real function on the half-open interval $(0, 1]$ the difference of two non-negative, non-decreasing continuous functions?

These problems can be abstracted and unified by considering classes of "τ-positive" functions on a semigroup S as follows.

Let S denote a commutative semigroup with identity 1 and involution $*$, and $\mathscr{F}(S)$ denote the class of all complex-valued functions on S. For each $s \in S$ define the shift operator $E_s | \mathscr{F}(S) \to \mathscr{F}(S)$ by

$$(E_s f)(t) = f(st) \quad (f \in \mathscr{F}(S), t \in S).$$

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Since $E_s E_t = E_{st}$, the linear span $\mathscr{A}$ of the set $\{E_s | s \in S\}$ of shift operators forms a commutative algebra with identity $I = E_1 (I = E_0$ if S is an additive semigroup) under Cauchy multiplication as defined by $(\Sigma a_i E_{s_i})(\Sigma b_j E_{s_j}) = \Sigma a_i b_j E_{s_i s_j}$. An involution is added to $\mathscr{A}$ by defining $(\Sigma c_i E_{s_i})^* = \Sigma \bar{c}_i E_{s_i^*}$. Following [8, p. 136], let P be the convex cone spanned by positive linear sums of finite products from a subset τ of $\mathscr{A}$ which satisfies:

(i) $T^* = T$ for each $T \in \tau$;

(ii) $I - T \in P$ for each $T \in \tau$;

(iii) $\mathscr{A}$ is spanned by linear sums of products of members of τ.

The linear functionals on $\mathscr{A}$ can be identified with $\mathscr{F}(S)$ via $T \to Tf(1)$. We call $f \in \mathscr{F}(S)$ *τ-positive* if $Tf(1) \geq 0$ for all $T \in P$, i.e. if f can be identified with a positive linear functional on $\mathscr{A}$ relative to the cone P.

The following illustrates how (1.1), (1.2) and (1.3) can be put in the preceding set up.

(1.1)′ Set $(S, \cdot, *) = (\{0, 1, \dots\}, +, n^* = n)$. If $\tau = \{E_1, I - E_1\}$ then the classical difference operator Δ_k (cf. [11, p. 101]) is given by

$$\Delta_k f(n) = E_1^n (I - E_1)^k f(0) \quad (f \in \mathscr{F}(S)).$$

Thus, f is τ-positive if and only if $\{f(n)\}_n$ is a completely monotonic sequence.

(1.2)′ Set $(S, \cdot, *) = (G, \cdot, s^* = s^{-1})$. If $\tau = \{T_{\sigma, s}\}_{\sigma, s}$ where σ is a fourth root of unity, $s \in G$ and

$$T_{\sigma, s} = \tfrac{1}{4}\left(I + \frac{\sigma}{2} E_s + \frac{\bar{\sigma}}{s} E_{s^*}\right)$$

then the τ-positive functions are precisely the positive definite functions [7, p. 141].

(1.3)′ Set $(S, \cdot, *) = ((0, 1], s \cdot t = \min[s, t], s^* = s)$. If

$$\tau = \{E_s, I - E_s | s \in (0, 1]\}$$

then it easily follows that the τ-positive functions are the non-negative, non-decreasing real functions.

Once we impose the discrete topology on the non-negative integers, the given topology on the topological group G and the usual open interval topology on $(0, 1]$ respectively, then all of the above semigroups are semitopological (i.e. multiplication is separately continuous) with a continuous involution. Moreover, we can abstract question (1.1), (1.2) and (1.3) by asking:

(1.4) When is a function on a semitopological semigroup S in the linear span of the continuous τ-positive functions on S?

As is well known, the answers to (1.1) and (1.3) are, respectively, that f be of bounded variation (BV) in the sense of Hausdorff and that f be continuous and BV in the classical real variable sense. Recall that a sequence $\{f(n)\}_n$ is BV in the sense of Hausdorff if and only if

$$\|f\| = \overline{\lim}_n \sum_{k=0}^{n} \binom{n}{k} |\Delta_{n-k} f(k)| < \infty .$$

While Eberlein's theorem (cf. [9, p. 32]) answers (1.2), it fails to extend in a natural way to the generality of (1.4). In fact since the constant 1-function is the only continuous semicharacter in the case of (1.3), Eberlein's theorem does not even carry over to this special case. As a corollary (Corollary 2.2) to the main theorem we answer (1.4) for bounded functions when the involution is continuous. In the case of (1.2), all Fourier transforms of measures are necessarily bounded and the involution is always continuous. Thus we have formulated a new answer to (1.2) in the flavor of the classical answers to (1.1) and (1.3). The Bochner-Herglotz-Weil theorem then implies that the BV-functions on a locally compact commutative topological group G are just the Fourier transforms of the regular Borel measures on the character group. We find it convenient to prove the above corollary in the next section before proving the theorem itself later in §3.

2. Functions of bounded variation on S

We recall the general notion of BV-functions introduced in [8] which subsumes all previous notions as mentioned in the introduction. Let Ω be a collection of finite subsets Λ of P such that:

(2.0) (i) $\Sigma_{T \in \Lambda} T = I$ for each $\Lambda \in \Omega$, i.e. each $\Lambda \in \Omega$ is a *partition of unity*.

(ii) Ω is a subsemigroup of the collection of all partitions of unity under Cauchy multiplication as defined by $\Lambda\Lambda' = \{TT' | T \in \Lambda, T' \in \Lambda'\}$.

(iii) Each $T \in \tau$ is a member of some partition of unity.

For each $f \in \mathscr{F}(S)$ and $\Lambda \in \Omega$, define $\|f\|_\Lambda = \Sigma_{T \in \Lambda} |Tf(1)|$ and impose the semigroup ordering on Ω; i.e., $\Lambda_1 \geq \Lambda_2$ whenever there exists $\Lambda_3 \in \Omega$ such that $\Lambda_1 = \Lambda_2 \Lambda_3$. Then the function $\Lambda \to \|f\|_\Lambda$ is nondecreasing and the total variation $\|f\|$ of $f \in \mathscr{F}(S)$ is defined by

(iv) $\|f\| = \lim_\Lambda \|f\|$.

The function f is said to be of *bounded variation* (BV) whenever $\|f\| < \infty$ and $BV(S)$ will denote the set of all BV-functions.

If in (1.1), Ω is selected as partitions of the form

$$\Lambda_n = \left\{ \binom{n}{i} E_1^i (I - E_1)^{n-i} | i = 0, 1, \ldots, n \right\} \quad \text{for } n = 1, 2, \ldots,$$

then the binomial theorem implies Ω satisfies conditions (2.0.i) through (2.0.iii) relative to $\tau = \{E_1, I - E_1\}$. In this event the total variation of Hausdorff referred to in §1 agrees with that introduced above. Similarly in (1.3), if Ω is selected as the set of partitions

$$\Lambda_n = \{E_{s_0}, E_{s_1} - E_{s_0}, \ldots, I - E_{s_n}\} \quad \text{where } 0 < s_0 < s_1 < \cdots < s_n < 1,$$

it follows that the abstract notion of total variation is in agreement with the classical real variable concept. As for question (1.2), for each member s of a group G let Λ_s be the partition defined by $\Lambda_s = \{T_{\sigma,s} | \sigma^4 = 1\}$ and Ω be the collection of all finite products Λ of partitions of the form Λ_s. Then (2.0.iv) can be taken as the definition of the total variation $\|f\|$ of a function f on G.

Let Γ be the set of all (not necessarily continuous) τ-positive semicharacters ρ on S; i.e., ρ is a complex function on S which is τ-positive such that $\rho \not\equiv 0$ and $\rho(st) = \rho(s)\rho(t)$ for all $s, t \in S$. It follows that $\rho(s^*) = \overline{\rho(s)}$ for each $\rho \in \Gamma$. If Γ is equipped with the topology of pointwise convergence then the following theorem is implicit in [8].

(2.1) THEOREM. *A complex-valued function f on S admits a (necessarily unique) integral representation of the form*

(i) $f(s) = \int_\Gamma \rho(s)\, d\mu_f(\rho)$

with respect to a regular Borel measure μ_f if and only if $f \in BV(S)$. Moreover the map $f \to \mu_f$ is a linear bijection of $BV(S)$ onto the regular Borel measures on Γ such that

(ii) *μ_f is non-negative if and only if f is τ-positive,*

(iii) *μ_f is real if and only if $\bar{f}(s) = f(s^*)$ for all $s \in S$ and*

(iv) $\|\mu_f\| = \|f\|$.

Assuming the main theorem as stated in the abstract and assuming continuity of the involution we can now answer question (1.4) for bounded functions.

(2.2) COROLLARY. *The span of the τ-positive bounded continuous functions on S is the collection of bounded continuous BV-functions.*

Proof. (i) A direct application of Theorem (2.1) shows that f is BV whenever it is in the span of the τ-positive functions.

(ii) Conversely assume f is bounded, continuous and BV. Suppose, moreover, that μ_f is real. Then the main theorem implies that the variation $|f|$, as already defined by $|f|(s) = \int_\Gamma \rho(s)\, d|\mu_f|$, is continuous and bounded. But then f is the difference of two τ-positive functions, since

$$f = \tfrac{1}{2}(|f| + f) - \tfrac{1}{2}(|f| - f).$$

In the general case f_* as defined by $f_*(s) = \overline{f(s^*)}$ is BV and bounded. Continuity of f_* follows from continuity of the involution. The converse assertion now follows since $f = f_1 + if_2$ where $f_1 = (f + f_*)/2$ and $f_2 = (f - f_*)/2i$ and since both f_1 and f_2 have real representing measures by (2.1.iii).

Remarks. (i) In general and contrary to all of the examples considered thus far, neither a BV-function nor its variation need be bounded. For let S be any commutative semigroup with trivial involution, which admits an unbounded τ-positive semicharacter ρ. If $\rho \in \Gamma$ then ρ itself may be represented by point mass at ρ and hence is an example of an unbounded BV-function. Note $|\rho| = \rho$. In particular one could take S to be the non-negative integers under addition with $n = n^*$ and with $\tau = \{\frac{1}{2}E_1, I - \frac{1}{2}E_1\}$ and define ρ by $\rho(n) = 2^n$.

(ii) If S is a group, or a semilattice as above when $S = (0, 1]$, or more generally an inverse semigroup (cf. [2, p. 27]) with $s^* = s^{-1}$, then every τ-positive and hence every BV-function is necessarily bounded (cf. [1]). In this event the boundedness can be removed from the corollary as well as from the main theorem. In as much as the involution is continuous for a locally compact group G as well as the semigroup $(0, 1]$ pertinent to (1.3), Corollary 2.2 provides the new answer to question (1.2) promised and reproves the known answer to (1.3).

In order to underscore the difference between Eberlein's result and ours for the group case, consider the group of integers under addition. Eberlein's result cf [5, p. 32] characterizes those functions f which are in the complex span of the positive definite functions as those f for which all finite sums $|\Sigma c_j f(j)|$ are bounded by a common multiple of $\sup_\theta |\Sigma c_j e^{-ij\theta}|$. Our characterization asserts that f is in this span if and only if

$$\lim_{n \to \infty} \sum \binom{n}{i_1, i_2, i_3, i_4} \left| (T_1)^{i_1} (T_{-1})^{i_2} (T_i)^{i_3} (T_{-i})^{i_4} f(0) \right| < \infty$$

where

$$T_\sigma = \tfrac{1}{4}\left(I + \frac{\sigma}{2} E_1 + \frac{\bar{\sigma}}{2} E_{-1}\right) \quad \text{and} \quad \sigma^4 = 1.$$

(iii) When S is not a group the functions which are τ-positive with respect to $\tau = \{T_{\sigma, s} | s \in S, \sigma^4 = 1\}$ are shown in [7, p. 141] to be the $*$-definite functions studied in [6] and are necessarily bounded. In this case, boundedness can also be removed from the corollary as well as from the main theorem. In fact the results of [1] show that the class of functions on S which are BV with respect to this τ is the largest possible collection of bounded BV-functions on S with respect to any τ.

3. Main theorem

We are indebted to J.P.R. Christensen for suggesting the proof of the following.

(3.1) LEMMA. *If $f \in BV(S)$ then f is bounded if and only if $|\rho(\cdot)| \leq 1$ for all ρ in the support of the representing measure μ_f. Consequently a BV-function f is bounded if and only if its variation $|f|(\cdot)$ is bounded.*

Proof. Assume f is bounded and fix $r, t \in S$. Let Q be the point wise open set

$$\{\rho \in \Gamma \mid |\rho(r)| > 1\}.$$

Assume Q is not empty and define ν to be the restriction of μ_f to Q. For each $s \in S$, set $|s| = \sup_\Gamma |\rho(s)|$ and note that compactness of Γ implies $0 \leq |s| < \infty$. Since

$$\left| \int_{\Gamma \setminus Q} |\rho(r)|^{2n} \rho(t)\, d\mu_f(\rho) \right| \leq |t| \cdot \|\mu_f\|,$$

it follows that the function $s \to \int_{\Gamma \setminus Q} \rho(s)\rho(t)\, d\mu_f(\rho)$ is bounded on the set $\{(rr^*)^n \mid n = 0, 1, \dots\}$ which implies boundedness of $s \to \int \rho(s)\rho(t)\, d\nu(\rho)$ on this same set. Thus the function $H(z)$ as defined by

$$\text{(i)} \qquad H(z) = \sum_{n=0}^{\infty} \left(\int |\rho(r)|^{2n} \rho(t)\, d\nu \right) z^n,$$

is analytic on the open unit disk U. But if $|z| < \frac{1}{2}(1/|r|)^2$ then

$$\left| \sum_{n=0}^{m} |\rho(r)|^{2n} z^n \rho(t) \right| \leq \frac{|\rho(t)|}{1 - |\rho(r)|^2 \cdot |z|} < 2|\rho(t)|.$$

Hence the dominated convergence theorem implies that the order of integration and summations in (i) can be interchanged obtaining

$$\text{(ii)} \qquad H(z) = \int \frac{1}{z - \left| \dfrac{1}{\rho(r)} \right|^2} \left(-\rho(t)/|\rho(r)|^2 \right) d\nu(\rho)$$

for z sufficiently close to zero.

Let m be the image of the measure $(-\rho(t)/|\rho(r)|^2)\, d\nu$ under the map

$$\rho \to 1/|\rho(r)|^2.$$

Then the support of m is contained in $[1/|r|^2, 1]$. Since $|r| < \infty$, the change of variables formula shows that

$$F(z) = \int \frac{1}{z - \xi}\, dm(\xi)$$

agrees with $H(z)$ on a sufficiently small neighborhood of the origin. But both $H(z)$ and $F(z)$ are analytic on $U \setminus [1/|r|^2, 1]$ so that $F(z) = H(z)$ on this set. Consequently $F(z)$ admits an analytic continuation, namely $H(z)$, to all of U which by the theory of Cauchy transforms [4, Corollary 1.3], implies

$$|m|([1/|r|, 1)) = 0.$$

Thus $\int(-\rho(t)/|\rho(r)|^2)\, d\nu(\rho) = 0$ for all $t \in S$ and hence the Stone-Weierstrass theorem implies $(1/|\rho(r)|^2)\, d\nu(\rho) = 0$ so that $|\nu| \equiv 0$. Consequently, if $|\rho(r)| > 1$ then there exists an open set, namely Q, containing ρ such that $|\mu_f|(Q) = 0$ so ρ cannot be in the support of μ_f. This establishes the "only if" part of the assertion. The converse is clear.

As in [6], we do not assume continuity of the involution in the following.

(3.2) THEOREM. *The variation of a BV-function f on a commutative semitopological semigroup S with identity and involution is continuous and bounded if and only if f is continuous and bounded.*

Proof. (i) If g is a continuous, bounded, τ-positive function on S and f is a BV-function such that μ_f is absolutely continuous with respect to μ_g then f is continuous.

Indeed, the Stone-Weierstrass theorem implies that the linear span of the functions on Γ of the form $\rho \to \rho(s)$ $(\rho \in \Gamma, s \in S)$ are dense in $L_1(\Gamma, \mu_g)$. But since $d\mu_f/d\mu_g \in L_1(\Gamma, \mu_g)$ and

$$(E_s g)(t) = \int_\Gamma \rho(t)\rho(s)\, d\mu_g(\rho) \quad (s, t \in S),$$

there exists a sequence $\{f_n\}_n$ of linear sums of translates of g such that

$$\lim_n \|f_n - f\| = 0.$$

Thus for each $s \in S$ we have

$$\begin{aligned} |f(s) - f_n(s)| &= \left| \int \rho(s) \left(\frac{d(\mu_{f-f_n})}{d\mu_g} \right) d\mu_g \right| \\ &\le \int \left| \frac{d(\mu_{f-f_n})}{d\mu_g} \right| d\mu_g \\ &= \|f - f_n\|, \end{aligned}$$

where the inequality follows from Lemma 3.1. Hence f_n converges in L_1-norm to f so that continuity of f_n implies continuity of f.

(ii) The "only if" part of the theorem now follows from (i) since μ_f is absolutely continuous with respect to $|\mu_f|$ and $|\mu_f| = \mu_{|f|}$ by definition.

(iii) Conversely, assume f is a continuous, bounded BV-function on S. By [10, p. 126], $|d\mu_f/d|\mu_f|| \equiv 1$ so that a complex-valued function h on Γ can be defined by

$$h(\rho) = 1/\left(d\mu_f/d|\mu_f|\right).$$

As in (i) the Stone-Weierstrass theorem implies the existence of a sequence $\{h_n\}_n$ of continuous functions on Γ of the form

$$h_n(\rho) = \sum_i \alpha_{i,n}\rho(s_{i,n}) \quad (\alpha_{i,n} \in \mathbf{C}, s_{i,n} \in \Gamma)$$

which converges uniformly to h. But h, $h_n \in L_1(\Gamma, |\mu|)$ for each n and $\lim_n \|h_n - h\|_1 = 0$. Define the continuous functions f_n on S by $f_n = \sum_i \alpha_{i,n} E_{s_{i,n}} f$. Then

$$\begin{aligned} |f_n(s) - |f|(s)| &= \left| \int \rho(s) \sum_i \alpha_{i,n}\rho(s_{i,n})\, d\mu_f - \int \rho(s)\, d|\mu_f| \right| \\ &= \left| \int \rho(s) \left[\sum_i \alpha_{i,n}\rho(s_{i,n})\, d\mu_f - h(\rho) \right] d\mu_f \right| \\ &\leq \int \left| \sum_i \alpha_{i,n}\rho(s_{i,n}) - h(\rho) \right| d|\mu_f| \qquad \text{(Lemma 3.1)} \\ &= \|h_n - h\|_1 \to 0. \end{aligned}$$

Therefore f_n converges uniformly to the variation $|f|$ so that continuity of f_n implies that of $|f|$.

Remarks. (i) The most well known application of this theorem is, as mentioned in the introduction, in connection with question (1.3). More generally it implies the known result that the variation $|f|$ of every continuous BV-function on the unit cube of $\mathbf{R}^n$ is "one-sided" continuous in each variable if and only if f is "the same-sided" continuous in each variable of [5, pp. 61–64].

(ii) We do not know if the main theorem or Cor. 2.2 is valid without boundedness.

References

1. C. Berg and P.H. Maserick, *Exponentially bounded positive definite functions*, Illinois J. Math., vol. 28 (1984), pp. 162–179.
2. A.H. Clifford and C.B. Preston, *The algebraic theory of semigroups*, (*I*), Math. Surveys, no. 7, Amer. Math. Soc., Providence, R.I., 1971.

3. N. DUNFORD and J. SCHWARTZ, *Linear operators (I)*, Interscience, New York, 1957.
4. J.B. GARNETT, *Analytic capacity and measure*, Lectures Notes in Math., no. 297, Springer, New York, 1972.
5. J. HAVNEN, *BV-functions on n-dimensional lattices*, Ph.D. dissertation, The Pennsylvania State University, 1970.
6. R.J. LINDAHL and P.H. MASERICK, *Positive definite functions on semigroups*, Duke Math. J., vol. 38 (1971), pp. 771–782.
7. P.H. MASERICK, *BV-functions, positive-definite functions and moment problems*, Trans. Amer. Math. Soc., vol. 214 (1975), pp. 137–152.
8. ______, *Moments of measures on convex bodies*. Pacific J. Math., vol. 68 (1977), pp. 135–152.
9. W. RUDIN, *Fourier analysis on groups*, Interscience, New York, 1962.
10. ______, *Real and complex analysis*, McGraw-Hill, New York, 1966.
11. D.V. WIDDER, *The Laplace transform*, Princeton Univ. Press, Princeton, N.J., 1941.

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