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SOME EXTREMAL PROBLEMS FOR CONTINUED FRACTIONS

BY

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1. Introduction

Consider the simple continued fraction

$$(1.1)\qquad [x_1,\ldots,x_n] = x_1 + \cfrac{1}{x_2 + \cfrac{1}{\ddots + \cfrac{1}{x_n}}} = \frac{p_n(x_1,\ldots,x_n)}{q_n(x_1,\ldots,x_n)}.$$

Mathematicians have traditionally looked at a number θ and determined its representation as an infinite continued fraction: $\theta = [x_1, x_2, \ldots]$, $x_i \in \mathbf{N}$. In this paper, we are exclusively interested in the properties of the *continuants*, $p_n(x_1,\ldots,x_n)$, viewed as polynomials. In 1891, E. Lucas [7, Ch. 24], in effect, posed the following problem. Suppose $t \in \mathbf{N}$ is given; determine the maximum value of $p_n(x_1,\ldots,x_n)$ subject to $x_i \in \mathbf{N}$ and $\sum_{i=1}^n x_i = t$. Lucas proved that $p_n(x_1,\ldots,x_n) \le p_t(1,\ldots,1) = F_{t+1}$, the $(t+1)$-*st* Fibonacci number, ($F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_n + F_{n+1}$). The first part of this paper considers this problem, but allows t and the x_i's to be non-negative reals, not just integers. In this case, it is not obvious that $p_n(x_1,\ldots,x_n)$ is bounded as a function of t; later in this introduction, we show that

$$p_n(x_1,\ldots,x_n) \le e^t.$$

With a more careful analysis, we show that the true supremum is the C^2 function $\phi(t)$:

$$(1.2)\qquad \phi(t) = \begin{cases} 1 + \frac{1}{4}t^2, & 0 < t \le 2,\\ 2e^{t/2-1}, & t \ge 2.\end{cases}$$

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We also look at the general continued fraction

$$(1.3)\qquad x_1 + \cfrac{y_1}{x_2 + \cfrac{y_2}{\ddots + \cfrac{y_n}{x_n}}} = \frac{p_n(x_1,\dots,x_n;\, y_2,\dots,y_n)}{q_n(x_1,\dots,x_n;\, y_2,\dots,y_n)}$$

and maximize

$$p_n(x_1,\dots,x_n;\, y_2,\dots,y_n)$$

subject to $x_i, y_i \geq 0$ and $\Sigma_{i=1}^n x_i + \Sigma_{i=2}^n y_i = t$. If the variables are real, the maximum is achieved for $y_i = 0$ and $x_i = t/n$;

$$p_n(\mathbf{x},\mathbf{y}) \leq \max_k (t/k)^k,$$

a function asymptotic to $e^{t/e}$. If they are non-negative integers, the maximum is asymptotically $c3^{t/3}$ and is achieved by again setting $y_i = 0$ and as many of the x_i's as possible equal to 3, with zero, one or two 2's depending on $t \bmod 3$. This problem is equivalent to the 1976 International Olympiad Problem 4 [8] or Putnam Competition Problem 1979 A1 [3]. If the x_i's and y_i's must be positive integers, then the maximum occurs when $y_i = 1$. The structure of the maximizing strings is somewhat more complicated, depending on $t \bmod 4$. The maximum grows like $c\alpha^{t/4}$ for $\alpha = (\sqrt{13} + 3)/2$. Note that in every instance, the growth is exponential in t, although the bases are different for the different cases.

Our interest in the last case revolves around its connection to the sequence $\{a_n\}$ defined below:

$$(1.4)\quad a_0 = 0,\quad a_1 = 1,\quad a_{2n} = a_n,\quad a_{2n+1} = a_{n+1} - a_n \quad \text{for } n \geq 1.$$

As a consequence of Corollary 6.13, $|a_n| \leq cn^{\beta}$, where $\beta = \log\alpha/\log 16$ and is best possible. See [15] for more properties of $\{a_n\}$.

Returning to (1.1), since $[x_1,\dots,x_n] = x_1 + [x_2,\dots,x_n]^{-1}$,

$$(1.5)\qquad \begin{aligned} p_n(x_1,\dots,x_n) &= x_1 p_{n-1}(x_2,\dots,x_n) + q_{n-1}(x_2,\dots,x_n),\\ q_n(x_1,\dots,x_n) &= p_{n-1}(x_2,\dots,x_n). \end{aligned}$$

It follows that the q_n's are superfluous, and if we adopt the convention that $p_{-1} = 0$ and $p_0 = 1$, the recurrence (1.6) can serve as a definition of the p_n's for $n \geq 1$:

$$(1.6)\qquad p_n(x_1,\dots,x_n) = x_1 p_{n-1}(x_2,\dots,x_n) + p_{n-2}(x_3,\dots,x_n).$$

The first few continuants are displayed below:

$$(1.7)\quad p_1(x) = x, \quad p_2(x, y) = xy + 1, \quad p_3(x, y, z) = xyz + x + z,$$
$$p_4(x_1, x_2, x_3, x_4) = x_1x_2x_3x_4 + x_1x_2 + x_1x_4 + x_3x_4 + 1.$$

Continuants have many nice properties; Section two is devoted to some of the more useful ones.

It is clear from (1.6) and (1.7) that $p_n(x_1, \dots, x_n) \le \Sigma x_{i_1} \cdots x_{i_k}$ where the sum is taken over *all* subsets of $\{1, \dots, n\}$, cf. (2.1), (2.2). With $\Sigma_{i=1}^n x_i = t$, we have, by the arithmetic-geometric inequality,

$$(1.8)\quad p_n(x_1, \dots, x_n) \le \prod_{i=1}^n (1 + x_i) \le \left(\frac{\sum_{i=1}^n (1 + x_i)}{n} \right)^n = \left(1 + \frac{t}{n}\right)^n \le e^t.$$

We close this introduction with some remarks. The following representation of continuants as determinants is fairly well-known, see [11, p. 8] for example:

$$(1.9)\qquad p_n(x_1, \dots, x_n) = \det \begin{vmatrix} x_1 & 1 & & \\ -1 & x_2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & -1 & x_n \end{vmatrix}.$$

Thus in the first two questions we are maximizing a determinant with fixed trace while varying the diagonal elements over non-negative reals or integers. Finally, note that these problems are not interesting if we allow negative variables: $p_3(-y, t + 2y, -y) = 2y^3 + y^2t - 2y$ which is unbounded as $y \to \infty$.

We should add that several other analytical questions about continuants have been addressed recently in the literature. First, define

$$E(m) = \left\{ t \in (0, 1)\colon t^{-1} = [x_1, x_2, \dots], 1 \le x_i \le m \right\}.$$

Good, Rogers, Cusick, Ramharter and others have studied the Hausdorff dimension of $E(m)$. Second, let $\{a_1, \dots, a_n\}$ be fixed positive integers. For which permutations $\{b_i\} = \{a_{\sigma(i)}\}$ does the continuant $p_n(b_1, \dots, b_n)$ achieve its maximum and minimum? Partial answers were given by Nicol, Motzkin and Straus and Cusick and a complete answer was given by Ramharter. See [13] for a complete bibliography. The work contained in this paper does not appear to be useful in addressing these questions.

2. Some properties of continuants

In this section, we collect some properties of $p_n(x_1, \ldots, x_n)$ that we will need in the rest of the paper. Most are well-known and easy to prove either by induction or direct examination. The first serious study of continuants was undertaken by L. Euler in 1764 [2], which contains special cases of most identities cited in this section. Since subscripts were not in widespread use at this time, he should perhaps be credited with (2.1) through (2.8). In any event, by 1853, Sylvester [16] had found them all, except as noted. For a more detailed history see Muir's encyclopedic works [9, pp. 413–444] and [10, pp. 393–422].

We have two equivalent general formulas for p_n:

$$p_n(x_1, \ldots, x_n) = \sum x_{i_1} \cdots x_{i_k}, \tag{2.1}$$

where the sum is taken over $1 \le i_1 \le \cdots \le i_k \le n$ with $i_j \equiv j \bmod 2$, $k \equiv n \bmod 2$, and 1 is included if n is even, and

$$p_n(x_1, \ldots, x_n) = x_1 \cdots x_n \sum (x_a x_{a+1} \cdots x_r x_{r+1})^{-1}, \tag{2.2}$$

where the sum is taken over all disjoint increasing pairs of consecutive indices. (For $n = 4$, these would be ϕ, $(1,2)$, $(2,3)$, $(3,4)$ and $(1,2,3,4)$, as may be checked against (1.7).) Both (2.1) and (2.2) imply (2.3), which, in turn, implies (2.4):

$$p_n(x_1, \ldots, x_n) = p_n(x_n, \ldots, x_1), \tag{2.3}$$

$$p_n(x_1, \ldots, x_n) = x_n p_{n-1}(x_1, \ldots, x_{n-1}) + p_{n-2}(x_1, \ldots, x_{n-2}). \tag{2.4}$$

At this point, we introduce a non-standard notation which saves us a lot of space. If $\mathbf{x} = (x_1, \ldots, x_n)$ then $\mathbf{x}' = (x_1, \ldots, x_{n-1})$ and $'\mathbf{x} = (x_2, \ldots, x_n)$; $\mathbf{x}''$, $'\mathbf{x}'$, etc. are self-explanatory. In case $n = 1$ and $\mathbf{x} = x$, we adopt the convention that $p_{n-1}(\mathbf{x}') = p_0 = 1$ and $p_{n-2}(\mathbf{x}'') = p_{-1} = 0$. We also write $(\mathbf{x}, \mathbf{y})$ for $(x_1, \ldots, x_n, y_1, \ldots, y_n)$, etc. In view of the foregoing, (1.3) and (2.4) can be rewritten as (2.5):

$$\begin{aligned} p_n(\mathbf{x}) &= x_n p_{n-1}(\mathbf{x}') + p_{n-2}(\mathbf{x}''), \qquad n \ge 1, \\ p_n(\mathbf{x}) &= x_1 p_{n-1}('\mathbf{x}) + p_{n-2}(''\mathbf{x}), \qquad n \ge 1. \end{aligned} \tag{2.5}$$

A crucial identity is (2.6):

$$p_{k+l}(\mathbf{x}, \mathbf{y}) = p_k(\mathbf{x}) p_l(\mathbf{y}) + p_{k-1}(\mathbf{x}') p_{l-1}('\mathbf{y}). \tag{2.6}$$

Perhaps the best way to prove this is by induction on l. By our conventions and (2.5), (2.6) is obvious for $l = 0$ and 1, and both sides satisfy the recurrence $c_l = y_l c_{l-1} + c_{l-2}$. Alternatively, we can derive (2.6) from (2.2) by considering whether or not the block $x_k y_1$ occurs in a term in $p_{k+l}(\mathbf{x}, \mathbf{y})$. An immediate corollary of (2.6) is (2.7):

(2.7)

$$p_{k+l+m}(\mathbf{x},\mathbf{y},\mathbf{z}) = p_k(\mathbf{x})p_l(\mathbf{y})p_m(\mathbf{z}) + p_{k-1}(\mathbf{x}')p_{l-1}('\mathbf{y})p_m(\mathbf{z}) + p_k(\mathbf{x})p_{l-1}(\mathbf{y}')p_{m-1}('\mathbf{z}) + p_{k-1}(\mathbf{x}')p_{l-2}('\mathbf{y}')p_{m-1}('\mathbf{z}).$$

In applying (2.6) we must have $l \geq 1$; when $l = 1$ we obtain

$$(2.8)\quad p_{k+1+m}(\mathbf{x}, y, \mathbf{z}) = yp_k(\mathbf{x})p_m(\mathbf{z}) + p_{k-1}(\mathbf{x}')p_m(\mathbf{z}) + p_k(\mathbf{x})p_{m-1}('\mathbf{z}).$$

After renaming the variables, (2.8) implies the following partial differential equation, useful in the application of Lagrange multipliers:

$$(2.9)\qquad \frac{\partial p_n}{\partial x_j}(x_1,\dots,x_n) = p_{j-1}(x_1,\dots,x_{j-1})p_{n-j}(x_{j+1},\dots,x_n).$$

This equation was apparently discovered by Onofrio Porcelli [12] and has a rather forlorn history explored more fully in [14]. Suffice it to say that [9] attributes (2.9) to P. Onofrio. We need three more special identities, which we isolate as a lemma; (b) and (c) appear as problems 9b and 9c in [4, p. 358].

LEMMA 2.10. (a) $p_{n+1}(\mathbf{x}, 1) = p_n(\mathbf{x}) + p_{n-1}(\mathbf{x}') = p_n(\mathbf{x}', x_n + 1)$
(b) $p_{n+1}(\mathbf{x}, 0) = p_{n-1}(\mathbf{x}')$
(c) $p_{k+l+1}(\mathbf{x}, 0, \mathbf{y}) = p_{k+l-1}(\mathbf{x}', x_k + y_1, '\mathbf{y})$

Proof. Both (a) and (b) follow directly from (2.5); for (c), applications of (2.5) and (2.8) to both sides show that each equals

$$(x_k + y_1)p_{k-1}(\mathbf{x}')p_{l-1}('\mathbf{y}) + p_{k-1}(\mathbf{x}')p_{l-2}(''\mathbf{y}) + p_{k-2}(\mathbf{x}'')p_{l-1}('\mathbf{y}).$$

We conclude this section with two lemmas whose import will be clearer in the next section.

LEMMA 2.11. *Suppose* $\mathbf{x} = (x_1,\dots,x_n) \geq 0$, $\mathbf{x} \neq 0$ *but* $x_k = 0$. *Then there exists* $\mathbf{y} = (y_1,\dots,y_m) > 0$ *with* $m < n$, $\Sigma y_i = \Sigma x_j$ *and* $p_m(\mathbf{y}) \geq p_n(\mathbf{x})$.

Proof. If $x_n = 0$ then by (2.10)(b),

$$p_n(x_1,\dots,x_n) = p_{n-2}(x_1,\dots,x_{n-2}) \leq p_{n-2}(x_1,\dots,x_{n-2} + x_{n-1}).$$

Continue in this way at both ends if necessary (cf. (2.3)) until the first and last components are positive. If any interior x_j's vanish, apply (2.10)(c) as necessary.

LEMMA 2.12. *Suppose*

$$\frac{\partial p_n}{\partial x_j}(\mathbf{y}) = \frac{\partial p_n}{\partial x_k}(\mathbf{y})$$

for $1 \le j \le k \le n$, $n \ge 3$ *and* $\mathbf{y} > 0$. *Then* $\mathbf{y}$ *may be parametrized as follows*: $y_1 = y_n = s$, $y_2 = \cdots = y_{n-1} = s - s^{-1}$. *Further*, $p_k(y_1, \dots, y_k) = s^k$ *for* $1 \le k \le n-1$, $p_n(\mathbf{y}) = s^n + s^{n-2}$ *and* $\Sigma_{i=1}^n y_i > 2$.

Proof. First note that the statements about $p_k(y_1, \dots, y_k)$ follow by induction from the claimed parametrization of the y_i's. Now let

$$a_i = p_i(y_1, \dots, y_i) \quad \text{and} \quad b_i = p_{n-i+1}(y_i, \dots, y_n)$$

for clarity. By (2.9) the hypothesis becomes

$$a_0 b_2 = a_1 b_3 = \cdots = a_{n-1} b_{n+1}. \tag{2.13}$$

By (2.5) we have the useful twin recurrences

$$a_j = y_j a_{j-1} + a_{j-2}, \qquad b_j = y_j b_{j+1} + b_{j+2}.$$

Rewriting the first three equations of (2.13) ($n \ge 3$), we have

$$1 \cdot (y_2 b_3 + b_4) = (y_1 y_2 + 1) b_4 = y_1 b_3.$$

From the first equality, $b_3 = y_1 b_4$ ($y_2 \ne 0$) and so from the second,

$$y_1 y_2 + 1 = y_1^2 \quad (b_4 \ne 0 \text{ as } n \ge 3).$$

Thus, setting $y_1 = s$, we have $y_2 = s - s^{-1}$ and $a_1 = s$, $a_2 = s^2$.

The general case is similar: suppose $y_2 = \cdots = y_j = s - s^{-1}$ for $j \le n-2$; then $a_k = s^k$ and $a_{j-1} b_{j+1} = a_{j+1} b_{j+3} = a_j b_{j+2}$ implies

$$s^{j-1}(y_{j+1} b_{j+2} + b_{j+3}) = \left(y_{j+1} s^j + s^{j-1}\right) b_{j+3} = s^j b_{j+2}.$$

As above, the first equality implies $b_{j+2} = s b_{j+3}$ and the second implies $1 + y_{j+1} s = s^2$, so $y_{j+1} = s - s^{-1}$. Since the procedure is symmetric, $s - s^{-1} = y_{n-1} = y_n - y_n^{-1}$ implies $y_n = s$. Finally, $s - s^{-1} > 0$ implies $s > 1$ so $\Sigma y_i = ns - (n-2)s^{-1} > 2$.

3. The first question

Let us now formally define

$$(3.1)\qquad a(t) = \sup\left\{ p_n(x_1,\dots,x_n) \colon n \ge 1,\ x_i \ge 0,\ \sum x_i = t \right\}.$$

From the introduction, we know that $e^t \ge a(t) \ge F_{[t]+1}$. A heuristic candidate for making $p_n(\mathbf{x})$ large is $\mathbf{x} = (t/n,\dots,t/n)$; let $r_n(x) = p_n(x,\dots,x)$. Then $r_{-1}(x) = 0$, $r_0(x) = 1$ and $r_n(x) = xr_{n-1}(x) + r_{n-2}(x)$, and it is easy to obtain two closed formulas for $r_n(x)$. Viewing x as a constant, r_n satisfies a second order linear recurrence. By standard methods,

$$(3.2)\quad r_n(x) = \frac{1}{\sqrt{x^2+4}}\left[\left(\frac{x+\sqrt{x^2+4}}{2}\right)^{n+1} - \left(\frac{x-\sqrt{x^2+4}}{2}\right)^{n+1}\right].$$

On the other hand, in (2.2), there are $\binom{n-i}{i}$ ways to place i disjoint consecutive pairs in $\{1,\dots,n\}$, hence

$$(3.3)\qquad r_n(x) = \sum \binom{n-i}{i} x^{n-2i}.$$

As a side note, if U_n is the Chebyshev polynomial defined by

$$U_n(\cos\theta) = \sin(n+1)\theta/\sin\theta,$$

then $r_n(x) = i^n U_n(-ix/2)$, see [1, p. 775]. In any event, the asymptotics of $r_n(t/n)$ are easy to handle and are summarized in (3.4):

$$(3.4)\qquad r_{2m}\left(\frac{t}{2m}\right) \to \cosh\frac{t}{2}, \qquad r_{2m+1}\left(\frac{t}{2m+1}\right) \to \sinh\frac{t}{2}.$$

As it happens, this distribution of weights is not asymptotically optimal, but the (implicit) lower bound for $a(t)$ is only off by a factor of $4/e$.

THEOREM 3.5.

$$(3.6)\qquad \phi(t) := \max\left(1 + t^2/4, 2e^{t/2-1}\right) = a(t).$$

Remark. Note that (3.6) agrees with (1.2) and that $\phi''(2)$ exists so that ϕ is quite smooth.

Proof. Let $a_n(t) = \max\{ p_n(x_1,\dots,x_n)\colon x_i \ge 0, \Sigma x_i = t\}$. (By continuity and compactness, the supremum is achieved.) Clearly $a(t) = \sup_n a_n(t)$. Consultation with (1.7) shows that $a_2(t) = 1 + t^2/4 \geq t = a_1(t)$ for all t. Suppose now that $n \geq 3$ and $a_n(t) = p_n(\mathbf{y})$. By Lagrange multipliers, either $\mathbf{y}$ lies on the boundary of $\{x_i \geq 0, \Sigma x_i = t\}$ or

$$\frac{\partial p_n}{\partial x_j}(\mathbf{y}) = \frac{\partial p_n}{\partial x_k}(\mathbf{y}) \quad \text{for } 1 \leq j \leq k \leq n.$$

In the first case, by Lemma 2.11, $a_n(t) = p_n(\mathbf{y}) \leq p_m(\mathbf{z})$ where $\Sigma z_i = t$, $z_i > 0$ and $m < n$. Possibly $p_m(\mathbf{z}) \leq a_m(t)$; in any event, by repeating if necessary, we get $a_n(t) \leq a_k(t)$ for $k = 1$ or 2 or $a_n(t) \leq a_k(t) = p_k(\mathbf{w})$ for $\mathbf{w} > 0$ and $k \geq 3$.

In the second case we must have, by Lemma 2.12,

$$\mathbf{y} = (s, s - s^{-1}, \dots, s - s^{-1}, s), \qquad ns - (n-2)s^{-1} = t$$

and

$$a_n(t) = p_n(\mathbf{y}) = s^n + s^{n-2}.$$

Solving for s, and assuming $t \geq 2$, we have

$$s = \frac{t + \sqrt{4n^2 - 8n + t^2}}{2n} := s_n(t). \tag{3.7}$$

Letting

$$h_n(t) = s_n^n(t) + s_n^{n-2}(t), \tag{3.8}$$

and noting that $h_2(t) = s_2^2(t) + 1 = t^2/4 + 1 = a_2(t)$, we have

$$a(t) = \sup_{n \geq 2} h_n(t). \tag{3.9}$$

(To prove that $h_n(t) = a_n(t)$, we would have to prove that the h_n's are monotonically increasing. Numerical evidence suggests this, but we have no proof.)

For fixed t, $s_n(t) = 1 + (t/2 - 1)n^{-1} + O(n^{-2})$ as $n \to \infty$, so $h_n(t) \to 2\exp(t/2 - 1)$. A more careful analysis shows that

$$h_n(t) = 2e^{t/2-1}\left(1 - \frac{1}{6}\left(\frac{t}{2} - 1\right)^3 n^{-2} + O(n^{-3})\right). \tag{3.10}$$

The relatively high powers of n^{-1} and $t/2 - 1$ in (3.10) suggest the delicacy of the asymptotics. Indeed, $s_n^n(t) > \exp(t/2 - 1) > s_n^{n-2}(t)$ as $n \to \infty$ and we will show $h_n(t) \leq \phi(t)$ for $n \geq 2$, $t \geq 2$ in a regrettably indirect way. This, with (3.9) and (3.10) will complete the proof.

Suppose (3.11) holds for $u \geq 2$:

$$\frac{1}{2} \geq \frac{h_n'(u)}{h_n(u)}. \tag{3.11}$$

Since $h_n(2) = 2$, integrating (3.11) from 2 to $t \geq 2$ gives $\phi(t) \geq h_n(t)$. Thus we consider

$$F_n(u) = h_n(u) - 2h_n'(u). \tag{3.12}$$

Letting $w_n(u) = (4n^2 - 8n + u^2)^{1/2}$, $s_n(u)$ and $w_n(u)$ satisfy a pair of convenient differential equations:

$$s_n'(u) = \frac{s_n(u)}{w_n(u)}, \qquad w_n'(u) = \frac{u}{w_n(u)}. \tag{3.13}$$

Keeping in mind that $w_n(2) = 2n - 2 > 0$ and $s_n(2) = 1$ and suppressing the arguments, we have

$$\begin{aligned} F_n &= s_n^n + s_n^{n-2} - 2ns_n^{n-1}s_n' - 2(n-2)s_n^{n-3}s_n' \\ &= \frac{s_n^n}{w_n}\left[(w_n - 2n)s_n^2 + w_n - (2n-4)\right] \\ &:= \frac{s_n^n}{w_n}G_n. \end{aligned} \tag{3.14}$$

Since $G_n(2) = 2w_n(2) - (4n - 4) = 0$, it suffices to prove that $G_n'(u) \geq 0$ for $u \geq 2$. But

$$\begin{aligned} G_n' &= w_n's_n^2 + (w_n - 2n)2s_ns_n' + w_n' \\ &= \frac{1}{w_n}\left[(2w_n + u - 4n)s_n^2 + u\right] \\ &:= \frac{1}{w_n}H_n. \end{aligned} \tag{3.15}$$

Again, $H_n(2) = 2w_n(2) + 4 - 4n = 0$ so we are finished if we can show that $H_n'(u) \geq 0$ for $u \geq 2$. Finally,

$$\begin{aligned} H_n' &= (2w_n' + 1)s_n^2 + (2w_n + u - 4n)2s_ns_n' + 1 \\ &= \frac{s_n^2}{w_n}[4u + 5w_n - 8n] + 1. \end{aligned} \tag{3.16}$$

We are done if we can show that $5w_n \geq |8n - 4u|$ for $u \geq 2$. However, by squaring both sides and subtracting,

$$\begin{aligned} 25w_n^2 - 64n^2 + 64nu - 16u^2 &= 36n^2 + 64nu + 9u^2 - 200n \\ &= 36(n-1)^2 + (u-2)(64n + 9u + 18) \\ &\geq 0. \end{aligned}$$

Working backwards, (3.11) has been verified and Theorem 3.5 is proved.

To compare this result with that of Lucas,

$$\phi(2) = 2 = F_3, \qquad \phi(3) \cong 3.2974 > 3 = F_4,$$

and

$$\phi(t+2) - \phi(t+1) - \phi(t) = (e - e^{1/2} - 1)\phi(t) \simeq .0696\phi(t) > 0,$$

hence $\phi(n) \geq F_{n+1}$ by induction.

4. The second question

Now we restrict the x_i's to be positive integers and define

$$b(t) = \sup\left\{ p_n(x_1, \ldots, x_n) \colon n \geq 1,\ x_i \in \mathbf{N},\ \sum x_i = t \right\}. \tag{4.1}$$

In view of Lemma 2.10, the supremum could just as well be over $\{x_i \geq 0, x_i \in \mathbf{Z}\}$. For the rest of this paper, we will suppress the suffix on the continuant when it is clear from the context. By a *string* we will mean a (possibly void) sequence $\mathbf{x} = (x_1, \ldots, x_n)$. If $n = 0$, $p(\mathbf{x}) = 1$; if $n = 1$, $p(\mathbf{x}') = 1$, $p('\mathbf{x}') = 0$, etc. By two applications of (2.7),

$$p(\mathbf{x}, u, 1, \mathbf{z}) - p(\mathbf{x}, u+1, \mathbf{z}) = (u-1)p(\mathbf{x})p('\mathbf{z}) + p(\mathbf{x}')p('\mathbf{z}). \tag{4.2}$$

By (4.2), the value of p can only be increased by splitting any x_i larger than 1 into $x_i - 1$ and 1. Proceeding in this way, if $x_i \in \mathbf{N}$ and $\sum_{i=1}^n x_i = t$ it follows immediately that $p(\mathbf{x}) \leq p(1, \ldots, 1) = r_t(1) = F_{t+1}$ (cf. (3.2)).

Lucas' proof, mentioned in the introduction, is pretty enough to outline. Consider an infinite array of rationals, the first four rows of which appear in (4.3):

$$\begin{array}{ccccccccc} \frac{0}{1} & & & & & & & & \frac{1}{1} \\ \frac{0}{1} & & & & \frac{1}{2} & & & & \frac{1}{1} \\ \frac{0}{1} & & \frac{1}{3} & & \frac{1}{2} & & \frac{2}{3} & & \frac{1}{1} \\ \frac{0}{1} & \frac{1}{4} & \frac{1}{3} & \frac{2}{5} & \frac{1}{2} & \frac{3}{5} & \frac{2}{3} & \frac{3}{4} & \frac{1}{1}. \end{array} \tag{4.3}$$

Each row is an increasing subset of the Farey sequence; if a/b and c/d are consecutive in row k then a/b, $(a+c)/(b+d)$ and c/d are consecutive in row $k+1$. This is called the *Stern-Brocot array*. Lucas proved that every reduced rational in $[0,1]$ occurs eventually in (4.3) and, in fact, if $a/b = [k_1, \dots, k_s]^{-1}$ in the usual simple continued fraction representation ($k_j \in \mathbf{N}$), then it occurs for the first time in row $\sum_{j=1}^{s} k_j$. Thus $b(t)$ is the largest (new) denominator in row t. Since any such denominator is the sum of two previous denominators, one of which is old, $b(t) \le b(t-1) + b(t-2)$. On the other hand, as (4.3) suggests, F_{t+1} *does* occur in the *t-th* row, next to an inherited F_t, and these leapfrog each other down the array, producing the Fibonacci sequence.

As an aside, consider the successive numerators of the *k-th* row of (4.3). There are $2^k + 1$ of them, and the first $2^{k-1} + 1$ appear to be the successive numerators of the $(k-1)$-*st* row. This observation, which is easily proved by induction, leads to the definition of the sequence s_n: $s_0 = 0$, $s_1 = 1$, $s_2 = 1$, $s_3 = 2$, $s_4 = 1$, $s_5 = 3, \dots$. This sequence satisfies the recurrence $s_{2n} = s_n$ and $s_{2n+1} = s_n + s_{n+1}$. Historically speaking, (4.3) is Brocot's array and s_n is Stern's sequence. A combinatorial interpretation of s_n and the related sequence a_n (cf. (1.4)) is given at the end of the paper. For some other remarkable properties, see [5] and [6].

5. The third question

We now turn to continued fractions with general numerators. By (1.3) and the reasoning of (1.5), we have

$$p_n(\mathbf{x}, \mathbf{y}) = x_1 p_{n-1}('\mathbf{x}; '\mathbf{y}) + y_2 p_{n-2}(''\mathbf{x}; ''\mathbf{y}), \tag{5.1}$$

with $p_{-1} = 0$, $p_0 = 1$ as before. Now define

$$\begin{aligned} c_m(t) &= \sup\left\{ p_n(\mathbf{x}; \mathbf{y}) \colon n \le m,\ x_i, y_i \ge 0,\ \sum_{i=1}^{n} x_i + \sum_{i=2}^{n} y_i = t \right\}, \\ c(t) &= \sup_m c_m(t). \end{aligned} \tag{5.2}$$

It is immediate that $c_1(t) = t$. Suppose

$$x_1 + x_2 + y_2 = t;$$

and let $x = (x_1 + x_2)/2$. Then

$$p_2(x_1, x_2; y_2) = x_1 x_2 + y_2 \le x^2 + y_2 = x^2 - 2x + t.$$

Since $x^2 - 2x + t$ is a convex function of x, it can only achieve its maximum at the endpoints of its range: $x = 0$ or $x = t/2$. Thus

$$c_2(t) = \max\{t, t^2/4\}.$$

As Theorem 5.3 shows, this behavior is typical.

THEOREM 5.3.

$$\text{(5.4)} \qquad c_m(t) = \sup\left\{ \prod_{i=1}^{k} u_i \colon u_i \geq 0,\ \sum_{i=1}^{k} u_i = t,\ k \leq m \right\}.$$

Proof. We have just shown (5.4) for $m = 1$ and 2. It is easy to see from (5.1) that $p_m(x_1, \ldots, x_m; 0, \ldots, 0) = \prod_{i=1}^m x_i$, so (5.4) gives a lower bound for $c_m(t)$. On the other hand, suppose (5.4) has been established for $m \leq n - 1$ and $(\mathbf{x}, \mathbf{y})$ is given with $\sum_{i=1}^n x_i + \sum_{i=2}^n y_i = t$. Then by (5.1),

$$\text{(5.5)} \quad \begin{aligned} p_n(\mathbf{x}; \mathbf{y}) &\leq x_1 c_{n-1}(t - x_1 - y_2) + y_2 c_{n-2}(t - x_1 - y_2 - x_2 - y_3) \\ &\leq (x_1 + y_2) c_{n-1}(t - x_1 - y_2) \end{aligned}$$

by the (obvious) monotonicity of $c_m(t)$ in both m and t. Taking the supremum first over all $(\mathbf{x}, \mathbf{y})$ with fixed $x_1 + y_2 = u_1$ and then over all u completes the proof.

COROLLARY 5.6.

$$\text{(5.7)} \qquad c(t) = \max_k \left(\frac{t}{k} \right)^k.$$

Proof. By the arithmetic-geometric inequality, it follows from (5.4) that

$$c_m(t) = \max\{(t/k)^k \colon 1 \leq k \leq m\}.$$

Let $f(x) = (t/x)^x$ where x is a continuous real variable. A calculus exercise shows that $\log f$ is concave with maximum at $x = t/e$. This justifies the "max" in (5.7).

For t large, choosing $k = [t/e]$ gives $c(t) \sim \exp(t/e)$. More specifically, let $r_k = k^k(k-1)^{-(k-1)}$ $(= (k + \frac{1}{2})e + o(1))$. Then $c(t) = (t/k)^k$ for $t \in [r_k, r_{k+1}]$. It can be shown that

$$c(t) = \exp(t/e)[1 + O(t^{-1})],$$

but we omit the (routine) calculation. If we restrict the x_i's and y_i's to be non-negative integers then Theorem 5.3 still applies and we seek to maximize a product of positive integers given their fixed sum. As noted in the introduction this is a recent Putnam problem. To maximize the product, we replace (m) with $(2, m-2)$ if $m \geq 4$ and $(1, k)$ with $(k+1)$ for any k. The maximum thus occurs on strings of 2's and 3's. Since $3 \cdot 3 > 2 \cdot 2 \cdot 2$, there are at most two 2's. The exact number is determined by $t \bmod 3$. The solution is unique up to the substitution of (4) for (2,2) and equals $c_i 3^{t/3}$ where $.92 \leq c_i \leq 1$ for $t \equiv i \bmod 3$.

6. The fourth question

For $t \geq 1$ define

(6.1)
$$d(t) = \max\left\{ p_n(x_1, \dots, x_n; y_2, \dots, y_n) \colon n \geq 1,\ x_i, y_i \in \mathbf{N},\ \textstyle\sum x_i + \sum y_i = t \right\}.$$

LEMMA 6.2. *If $d(t) = p(\mathbf{x}; \mathbf{y})$ then all y_i's equal* 1. *Hence*

$$d(t) = \max\left\{ p_n(x_1, \dots, x_n) \colon n \geq 1,\ x_i \in \mathbf{N}^+,\ \sum_{i=1}^{n} x_i + n - 1 = t \right\}. \tag{6.3}$$

Proof. If $n = 1$, the lemma is vacuous; if $n = 2$ then $t \geq 3$ and the lemma is true by the argument of the last section. In any case, (5.1) and $x_i, y_i \geq 1$ imply that $p_n(\mathbf{x}, \mathbf{y}) \geq p_{n-1}('\mathbf{x}, '\mathbf{y})$. Keeping $w_j = x_j + y_{j+1}$ fixed for $1 \leq j \leq n-1$, it follows by repeated application of (5.1) that $p_n(\mathbf{x}, \mathbf{y})$ is only increased by setting x_j to $w_j - 1$ and y_j to 1.

It follows from this lemma that we only need consider simple continued fractions. Our analysis is based on the systematic elimination of strings $\mathbf{x} = (x_1, \dots, x_n)$ from consideration in (6.3) and is an extension of the ideas in the last two sections. If we write $\|\mathbf{x}\| = \sum_{i=1}^{n} x_i + n - 1$, the *norm* of x, then Lemma 6.2 is, more succinctly,

$$d(t) = \max\{ p(\mathbf{x}) \colon \|\mathbf{x}\| = t \}. \tag{6.4}$$

Suppose $\|\mathbf{u}\| = \|\mathbf{v}\|$ and for all strings $\mathbf{w}$ and $\mathbf{z}$, $p(\mathbf{w}, \mathbf{u}, \mathbf{z}) \geq p(\mathbf{w}, \mathbf{v}, \mathbf{z})$; as strings may be void this statement includes $p(\mathbf{w}, \mathbf{u}) \geq p(\mathbf{w}, \mathbf{v})$, etc. Since $\|(\mathbf{w}, \mathbf{u}, \mathbf{z})\| = \|(\mathbf{w}, \mathbf{v}, \mathbf{z})\|$, we may take the maximum in (6.4) over all strings which do not contain $\mathbf{v}$ as a substring. In this case we say that $\mathbf{u}$ *supersedes* $\mathbf{v}$, or $\mathbf{u} \succ \mathbf{v}$. We call a set of strings A *sufficient* if, for $t \geq 2$ there exists $\mathbf{x} \in A$, $\|\mathbf{x}\| = t$ with $p(\mathbf{x}) = d(t)$. Note that A need not contain *all* such maximizing strings. We take $t \geq 2$ to avoid the case $t = 1$, since $\|\mathbf{x}\| = 1$ implies $x = (1)$ and $p(\mathbf{x}) = 1$, and it is tedious to include this as a special case throughout.

Our first task is to find strings which are superseded. It is essential to keep (2.3) in mind; let $\tilde{\mathbf{x}} = (x_n, \dots, x_1)$. If $\mathbf{u} \succ \mathbf{v}$ then $\tilde{\mathbf{u}} \succ \tilde{\mathbf{v}}$; this reduces our work by half. The first lemma may be proved by two applications of (2.7).

LEMMA 6.5.

$$\begin{aligned} p(\mathbf{w},\mathbf{u},\mathbf{z}) - p(\mathbf{w},\mathbf{v},\mathbf{z}) &= p(\mathbf{w})p(\mathbf{z})(p(\mathbf{u}) - p(\mathbf{v})) \\ &\quad + p(\mathbf{w})p('\mathbf{z})(p(\mathbf{u}') - p(\mathbf{v}')) + p(\mathbf{w}')p(\mathbf{z})(p('\mathbf{u}) - p('\mathbf{v})) \\ &\quad + p(\mathbf{w}')p('\mathbf{z})(p('\mathbf{u}') - p('\mathbf{v}')). \end{aligned}$$

For given $\mathbf{u}$ and $\mathbf{v}$ let $D = p(\mathbf{u}) - p(\mathbf{v})$, $D' = p(\mathbf{u}') - p(\mathbf{v}')$, $'D = p('\mathbf{u}) - p('\mathbf{v})$ and $'D' = p('\mathbf{u}') - p('\mathbf{v}')$; $\mathbf{D} = (D, D', 'D, 'D')$ is the *difference vector* of $\mathbf{u}$ and $\mathbf{v}$.

LEMMA 6.6. *If* $\|\mathbf{u}\| = \|\mathbf{v}\|$ *and* (i), (ii) *or* (iii) *hold then* $\mathbf{u} \succ \mathbf{v}$:

(i) $D, D', 'D, 'D' \geq 0$;
(ii) $D, D + D', 'D, 'D + 'D' \geq 0$;
(iii) $D, D + 'D, D', D' + 'D' \geq 0$.

Proof. We prove (ii), which is equivalent to (iii) and implied by (i). For any $\mathbf{w}$ and $\mathbf{z}$, $p(\mathbf{w}) \geq p(\mathbf{w}') \geq 0$ and $p(\mathbf{z}) \geq p('\mathbf{z}) \geq 0$ since all components are at least 1. By Lemma 6.5, then,

$$p(\mathbf{w},\mathbf{u},\mathbf{z}) - p(\mathbf{w},\mathbf{v},\mathbf{z}) \geq (D + D')p(\mathbf{w})p('\mathbf{z}) + ('D + 'D')p(\mathbf{w}')p('\mathbf{z}) \geq 0.$$

It is now convenient to introduce some more notation which we will use extensively in this section. Let $(i^a, j^b, k^c, \dots)$ denote the string of a i's followed by b j's, c k's, etc., and let $r_n(3) = p(3^n)$. Given a set of integers I, let $A(I)$ denote the set of all strings $\mathbf{x}$ with $x_i \in I$. (Thus the maximum in (6.4) is taken over $A(\mathbf{N})$.) For a string $\mathbf{v}$ let $B(\mathbf{v})$ denote the set of all strings which do *not* contain $\mathbf{v}$ as a substring. Our last lemma for a while is a direct consequence of (2.5), (2.7) and such identities as $r_m - r_{m-2} = 3r_{m-1}$.

LEMMA 6.7.

(i) $p(3^n, x) = xr_n + r_{n-1} = r_{n+1} + (x - 3)r_n$.
(ii) $p(3^n, x, y) = (xy + 1)r_n + yr_{n-1}$.
(iii) $p(x, 3^n, y) = (xy + 1)r_n + (x + y - 3)r_{n-1}$.

PROPOSITION 6.8. *Let* $C = \{(3^a, 4, 3^b)\colon a, b \geq 0\}$; *then*

$$\big(A(\{2,3\}) \cap B(\{2^4\})\big) \cup C$$

is sufficient.

Remark. In other words, we may restrict out attention to strings of 2's and 3's with no four consecutive 2's and strings that are all 3's except for one 4.

Proof. We first show that no string containing a 1 can be maximal. Observe that $(\mathbf{x}, 1)$ may be written $(\mathbf{u}, s, 1)$ for $n \geq 2$ (since $t \geq 1$) and that

$$\|(\mathbf{u}, s, 1)\| = \|(\mathbf{u}, s+2)\| = \|\mathbf{u}\| + s + 3.$$

By Lemma 2.10(ii), $p(\mathbf{u}, s, 1) = p(\mathbf{u}, s+1) < p(\mathbf{u}, s+2)$. Now suppose $s \geq r$ and consider the difference vector of $\mathbf{u} = (r+2, s)$ and $\mathbf{v} = (r, 1, s)$, ($\|u\| = \|v\| = r + s + 3$):

$$\begin{aligned}\mathbf{D} &= (rs + 2s + 1, r + 2, s, 1) - (rs + r + s, r + 1, s + 1, 1)\\ &= (s - r + 1, 1, -1, 1).\end{aligned}$$

By Lemma 6.6(iii), $\mathbf{u} \succ \mathbf{v}$ and, as noted earlier, $(s, r+2) \succ (s, 1, r)$. Therefore $A(\mathbf{N}) \cap B(\{1\})$ is sufficient. If $s \geq 5$, then $(s-3, 2) \succ (s)$ with difference vector

$$(2s - 5, s - 3, 2, 1) - (s, 1, 1, 0) = (s - 5, s - 4, 1, 1) \geq 0.$$

Thus $A(\{2, 3, 4\})$ is sufficient.

Now we consider strings in $A(\{2, 3, 4\})$ which contain a 4; we show that

$$(3^{m+1}, 2, 2) \succ (4, 3^m, 4) \quad \text{and} \quad (3^{m+2}) \succ (4, 3^m, 2)$$

for $m \geq 0$. These two supersedures show that we can replace any string with a 4 if it contains any other non-3's. Let $\mathbf{u} = (3^{m+1}, 2, 2)$ and $\mathbf{v} = (4, 3^m, 4)$, then $\|\mathbf{u}\| = \|\mathbf{v}\| = 4m + 9$ and by Lemma 6.7,

$$\begin{aligned}\mathbf{D} &= (5r_{m+1} + 2r_m, 2r_{m+1} + r_m, 5r_m + 2r_{m-1}, 2r_m + r_{m-1})\\ &\quad - (17r_m + 5r_{m-1}, 4r_m + r_{m-1}, 4r_m + r_{m-1}, r_m)\\ &= (0, r_{m+1}, r_m + r_{m-1}, r_m + r_{m-1}) \geq 0.\end{aligned}$$

The skeptical reader may check this formula for $m = 0$: $\mathbf{u} = (3, 2, 2)$, $\mathbf{v} = (4, 4)$ and $\mathbf{D} = (0, 3, 1, 1)$. Now let $\mathbf{u} = (3^{m+2})$ and $\mathbf{v} = (4, 3^m, 2)$. Then $\|\mathbf{u}\| = \|\mathbf{v}\| = 4m + 7$, and, by more of Lemma 6.7,

$$\begin{aligned}\mathbf{D} &= (10r_m + 3r_{m-1}, 3r_m + r_{m-1}, 3r_m + r_{m-1}, r_m)\\ &\quad - (9r_m + 3r_{m-1}, 4r_m + r_{m-1}, 2r_m + r_{m-1}, r_m)\\ &= (r_m, -r_m, r_m, 0).\end{aligned}$$

By Lemma 6.6(ii), $\mathbf{u} \succ \mathbf{v}$, and by symmetry, $(3^{m+2}) \succ (2, 3^m, 4)$. Combining the above, $A(\{2, 3\}) \cup C$ is sufficient.

Finally, we exclude strings with four consecutive 2's. Note that

$$\|(3^3)\| = \|(2^4)\| = 11 \quad \text{and} \quad p(3^3) = 33 > 29 = p(2^4).$$

Further, we claim that $(3^3, 2) \succ (2^5)$ and $(3^4) \succ (2^4, 3)$. The norms on both sides of the supersedures are 14 and 15 respectively and the two difference vectors are

$$(109, 33, 33, 10) - (99, 29, 41, 12) = (10, 4, -8, -2)$$

and

$$(76, 33, 23, 10) - (70, 29, 29, 12) = (6, 4, -6, -2)$$

respectively. Two more applications of Lemma 6.6(iii) complete the proof of the claim. Since we are now restricting our attention to $A(\{2,3\}) \cup C$, we have handled all possible appearances of 2^4 and the proposition is proved.

In the last part of the proof, it is tempting to write $(3^3) \succ (2^4)$. This is false, however as $p(1, 2^4, 1) = 58 > 56 = p(1, 3^3, 1)$. In order to pare down $A(\{2,3\})$ one step further we need one more result, which we isolate as a lemma.

Lemma 6.9. $(3^{m+1}, 2^{n+1}) \succ (3, 2^n, 3^m, 2)$ *for* $n, m \geq 1$.

Proof. We compute, in turn, D, D', $'D$ and $'D'$. First,

$$D = p(3^{m+1}, 2^{n+1}) - p(3, 2^n, 3^m, 2) = p(2, 2^n, 3^m, 3) - p(3, 2^n, 3^m, 2).$$

By two applications of (2.7) with $\mathbf{y} = (2^n, 3^m)$,

$$\begin{aligned} D &= p(2^{n-1}, 3^m) - p(2^n, 3^{m-1}) = p(2^{n-1}, 3, 3^{m-1}) - p(2^{n-1}, 2, 3^{m-1}) \\ &= p(2^{n-1})p(3^{m-1}) > 0 \end{aligned}$$

by Lemma 6.5. Next, using (2.5) twice,

$$\begin{aligned} D' &= p(3^{m+1}, 2^n) - p(3, 2^n, 3^m) \\ &= 3p(3^m, 2^n) + p(3^{m-1}, 2^n) - 3p(2^n, 3^m) - p(2^{n-1}, 3^m) = -D. \end{aligned}$$

Similarly,

$$\begin{aligned} 'D &= p(3^m, 2^{n+1}) - p(2^n, 3^m, 2) \\ &= 2p(3^m, 2^n) + p(3^m, 2^{n-1}) - 2p(2^n, 3^m) - p(2^n, 3^{m-1}) = D. \end{aligned}$$

Since $'D' = p(3^m, 2^n) - p(2^n, 3^m) = 0$, $\mathbf{D} = (D, -D, D, 0)$ and the lemma is proved by another application of (6.6)(ii).

The import of this lemma is that we need only consider strings of 2's and 3's with at most two "changes". The delicacy of the argument is suggested by the fact that $(3^{m+1}, 2^{n+1})$ does not supersede $(2, 3^m, 2^n, 3)$; of course $(2^{n+1}, 3^{m+1})$ does, by symmetry. We are now done with " $\succ$ "; we can calculate $p(\mathbf{x})$ for the last sufficient set. The next lemma reduces us to four cases.

LEMMA 6.10.

$$p(x^l, y^m, x^n) - p(x^{l+n}, y^m) = (x-y)p(x^{l-1})p(y^{m-1})p(x^{n-1}).$$

Proof. By two applications of (2.7), the left hand side equals

$$\begin{aligned}
&p(x^l, y^m, x^n) - p(x^l, x^n, y^m)\\
&\quad = p(x^l)\big[p(y^m)p(x^n) + p(y^{m-1})p(x^{n-1})\\
&\qquad -p(x^n)p(y^m) - p(x^{n-1})p(y^{m-1})\big]\\
&\qquad +p(x^{l-1})\big[p(y^{m-1})p(x^n) + p(y^{m-2})p(x^{n-1})\\
&\qquad -p(x^{n-1})p(y^m) - p(x^{n-2})p(y^{m-1})\big]\\
&\quad = p(x^{l-1})\big[p(y^{m-1})(p(x^n) - p(x^{n-2}))\\
&\qquad -p(x^{n-1})(p(y^m) - p(y^{m-2}))\big]\\
&\quad = p(x^{l-1})\big[xp(x^{n-1})p(y^{m-1}) - yp(y^{m-1})p(x^{n-1})\big].
\end{aligned}$$

THEOREM 6.11. *The set of strings of the form* (2^m), (3^a), $(4, 3^a)$ *and* $(3^a, 2^m, 3)$ *with* $a \geq 1$ *and* $3 \geq m \geq 1$ *is sufficient.*

Proof. By Proposition 6.8 and Lemma 6.10, the named strings plus those of the form $(2^m, 3^a)$, $(3^a, 4, 3^b)$, $(2^m, 3^a, 2^n)$ and $(3^a, 2^m, 3^b)$ form a sufficient set. By Lemma 6.10, however,

$$p(3^{a+b}, 4) \geq p(3^a, 4, 3^b),\ p(3^a, 2^{m+n}) \geq p(2^m, 3^a, 2^n)$$

and

$$p(3^{a-1}, 2^m, 3) \geq p(3^a, 2^m),$$

$a \geq 1$, so we may exclude these strings. Further,

$$\begin{aligned}
p(3^a, 2^m, 3^b) &= p(3^{a+b}, 2^m) + p(3^{a-1})p(3^{b-1})p(2^{m-1})\\
&= p(3^{a+b}, 2^m) + p(2^{m-1})r_{a-1}r_{b-1}.
\end{aligned}$$

Hence the theorem will be proved if we can show that $r_0 r_{a+b-2} \geq r_{a-1}r_{b-1}$ for

$a \geq 1$, $b \geq 1$. Fix $c \geq 1$ and let $t_d = r_0 r_{d+c} - r_c r_d$. Then $t_0 = 0$, $t_1 = r_{c+1} - 3r_c = r_{c-1} \geq 0$ and $t_{d+2} = 3t_{d+1} + t_d$ so $t_d \geq 0$ for all $d \geq 0$ and we are done.

The following table presents the remaining strings with the relevant parameters computed. We separate out the three possible values for m.

Table 6.12

$\mathbf{x}$	$\lVert\mathbf{x}\rVert$	$p(\mathbf{x})$
(2)	2	2
(2^2)	5	5
(2^3)	8	12
(3^a), $a \geq 1$	$4a - 1$	r_a
$(4, 3^a)$, $a \geq 0$	$4a + 4$	$4r_a + r_{a-1}$
$(3^a, 2, 3)$, $a \geq 0$	$4a + 6$	$7r_a + 3r_{a-1}$
$(3^a, 2^2, 3)$, $a \geq 0$	$4a + 9$	$17r_a + 7r_{a-1}$
$(3^a, 2^3, 3)$, $a \geq 0$	$4a + 12$	$41r_a + 17r_{a-1}$

Observe that the strings in this table have norms which cover $\mathbf{N} - \{1\}$ with duplication only at multiples of 4. First,

$$\lVert(2^3)\rVert = \lVert(4,3)\rVert = 8 \quad \text{and} \quad p(4,3) = 13 > p(2^3),$$

so $d(8) = 13$. Otherwise, suppose $t = 4m$, $m \geq 3$; then

$$\lVert(4, 3^{m-1})\rVert = \lVert(3^{m-3}, 2^3, 3)\rVert = t$$

and

$$\begin{aligned} p(4, 3^{m-1}) - p(3^{m-3}, 2^3, 3) &= 4r_{m-1} + r_{m-2} - 41r_{m-3} - 17r_{m-4} \\ &= 2r_{m-3} - 4r_{m-4} = u_m. \end{aligned}$$

Again, $u_3 = 2$, $u_4 = 2$ and $u_{m+2} = 3u_{m+1} + u_m$ so $u_m > 0$. Putting it all together, with a relabeling of indices, and the usual manipulations with the r_k's, we have the following corollary.

COROLLARY 6.13. *If $t \leq 5$, $d(t) = t$; otherwise,*

$$d(4m) = r_m + r_{m-1}, \quad d(4m+1) = -4r_m + 19r_{m-1},$$
$$d(4m+2) = 3r_m - 2r_{m-1} \quad \textit{and} \quad d(4m+3) = 3r_m + r_{m-1}.$$

Let $\alpha = (\sqrt{13} + 3)/2$; $r_m \sim 13^{-1/2}\alpha^{m+1}$ from (3.2) so that $d(t) \sim c_i \alpha^{t/4}$, for $t \equiv i \bmod 4$ with $1.19 \leq c_i \leq 1.24$.

Recall the sequence $\{a_n\}$ defined in (1.4). It is a consequence of the recurrence that $a_{4n+3} = a_n$, so the candidate n's to make a_n large quickly have the binary form $n = [10 \cdots 010 \cdots 010 \cdots 1]_2$. That is, there are no two consecutive 1's in the base 2 representation of n. In fact, if there are blocks of 0's of lengths $k_1, \ldots, k_r$ in this representation, then $|a_n| = p_r(k_1, \ldots, k_r)$. Since

$$2^m > n \geq 2^{m-1} \quad \text{for } m = \|(k_1, \ldots, k_r)\|,$$

Corollary 6.13 proves the claim made in the introduction. See [14] for further details.

The sequence $\{a_n\}$ can be derived in a way similar to the Stern sequence, except that you take the difference (rather than the sum) of two consecutive terms to define the $(k+1)$-*st* row from the k-*th*. This sequence and the Stern sequence have analogous combinatorial interpretations. Let $a(m,i)$ denote the number of ways to write m as $\Sigma\varepsilon_k 2^k$ with $\varepsilon_k \in \{0,1,2\}$. Then $s_n = \Sigma a(n-1,i)$ and $a_n = \Sigma(-1)^i a(n-1,i)$. I thank Professor A. Garsia for prodding me to find a combinatorial interpretation of $\{a_n\}$.

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