ILLINOIS JOURNAL OF MATHEMATICS
Volume 29, Number 4, Winter 1985

INNER PRODUCTS ON A GREEN RING FOR FINITE GROUPS WITH A CYCLIC P-SYLOW SUBGROUP

BY

I. REITEN AND K.W. ROGGENKAMP

Introduction

Let G be a finite group with a cyclic p-Sylow subgroup and let R be an unramified extension of the p-adic integers, for some prime number p. Denote by $\mathfrak{p}$ the radical of R and by K its field of quotients. Then L will be either R or $R/\mathfrak{p} = k$. In addition we assume k to be a splitting field for G. (This is a technical assumption which is only used in Lemma 1.2 to guarantee that the projectives in a minimal projective resolution of R over RG are indecomposable. It is superfluous when $L = k$ (see [10]), and if the p-Sylow subgroup has order p [6], [14].) Let ${}_{LG}M^0$ be the category of L-free finitely generated left LG-modules, and $\mathfrak{A}_L(G)$ the Green ring of the LG-modules in ${}_{LG}M^0$, that is, the elements in $\mathfrak{A}_L(G)$ are generated by the isomorphism classes of modules in ${}_{LG}M^0$. Addition is induced from the direct sum and multiplication from the tensor product over L. We often do not distinguish carefully between the modules in ${}_{LG}M^0$ and the objects in $\mathfrak{A}_L(G)$.

Denote by L_0 the trivial LG-module, and consider

$$\mathscr{P}_{L_0}\colon \cdots \to Q_i \to Q_{i-1} \to \cdots \to Q_0 \to L_0 \to 0,$$

a minimal projective resolution of L_0. We note that if $L = R$ and the p-Sylow subgroup of G has order p, then all nonprojective indecomposable R-free RG-modules in the principal block occur as syzygies in $\mathscr{P}_{R_0}$ [6], [14]. Let $\mathfrak{A}_L^0(G)$ be the subring of $\mathfrak{A}_L(G)$ generated by the finitely generated projective LG-modules and the syzygies in $\mathscr{P}_{L_0}$. If Ω_i is such a syzygy, then $\mathscr{P}_{L_0} \otimes_L \Omega_i$ gives a projective resolution of Ω_i, so that $\Omega_j \otimes_L \Omega_i$ decomposes into a direct sum of a projective and a syzygy module of Ω_i, which is also a syzygy of L_0. $\mathfrak{A}_L^1(G)$ denotes the ideal in $\mathfrak{A}_L^0(G)$ generated by the finitely generated projective modules.

In this note we study a bilinear form $[\ \ ,\ \ ]$ on $\mathfrak{A}_L^0(G)$, and we show that this form is nondegenerate unless $L = R$ and the p-Sylow subgroup of G has order 2. To prove this, denoting by Q the rational numbers, we consider the associated ring $\tilde{\mathfrak{A}}_L^0(G) = Q \otimes_Z \mathfrak{A}_L^0(G)$ with corresponding ideal $\tilde{\mathfrak{A}}_L^1(G)$ and

Received April 27, 1983.

define a related bilinear form $\langle\ ,\ \rangle$ on the quotient $\overline{\tilde{\mathfrak{A}}^0_L(G)} = \tilde{\mathfrak{A}}^0_L(G)/\tilde{\mathfrak{A}}^1_L(G)$.

The bilinear forms we consider are described as follows.

For M, N in ${}_{LG}M^0$, denote by $P(M, N)$ the projective homomorphisms, that is, the $\phi \in \operatorname{Hom}_{LG}(M, N)$ such that there exists a commutative diagram

$$\begin{tikzcd} & P \arrow[dr, "v"] & \\ M \arrow[ur, "u"] \arrow[rr, "\varphi"] & & N \end{tikzcd}$$

with P projective. It was shown in [4] using almost split sequences that $[\ ,\ ] = \dim_k P(\ ,\ )$ is a symmetric nondegenerate bilinear form on $\mathfrak{A}_k(G)$, and there is a generalization to symmetric algebras in [3]. This is the form we consider when $L = k$. If X is indecomposable and in $\mathfrak{A}^0_k(G)$, then the dual $\hat{X}$ of X as constructed in [4] using almost split sequences does not in general lie in $\mathfrak{A}^0_k(G)$, so the result for $\mathfrak{A}_k(G)$ can not be applied. If $L = R$ we define the form $[\ ,\ ]$ by

$$[M, N] = \dim_k\big(P(M, N) + \mathfrak{p}\operatorname{Hom}_{RG}(M, N)\big)/\mathfrak{p}\operatorname{Hom}_{RG}(M, N),$$

for M, N in ${}_{RG}M^0$. As for $L = k$ [4], we show that $[M, N]$ is the number of times Q_0, the projective cover of the trivial module R_0, occurs as a summand in a direct sum decomposition of $\operatorname{Hom}_R(M, N)$. We reduce the problem of showing that $[\ ,\ ]$ is nondegenerate on $\mathfrak{A}^0_R(G)$ (with the exceptions mentioned before), to the corresponding problem for $\mathfrak{A}^0_k(G)$.

Let $P_1, \ldots, P_e$ be the nonisomorphic indecomposable projective kG-modules. Since the Cartan matrix

$$C = (c_{ij})_{1 \le i, j \le e}, \text{ where } c_{ij} = \dim_k \operatorname{Hom}_{kG}(P_i, P_j),$$

is known to have nonzero determinant, there is a dual basis $\{P_1^{\perp}, \ldots, P_e^{\perp}\}$ to $\{P_1, \ldots, P_e\}$ in $\tilde{\mathfrak{A}}^1_k(G)$ with respect to $[\ ,\ ]$. For X, Y in $\tilde{\mathfrak{A}}^0_k(G)$ we define

$$\langle X, Y\rangle' = [X, Y] - \sum_{i=1}^{e} [P_i, Y][X, P_i^{\perp}].$$

Then $\langle\ ,\ \rangle'$ vanishes on $\tilde{\mathfrak{A}}^1_k(G)$, and hence it induces a bilinear form $\langle\ ,\ \rangle$ on $\overline{\tilde{\mathfrak{A}}^0_k(G)}$. We also consider $\langle\ ,\ \rangle$ as a form on the subgroup $\mathfrak{A}^2_k(G)$ of $\mathfrak{A}^0_k(G)$ generated by the indecomposable nonprojective modules. We prove that $\langle\ ,\ \rangle$ is nondegenerate on $\overline{\tilde{\mathfrak{A}}^0_k(G)}$.

Two algebras B and B' are said to be stably equivalent if the module categories modulo projectives are equivalent categories. The form $[\ ,\ ]$ is not invariant under stable equivalence. We can, however, prove along the way that the form $\langle\ ,\ \rangle$ on $\overline{\tilde{\mathfrak{A}}^0_k(G)}$ is invariant under stable equivalence. We found this fact, which is of interest in itself, surprising since $\langle\ ,\ \rangle$ is defined entirely in terms of projective homomorphisms.

The proofs will be carried out so that they apply to blocks with a cyclic defect group and k_0 replaced by a suitably chosen irreducible representation. We hope that our results can be used to get orthogonality relations like in [4], [16].

The organization of the paper is as follows. In Section 1 we reduce the case $L = R$ to the case $L = k$, and we show that $\langle\ ,\ \rangle$ being nondegenerate on $\overline{\tilde{\mathfrak{A}}_k^0(G)}$ implies that $[\ ,\ ]$ is nondegenerate on $\overline{\mathfrak{A}_k^0(G)}$. In Section 2 we show that $\langle\ ,\ \rangle$ is invariant under stable equivalence and prove that it is nondegenerate on $\overline{\tilde{\mathfrak{A}}_k^0(G)}$. In Section 3 we give some examples, in particular showing that our results on invariance under stable equivalence do not have an obvious generalization. In Section 4 we consider a Brauer tree T and show that there is a Bäckström order Λ such that $\Lambda/\mathfrak{p}\Lambda = S$ is associated with the Brauer tree T, and the indecomposable Λ-lattices reduce modulo $\mathfrak{p}$ exactly to the indecomposables occurring as syzygies in a minimal projective resolution of a simple module corresponding to an edge having one vertex which is a nonexceptional end point of the tree. Alternatively, we could prove the results on the forms working with this Λ, but this turned out not to be necessary. We include the construction since it seems interesting in itself, and may be thought of as an analogue of the result that for every Brauer tree there is some associated symmetric algebra [8] [9].

We would like to thank M. Auslander and A. Wiedemann for valuable discussions on some of the questions involved.

1. Connection between nondegeneracy of two forms

In this section we reduce the problem for orders and for algebras to a common setting. Then we show that if $\langle\ ,\ \rangle$ is nondegenerate on $\overline{\tilde{\mathfrak{A}}_k^0(G)}$, then $[\ ,\ ]$ is nondegenerate on $\overline{\mathfrak{A}_k^0(G)}$.

Let R and K be as before and Λ an R-order in the semisimple K-algebra A. We write $S = \Lambda/\mathfrak{p}\Lambda$, and let $[\ ,\ ]_\Lambda$ denote our form on ${}_\Lambda M^0$, $[\ ,\ ]_S$ the form on ${}_S M^0$, as defined in the introduction for RG and kG. The following fact was pointed out to us by M. Auslander.

LEMMA 1.1. *If Λ is a Gorenstein order, then for M, N in ${}_\Lambda M^0$,*

$$[M, N]_\Lambda = [M/\mathfrak{p}M, N/\mathfrak{p}N]_S.$$

Proof. We recall that Λ is said to be a Gorenstein order provided Λ is an injective object in ${}_\Lambda M^0$.

Reduction modulo $\mathfrak{p}$ induces an R-linear map

$$\rho:\mathrm{Hom}_\Lambda(M,N) \to \mathrm{Hom}_S(M/\mathfrak{p}M, N/\mathfrak{p}N).$$

If $\phi \in P_\Lambda(M,N)$, then clearly $\rho(\phi) \in P_S(M/\mathfrak{p}M, N/\mathfrak{p}N)$, so there is an induced map ρ':

$$P_\Lambda(M,N) \to P_S(M/\mathfrak{p}M, N/\mathfrak{p}N).$$

If M is projective, then ρ is surjective, and dually, if N is an injective object in ${}_\Lambda M^0$, then ρ is surjective. Since Λ is a Gorenstein order, ρ is surjective when N is projective. This implies that ρ' is surjective. For given a factorization

$$\begin{tikzcd} & \bar{P} \arrow[dr, "\tilde{\beta}"] & \\ M/\mathfrak{p}M \arrow[ur, "\tilde{\alpha}"] \arrow[rr, "\tilde{\phi}"] & & N/\mathfrak{p}N, \end{tikzcd}$$

where $\bar{P}$ is projective in ${}_S M^0$, there is a projective Λ-module P with $P/\mathfrak{p}P \simeq \bar{P}$, and by the above $\tilde{\alpha}$ and $\tilde{\beta}$ can be lifted to $\alpha \in P_\Lambda(M,P)$ and $\beta \in P_\Lambda(P,N)$. Hence $\beta\alpha$ lifts $\tilde{\phi}$. But then the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{p}\,\mathrm{Hom}_\Lambda(M,N) \arrow[r] & \mathrm{Hom}_\Lambda(M,N) \arrow[r, "\rho"] & \mathrm{Hom}_S(M/\mathfrak{p}M, N/\mathfrak{p}N) \arrow[r] & 0 \\ 0 \arrow[r] & \mathrm{Ker}\,\rho' \arrow[r] \arrow[u, hook] & P_\Lambda(M,N) \arrow[r, "\rho'"] \arrow[u, hook] & P_S(M/\mathfrak{p}M, N/\mathfrak{p}N) \arrow[r] \arrow[u, hook] & 0 \end{tikzcd}$$

shows that $\mathrm{Ker}\,\rho' = P_\Lambda(M,N) \cap \mathfrak{p}\,\mathrm{Hom}_\Lambda(M,N)$, and hence

$$\begin{aligned} P_S(M/\mathfrak{p}M, N/\mathfrak{p}N) &\simeq P_\Lambda(M,N)/(P_\Lambda(M,N) \cap \mathfrak{p}\,\mathrm{Hom}_\Lambda(M,N)) \\ &\simeq (P_\Lambda(M,N) + \mathfrak{p}\,\mathrm{Hom}_\Lambda(M,N))/\mathfrak{p}\,\mathrm{Hom}_\Lambda(M,N). \end{aligned}$$

This finishes the proof of the lemma.

LEMMA 1.2. *With the notation of the introduction,*

$$R/\mathfrak{p} \otimes_R \mathscr{P}_{R_0} \simeq \mathscr{P}_{k_0};$$

in particular,

$$\mathfrak{A}_R^0(G) \simeq \mathfrak{A}_k^0(G),\ \mathfrak{A}_R^1(G) \simeq \mathfrak{A}_k^1(G),$$

unless $p = 2$ and the 2-Sylow subgroup of G has order 2.

Proof. For an R-free RG-module X we write $\overline{X} = X/\mathfrak{p}X$. Since

$$\overline{X}/\operatorname{rad}\overline{X} \simeq X/\operatorname{rad} X,$$

X has a simple top if and only if the same holds for $\overline{X}$. Since the p-Sylow subgroup of G is cyclic, it follows from [5] that all syzygies of R_0 have a simple top. Since $\overline{R}_0 \simeq k_0$, it follows that $R/\mathfrak{p} \otimes_R \mathscr{P}_{R_0}$ is a minimal projective resolution of k_0.

Let $\{\Omega_i\}$ be the syzygies in $\mathscr{P}_{R_0}$. We claim that $\Omega_i \simeq \Omega_j$ if and only if $\overline{\Omega}_i \simeq \overline{\Omega}_j$, unless $p = 2$ and the 2-Sylow subgroup of G has order 2. To see this, let P be a p-Sylow subgroup of G and N the normalizer of P in G. Unless $p = 2$ and P has order 2, the RP-modules R_0, RP and the augmentation ideal $I_R(P)$ satisfy the hypothesis of [11, Theorem 1]. Since a syzygy in an RN-minimal resolution of R_0 is a direct summand of one of the induced modules $RN \otimes_{RP} R_0$, RN or $RN \otimes_{RP} I_R(P)$, the result follows for N. Using the first part of the proof we can pass from N to G with Green correspondence.

To complete the proof we use that Green correspondence from N to G commutes with tensor products and that for N the result follows from [11, Theorem 2].

We note that Lemma 1.2 reduces for normal P with more than two elements to [12, Theorem 5]. Observe also that the result is definitely false for $p = 2$ and P of order 2.

We have the following consequence of Lemmas 1.1 and 1.2.

PROPOSITION 1.3. *Let the notation be as before and assume that the p-Sylow subgroup P of G is not of order 2. Then* $\langle\ ,\ \rangle$ *is nondegenerate on* $\overline{\tilde{\mathfrak{A}}_R^0(G)}$ *if and only if it is nondegenerate on* $\overline{\tilde{\mathfrak{A}}_k^0(G)}$.

As for $L = k$ [4], we have the following description of the form $[\ ,\ ]$ for ${}_{RG}M^0$.

PROPOSITION 1.4. *For* M, N *in* ${}_{RG}M^0$, $[M, N]$ *is the number of times the projective cover* P_0 *of the trivial module* L_0 *occurs as a summand in a direct sum decomposition of* $\operatorname{Hom}_R(M, N)$.

Proof. The corresponding result with R replaced by k was proved in [4]. Since RG is a Gorenstein order, we want to use Lemma 1.1 to reduce to this case. If $P_0^{(s)} | \operatorname{Hom}_R(M, N)$, then clearly

$$(P_0/\mathfrak{p}P_0)^{(s)} | \operatorname{Hom}_k(M/\mathfrak{p}M, N/\mathfrak{p}N),$$

and $P_0/\mathfrak{p}P_0 = \overline{P_0}$ is the projective cover of the trivial module k. Conversely, assume that

$$\overline{P_0} | \operatorname{Hom}_k(M/\mathfrak{p}M, N/\mathfrak{p}N) \simeq \overline{\operatorname{Hom}_R(M, N)}$$

and let $\operatorname{Hom}_R(M, N) = X$. We then have maps $\bar{\pi}: \bar{X} \to \overline{P_0}$ and $\bar{i}: \overline{P_0} \to \bar{X}$ such that $\bar{\pi}\bar{i} = \bar{e} = \bar{e}^2$. By the proof of Lemma 1.1, $\bar{\pi}$ can be lifted to $\pi: X \to P_0$ and $\bar{i}$ to $i: P_0 \to X$. Then $\pi i: X \to P_0 \to X$ is an idempotent modulo $\mathfrak{p}$, and hence π is surjective, so that P_0 is a summand of $X = \operatorname{Hom}_R(M, N)$.

We now show that it is sufficient for our problem to show that $\langle \ , \ \rangle$ is nondegenerate on $\overline{\tilde{\mathfrak{A}}_k^0(G)}$, as a consequence of the following more general result.

PROPOSITION 1.5. *Let S be a k-algebra where all simples have k as endomorphism ring, whose Cartan matrix has nonzero determinant and where $[\ ,\ ]$ is symmetric. Let $\mathscr{D}$ be an additive subcategory of ${}_SM^0$ containing the projectives, $\mathfrak{A}(\mathscr{D})$ the free abelian group having the isomorphism classes of indecomposable modules in $\mathscr{D}$ as basis. If the form $\langle\ ,\ \rangle$ is nondegenerate on $\tilde{\mathfrak{A}}(\mathscr{D})/\tilde{\mathfrak{A}}^1(S)$(where $\mathfrak{A}^1(S)$ is generated by the projectives), then $[\ ,\ ]$ is nondegenerate on $\mathfrak{A}(\mathscr{D})$.*

Proof. Write X in $\tilde{\mathfrak{A}}(\mathscr{D})$ as $X = \sum_{i=1}^e a_i P_i + \sum_{i=1}^t b_i M_i$, where the M_i are the indecomposable nonprojective objects in $\mathscr{D}$ and the a_i and b_i are in Q. Assume that $[X, Y] = 0$ for all Y in $\tilde{\mathfrak{A}}(\mathscr{D})$. Consider

$$\langle X, Y\rangle = [X, Y] - \sum_{i=1}^{e} [X, P_i^{\perp}][P_i, Y].$$

We must then have that $\langle X, Y\rangle = 0$. If P is indecomposable projective, then $\langle P, Y\rangle = 0$. Hence we get that $\langle \sum_{i=1}^t b_i M_i, Y\rangle = 0$ for all Y in $\tilde{\mathfrak{A}}(\mathscr{D})/\tilde{\mathfrak{A}}^1(S)$. Since $\langle\ ,\ \rangle$ is assumed to be nondegenerate, we must have that all b_i are zero. Since $[\ ,\ ]$ is nondegenerate on $\tilde{\mathfrak{A}}^1(S)$, we conclude that also all the a_i are zero. Since $[\ ,\ ]$ is symmetric, this shows that $[\ ,\ ]$ is nondegenerate on $\tilde{\mathfrak{A}}(\mathscr{D})$ and hence on $\mathfrak{A}(\mathscr{D})$.

2. Nondegeneracy of forms for Brauer trees

Let k be a field, T a Brauer tree with e edges and multiplicity m at the exceptional vertex, and S a corresponding k-algebra. For example the blocks of group algebras with cyclic defect group are given by a Brauer tree when k is a splitting field for the group [8], [9]. (See [10] for arbitrary k.) The edges are in one-one correspondence with the indecomposable projective S-modules $P_1, \ldots, P_e$. Consider an edge having a vertex which is a nonexceptional end point of the tree and the associated projective module Q_0. Then there is an exact sequence

$$0 \to S_0 \to Q_{2e-1} \to \cdots \to Q_1 \to Q_0 \to S_0 \to 0,$$

where S_0 is simple, all Q_i are indecomposable projective, and the resolution is obtained by walking around the tree T [1], [5]. Denote by $\mathfrak{A}(S)$ the free abelian group whose elements are the isomorphism classes of finitely generated modules, by $\mathfrak{A}^0(S)$ the subgroup generated by the syzygy modules $\Omega^i S_0$, $0 \leq i < 2e$, and the indecomposable projectives and by $\mathfrak{A}^1(S)$ the subgroup generated by the indecomposable projectives.

Let $[\ ,\ ]$ on $\tilde{\mathfrak{A}}^0(S)$ and $\langle\ ,\ \rangle$ on $\tilde{\mathfrak{A}}^0(S)/\tilde{\mathfrak{A}}^1(S)$ be the bilinear forms as defined before, where we use the fact that the Cartan matrix for S has nonzero determinant. We find a more suitable expression for $\langle\ ,\ \rangle$, enabling us to show that it is nondegenerate on $\tilde{\mathfrak{A}}^0(S)/\tilde{\mathfrak{A}}^1(S)$. Along the way we show the curious fact that $\langle\ ,\ \rangle$ is invariant under stable equivalence.

Let $T_1, \ldots, T_{e+1}$ denote the vertices of the Brauer tree T and assume for all the lemmas that T is not

$$\overset{1}{\bullet}\!\!\frac{\qquad\quad}{}\!\!\overset{1}{\bullet}\,.$$

Then the modules $\Omega^i S_0$, $0 \leq i < 2e$, are pairwise nonisomorphic, and we can think of them as belonging to exactly one of these vertices in the following way, which has to do with how the resolution is obtained by walking around the tree. Let S_0 belong to the end point we start with. We associate $\Omega^1 S_0$ with the other vertex of this edge. $\Omega^2 S_0$ is placed at the other vertex of the edge of the projective cover of $\Omega^1 S_0$, and so on. For $M = \Omega^i S$ we define $\operatorname{sig}(M) = (-1)^i$. If M belongs to the vertex T_j, we define $\operatorname{sig} T_j = \operatorname{sig} M$. This is clearly well defined since T is a tree.

We have the following description of the modules belonging to a given vertex in the above sense.

LEMMA 2.1. *The indecomposable modules belonging to a given vertex T_i are the following: For each edge E with T_i as a vertex, take the uniserial module corresponding to winding around T_i m_i times, starting with E and ending with the edge preceding E, where the composition factors are given from top to bottom. Here m_i is m at the exceptional vertex and* 1 *otherwise.*

Proof. Assume some $\Omega^t S_0$ at T_j has this form. Let E be the edge corresponding to the projective cover of $\Omega^t S_0$. From the structure of indecomposable projectives $\Omega^{t+1} S_0$ is then of the desired form. It is associated with the other vertex of E, and the structure as uniserial module is given by starting with the edge following E. Since S_0 itself has the desired form, these considerations prove the lemma.

The values $[P, M]$ and $[M, N]$, for M and N indecomposable in $\mathfrak{A}^0(S)$, depend heavily on the vertex to which the modules belong.

LEMMA 2.2. *Let M and N be syzygies of S_0.*

(a) $[P_u, P_v]$ *is* m_i *if* $P_u \not\simeq P_v$ *have* T_i *as a common vertex, is* $\max(m_i + 1, m_j + 1)$ *if* $P_u \simeq P_v$ *and* T_i *and* T_j *are the corresponding vertices, and is* 0 *otherwise.*

(b) $[P, M] = [M, P]$ *is equal to* m_i *if* M *belongs to a vertex* T_i *of the edge corresponding to* P, *and is* 0 *otherwise.*

(c) $[M, N]$ *is* $m_i - 1$ *if* $M \simeq N$, *is* m_i *if* $M \not\simeq N$, *but* M *and* N *belong to the same vertex* T_i, *and is* 0 *otherwise.*

Proof. (a) This follows directly from the description of the indecomposable projectives, since $[P_u, P_v]$ equals the number of times $P_u/\mathfrak{r}P_u$ where $\mathfrak{r}$ is the radical of S, occurs as a composition factor in P_v.

(b) This follows similarly, by counting composition factors.

(c) If M and N belong to the same vertex T_i, then $M/\mathfrak{r}M$ occurs m_i times as a composition factor in N. From the description of M and N given in Lemma 2.1 it follows that each map from M to N which is not an isomorphism must factor through a projective module.

Let M and N belong to different vertices. If there is no edge connecting these vertices, then M and N have no common composition factors, so that there are no nonzero maps from M to N. If there is an edge connecting the vertices, M and N have one composition factor in common. But it is easy to see that any corresponding nonzero map can not factor through a projective module.

We shall need the following matrices associated with a Brauer tree, in addition to the Cartan matrix. The $(e+1, e)$ matrix $D = (d_{ij}), 1 \le i \le e+1, 1 \le j \le e$, is defined as follows:

$$d_{ij} = \begin{cases} m_i & \text{if } T_i \text{ is a vertex of the edge corresponding to } P_j. \\ 0 & \text{otherwise.} \end{cases}$$

If for each $i, 1 \le i \le e+1$, we choose a module Ω_i at T_i, we have $d_{ij} = [P_j, \Omega_i]$. The matrix $\tilde{D} = (\tilde{d}_{ij})$ is defined by

$$\tilde{d}_{ij} = \begin{cases} 1 & \text{if } T_i \text{ is a vertex of the edge corresponding to } P_j \\ 0 & \text{otherwise.} \end{cases}$$

There is the following relationship with the Cartan matrix C.

LEMMA 2.3. $C = D^{tr}\tilde{D}$.

Proof. We have $C = (c_{ij})$, where c_{ij} is m_u if $P_i \not\simeq P_j$ have a common vertex T_u, is $\max(m_u + 1, m_v + 1)$ if $P_i \simeq P_j$ has vertices T_u and T_v, and is 0 otherwise. So clearly $\sum_{i=1}^{e+1} d_{iv}\tilde{d}_{iu} = c_{vu}$.

The next lemma provides an essential step in our proof.

LEMMA 2.4.

$$\sum_{i=1}^{e} [P_i, \Omega_u][\Omega_v, P_i^{\perp}] = m_u \delta_{uv} - \operatorname{sig} T_u \operatorname{sig} T_v \frac{m}{me+1}.$$

Proof. Let $X = (x_{uv})_{1 \le u, v \le e+1}$ be the $(e+1, e+1)$ matrix defined by

$$x_{uv} = \sum_{i=1}^{e} [P_i, \Omega_u][\Omega_v, P_i^{\perp}].$$

Let $C^{-1} = (\tilde{c}_{ij})$. We then have

$$x_{uv} = \sum_{i=1}^{e} [P_i, \Omega_u] \sum_{j=1}^{e} \tilde{c}_{ij} [\Omega_v, P_j].$$

Since $[P_i, \Omega_u] = d_{ui}$ and $[\Omega_v, P_j] = d_{vj}$, this shows that $X = DC^{-1}D^{tr}$.
Define the $(e+1, e+1)$ matrix $Y = (y_{uv})_{1 \le u, v \le e+1}$ by

$$y_{uv} = \frac{1}{me+1}\left(\delta_{uv}(me+1)m_u - \operatorname{sig} T_u \operatorname{sig} T_v\right).$$

We want to show that $X = Y$. To do this we first show that $Y\tilde{D} = D$, as we obviously have $X\tilde{D} = D$. For this, we have to show $\sum_{i=1}^{e+1} y_{ui}\tilde{d}_{iv} = d_{uv}$. Let v be fixed, and consider the corresponding projective module P_v. Let T_{v_1} and T_{v_2} be the vertices of the edge corresponding to P_v. We clearly have $\operatorname{sig} T_{v_1} = -\operatorname{sig} T_{v_2}$. If $u \neq v_1, v_2$, we have

$$\sum_{i=1}^{e+1} y_{ui}\tilde{d}_{iv} = \frac{m}{me+1}\left(-\operatorname{sig} T_u \operatorname{sig} T_{v_1} - \operatorname{sig} T_u \operatorname{sig} T_{v_2}\right) = 0,$$

and in this case also $d_{uv} = 0$. If $u = v_1$ we have

$$\begin{aligned}\sum_{i=1}^{e+1} y_{v_1 i}\tilde{d}_{iv} &= m_{v_1} + \frac{m}{me+1}\left(-\operatorname{sig} T_{v_1} \operatorname{sig} T_{v_1} - \operatorname{sig} T_{v_1} \operatorname{sig} T_{v_2}\right)\\ &= m_{v_1}\\ &= d_{v_1 v}.\end{aligned}$$

The calculation for $u = v_2$ is the same, so that we have $Y\tilde{D} = D$.
Interpreting X and Y as maps from an $(e+1)$-dimensional rational vector space V to itself relative to the natural basis $n_1, \ldots, n_{e+1}$, we next want to prove that $\operatorname{Ker} X = \operatorname{Ker} Y$.

D and $\tilde{D}$ have rank e since $C = D^{tr}\tilde{D}$. X has rank e since $X = DC^{-1}D^{tr}$. Since $X\tilde{D} = Y\tilde{D}$, we have rank $Y \geq$ rank $X = e$. Since

$$r_0 = \left(\operatorname{sig} T_j \frac{1}{m_j}\right)_{1 \leq j \leq e+1}$$

is in $\operatorname{Ker} D$, hence in $\operatorname{Ker} X$, we need to show that r_0 is in $\operatorname{Ker} Y$. We have

$$\begin{aligned}
(r_0 Y)_j &= \sum_{i=1}^{e+1} \frac{1}{m_i} \operatorname{sig} T_i \left(\delta_{ij} m_i - \frac{m}{me+1} \operatorname{sig} T_i \operatorname{sig} T_j\right) \\
&= \operatorname{sig} T_j - \sum_{i=1}^{e+1} \operatorname{sig} T_j \frac{m}{me+1} \cdot \frac{1}{m_i} \\
&= \operatorname{sig} T_j \left(1 - \frac{m}{me+1} \sum_{i=1}^{e+1} \frac{1}{m_i}\right) \\
&= \operatorname{sig} T_j \left(1 - \frac{1}{me+1} \sum_{i=1}^{e+1} \frac{m}{m_i}\right) \\
&= 0.
\end{aligned}$$

Hence we conclude that $\operatorname{Ker} X = \operatorname{Ker} Y$.

Since $\tilde{D}$ has rank e, $\operatorname{Ker} \tilde{D}$ has dimension 1. Clearly the vector $(\operatorname{sig} T_j)_{1 \leq j \leq e+1}$ is in $\operatorname{Ker} \tilde{D}$. Let $y_i = n_i(X - Y)$. Since we have $X\tilde{D} = D = Y\tilde{D}$, then $y_i \tilde{D} = 0$, hence $y_i = (\alpha(i) \operatorname{sig} T_j)_{1 \leq j \leq e+1}$. Then we have

$$X = Y + \left(\alpha(i) \operatorname{sig} T_j\right)_{1 \leq i, j \leq e+1}.$$

Since C is a symmetric matrix, X and Y are symmetric matrices. Hence

$$\alpha(i) \operatorname{sig} T_j = \alpha(j) \operatorname{sig}(T_i),$$

so that $\alpha(i) = \alpha \operatorname{sig} T_i$. This shows that

$$X = Y + \alpha\left(\operatorname{sig} T_i \operatorname{sig} T_j\right)_{1 \leq i, j \leq e+1}.$$

We now have

$$\begin{aligned}
0 &= r_0 \alpha\left(\operatorname{sig} T_i \operatorname{sig} T_j\right)_j \\
&= \sum_{i=1}^{e+1} \frac{1}{m_i} \operatorname{sig} T_i \alpha \operatorname{sig} T_i \operatorname{sig} T_j \\
&= \left(\sum_{i=1}^{e+1} \frac{1}{m_i}\right) \alpha \operatorname{sig} T_j,
\end{aligned}$$

so that $\alpha = 0$, and hence $X = Y$. This finishes the proof of the lemma.

We now have the following expression for our form.

PROPOSITION 2.5. *Let S be a k-algebra given by a Brauer tree T different from*

$$\overset{1}{\bullet}\text{———}\overset{1}{\bullet}\,.$$

If M and N are indecomposable in $\mathfrak{A}^0(S)$, then

$$\langle M, N\rangle = -\delta_{M,N} + \frac{m}{me+1}\operatorname{sig} M \operatorname{sig} N.$$

Proof. We have shown that

$$\sum_{i=1}^{e} [P_i, N][M, P_i^{\perp}] = \delta_{lj} m_l - \frac{m}{me+1}\operatorname{sig} M \operatorname{sig} N,$$

if M belongs to T_l and N belongs to T_j. The result then follows from Lemma 2.2(c).

We have the following consequence.

PROPOSITION 2.6. *If T is a Brauer tree and S an associated k-algebra, then $\langle\ ,\ \rangle$ is nondegenerate on $\tilde{\mathfrak{A}}^0(S)/\tilde{\mathfrak{A}}^1(S)$.*

Proof. If T is

$$\overset{1}{\bullet}\text{———}\overset{1}{\bullet}\,,$$

there is only one indecomposable nonprojective module in $\mathfrak{A}^0(S)$, and it is easy to see that $\langle\ ,\ \rangle$ is nondegenerate on $\tilde{\mathfrak{A}}^0(S)/\tilde{\mathfrak{A}}^1(S)$. For T otherwise, we arrange the $2e$ indecomposable nonprojectives in $\mathfrak{A}^0(S)$ such that sig is $+1$ for the first e ones and -1 for the others, and get the following associated matrix by using Proposition 2.5:

$$\frac{m}{me+1}\begin{pmatrix} 1 & & 1 & -1 & & -1 \\ & \ddots & & & & \\ 1 & & 1 & -1 & & -1 \\ -1 & & -1 & 1 & & 1 \\ & & & & \ddots & \\ -1 & & -1 & 1 & & 1 \end{pmatrix} - \begin{pmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ 0 & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

By elementary operations this is reduced to:

$$\begin{pmatrix} -1 & & & * \\ & \ddots & & \\ & & -1 & \\ & 0 & & -1+\dfrac{2m}{me+1} \end{pmatrix}$$

Since we have assumed that $(e, m) \neq (1,1)$, the determinant is not zero.

We get the following main result by combining Propositions 1.5 and 2.6.

THEOREM 2.7. *Let T be a Brauer tree and S an associated k-algebra. Then* $[\ ,\ ]$ *is nondegenerate on* $\mathfrak{A}^0(S)$.

We end this section by pointing out the relationship with stable equivalence. Let S be a k-algebra over an algebraically closed field k given by a Brauer tree T. We recall that the indecomposable nonprojective objects M in $\mathfrak{A}^0(S)$ are uniquely determined by $\beta(M) \leq 1$, where $\beta(M)$ denotes the number of nonprojective indecomposable summands in the middle term L of an almost split sequence $0 \to N \to L \to M \to 0$ [13]. Since β is an invariant of stable equivalence [2], a correspondence is induced between the nonprojective indecomposables in $\mathfrak{A}^0(S)$ and $\mathfrak{A}^0(S')$ when S and S' are stably equivalent k-algebras given by Brauer trees. Now S and S' are known to be stably equivalent if and only if $e = e'$ and $m = m'$, where e' is the number of simples for S' and m' the multiplicity of the exceptional vertex [7]. In particular, any k-algebra given by a Brauer tree is stably equivalent to a k-algebra given by a star with the exceptional vertex in the middle, that is, to a Nakayama algebra. Combining with Proposition 2.5 we therefore have the following.

THEOREM 2.8. *Let S and S' be stably equivalent algebras over an algebraically closed field k, given by Brauer trees. The stable equivalence induces an isomorphism* $\tilde{\mathfrak{A}}^0(S)/\tilde{\mathfrak{A}}^1(S) \to \tilde{\mathfrak{A}}^0(S')/\tilde{\mathfrak{A}}^1(S')$, *and the isomorphism commutes with* $\langle\ ,\ \rangle$.

3. Examples

Let k be an algebraically closed field and S and S' stably equivalent k-algebras given by Brauer trees. In Section 2 we showed that the form $\langle\ ,\ \rangle$ is invariant under stable equivalence when restricted to the additive subcategory generated by a special Ω-orbit of indecomposable modules. In the proof we used strongly properties of this Ω-orbit. We show here that the result is not necessarily true for any Ω-orbit, in particular, it does not necessarily hold for the whole module category. We also show that the form [,] being nondegenerate may fail on arbitrary Ω-orbits by studying the situation for Nakayama algebras.

Let T be the Brauer tree

$$\cdot\!\!-\!\!-\!\!-\!\!-\!\!\cdot\!\!-\!\!-\!\!-\!\!-\!\!\cdot\!\!-\!\!-\!\!-\!\!-\!\!\cdot\ 2.$$

(It should be noted that T is the Brauer tree of the principal 7-block of PSL (2.7).) Then the Cartan matrix C_T of an associated algebra is

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix},$$

so that

$$C_T^{-1} = 1/7\begin{pmatrix} 5 & -3 & 1 \\ -3 & 6 & -2 \\ 1 & -2 & 3 \end{pmatrix}.$$

We then have

$$P_1^{\perp} = 1/7(5P_1 - 3P_2 + P_3), \quad P_2^{\perp} = 1/7(-3P_1 + 6P_2 - 2P_3),$$

and

$$P_3^{\perp} = 1/7(P_1 - 2P_2 + P_3),$$

where P_1, P_2, P_3 are the indecomposable projectives corresponding to the edges, from left to right. We have a stable equivalence with an algebra given by the tree T',

2,

since $e = 3$ in both cases, and the m_i are the same. Here we have

$$C_{T'} = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix},$$

so that

$$C_{T'}^{-1} = 1/7\begin{pmatrix} 5 & -2 & -2 \\ -2 & 5 & -2 \\ -2 & -2 & 5 \end{pmatrix}.$$

If Q_1, Q_2, Q_3 denote the indecomposable projectives, given in anticlockwise order, we have

$$Q_1^{\perp} = 1/7(5Q_1 - 2Q_2 - 2Q_3), \quad Q_2^{\perp} = 1/7(-2Q_1 + 5Q_2 - 2Q_3)$$

and

$$Q_3^{\perp} = 1/7(-2Q_1 - 2Q_2 + 5Q_3).$$

Let $P_i/\mathfrak{r}P_i = U_i$ and $Q_i/\mathfrak{r}Q_i = V_i$. It is not hard to see that if F is a stable equivalence between algebras given by T' and T, and we also denote by F the induced correspondence between the indecomposable modules, then we can have the following:

$$F(V_1) = U_1, \quad F(\Omega V_1) = \Omega U_1 = \begin{pmatrix} U_2 \\ U_1 \end{pmatrix},$$

$$F(V_2) = F(\Omega^2 V_1) = \Omega^2 F(V_1) = \begin{pmatrix} U_3 \\ U_2 \end{pmatrix},$$

$$F\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \mathfrak{r}P_2 = \begin{array}{ccccc} U_1 & & & & U_3 \\ & \diagdown & & \diagup & \\ & & U_2 & & \end{array},$$

$$F\begin{pmatrix} V_3 \\ V_1 \\ V_2 \\ V_2 \\ V_1 \end{pmatrix} = \begin{array}{ccccccc} & & U_2 & & & & U_3 \\ & \diagup & & \diagdown & & \diagup & \\ U_1 & & & & U_3 & & \end{array}.$$

We have the following:

$$\left\langle \begin{pmatrix} V_3 \\ V_1 \\ V_2 \\ V_3 \\ V_1 \end{pmatrix}, \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \right\rangle = 0 - 1/7(5.2 - 2 - 2.2 - 2.2 + 5 - 2.2) = -1/7,$$

$$\left\langle \begin{array}{ccccccc} & & U_2 & & & & U_3 \\ & \diagup & & \diagdown & & \diagup & \\ U_1 & & & & U_3 & & \end{array}, \begin{array}{ccccc} U_1 & & & & U_3 \\ & \diagdown & & \diagup & \\ & & U_2 & & \end{array} \right\rangle$$

$$= 1 - 1/7(5 - 3 + 2 \cdot 1 - 3 + 6 - 2 \cdot 2 + 1 - 2 + 2 \cdot 1)$$
$$= 3/7.$$

Hence $\langle \ , \ \rangle$ is not invariant under stable equivalence on the Ω-orbit of

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}.$$

Now let Λ be an algebra given by a star with the exceptional vertex in the middle, that is, Λ is a basic symmetric Nakayama algebra. Let $P_1, \dots, P_e$ be the indecomposable projective modules and $U_1, \dots, U_e$ the corresponding simple modules. Let C be an indecomposable module and $\mathscr{C}$ the additive category generated by the P_i and the syzygies $\Omega^i C$ of C. We can clearly assume that $l(C) \leq (me + 1)/2$, where l denotes length, and we write $l(C) = te + \alpha$, where $t < m/2$, $1 \leq \alpha \leq e$. Let A_i be the indecomposable Λ-module of length $te + \alpha$ with $A_i/\mathfrak{r}A_i = U_i$, and B_i the indecomposable Λ-module of length $(m - t - l)e + (e + 1 - \alpha)$, with $B_i/\mathfrak{r}B_i = U_i$. Then the indecomposable objects of $\mathscr{C}$ are the P_i, $A_i = B_i$ if e and m are odd and $t = (m-1)/2$, $\alpha = (e+1)/2$, and the P_i, A_i, B_i otherwise, $1 \leq i \leq e$. Using the structure of indecomposable modules over Nakayama algebras and that length considerations determine whether a map factors through a projective module, we get the following values for [,]:

$$[A_i, A_j] = 0 \quad \text{for all } i, j,$$

$$[P_i, P_j] = \begin{cases} m+1 & \text{for } i = j \\ m & \text{for } i \neq j, \end{cases}$$

$$[P_i, A_j] = [A_j, P_i] = \begin{cases} t+1 & \text{if } i \in [j, j+\alpha+1] \\ t & \text{otherwise.} \end{cases}$$

Furthermore, if $A_i \not\cong B_i$, we have

$$[P_i, B_j] = \begin{cases} m-t & \text{for } i \in [j, j+e-\alpha] \\ m-t-1 & \text{otherwise} \end{cases},$$

$$[B_i, A_j] = 0 = [A_j, B_i] \quad \text{for all } i, j,$$

$$[B_i, B_j] = m - 2t + x + y,$$

where $x = 0$ if $j \in [i + \alpha, i + e]$ and $x = -1$ otherwise, $y = 0$ if $j \in [i+1, i + (e - \alpha)]$ and $y = -1$ otherwise, and all additions are considered modulo e.

Denoting the P_i by $\mathscr{P}$, the A_i by $\mathscr{A}$ and the B_i by $\mathscr{B}$, we get the following matrices associated with the form in the two cases

$$M_1 = \left(\begin{array}{c|c} \mathscr{M}_{\mathscr{P}}^{\mathscr{P}} & \mathscr{M}_{\mathscr{P}}^{\mathscr{A}} \\ \hline \mathscr{M}_{\mathscr{P}}^{\mathscr{A}} & 0 \end{array}\right),$$

$$M_2 = \left(\begin{array}{c|c|c} \mathscr{M}_{\mathscr{P}}^{\mathscr{P}} & \mathscr{M}_{\mathscr{P}}^{\mathscr{B}} & \mathscr{M}_{\mathscr{P}}^{\mathscr{A}} \\ \hline \mathscr{M}_{\mathscr{P}}^{\mathscr{B}} & \mathscr{M}_{\mathscr{B}}^{\mathscr{B}} & 0 \\ \hline \mathscr{M}_{\mathscr{P}}^{\mathscr{A}} & 0 & 0 \end{array}\right)$$

where $\mathcal{M}_y^x$ denotes the matrix relative to x and y. Hence $\det M_1 \neq 0$ if and only if $\det \mathcal{M}_{\mathcal{P}}^{\mathcal{A}} \neq 0$ and $\det M_2 \neq 0$ if and only if $\det \mathcal{M}_{\mathcal{P}}^{\mathcal{A}} \neq 0$ and $\det \mathcal{M}_{\mathcal{B}}^{\mathcal{B}} \neq 0$.

We see that if $\alpha = e$, then

$$\mathcal{M}_{\mathcal{P}}^{\mathcal{A}} = \begin{pmatrix} t+1 & t+1 \\ t+1 & t+1 \end{pmatrix},$$

so that $\det \mathcal{M}_{\mathcal{P}}^{\mathcal{A}} = 0$. So in this case the form is degenerate.

The case we have studied before is $t = 0$, $\alpha = 1$. Assume more generally that $\alpha = 1$ and $t \langle m/2$. Then

$$\mathcal{M}_{\mathcal{P}}^{\mathcal{A}} = \begin{pmatrix} t+1 & t & \cdots & t \\ t & \ddots & & \vdots \\ & & \ddots & t \\ t & \cdots & & t+1 \end{pmatrix}$$

has determinant $te + 1 \neq 0$ so that $\det M_1 \neq 0$ when $t = (m-1)/2$ and $\alpha = 1 = (e+1)/2$. If

$$(t, e) \neq ((m-1)/2, 1)$$

then

$$\mathcal{M}_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} (m-2t-1)(m-2t) & & (m-2t) \\ (m-2t)(m-2t-1) & & (m-2t) \\ & \ddots & \\ (m-2t)(m-2t) & & (m-2t-1) \end{pmatrix}$$

has determinant

$$\begin{aligned} &(m-2t-1) + (e-1)(m-2t) \quad \text{if } e \text{ is odd} \\ -&(2m-4t-1) - (e-2)(m-2t) \quad \text{if } e \text{ is even.} \end{aligned}$$

Therefore $\det \mathcal{M}_{\mathcal{B}}^{\mathcal{B}}$ is not zero.

4. Construction of Bäckström orders associated with a given Brauer tree

Let T unequal to

$$\overset{1}{\bullet}\!\!-\!\!-\!\!-\!\!-\!\!\overset{1}{\bullet}$$

be a Brauer tree with e edges and $e+1$ vertices, and let m be the multiplicity of the exceptional vertex. Let $k = R/\mathfrak{p}$ with R and $\mathfrak{p}$ as before. We know that there is some k-algebra S where the structure of the indecomposable projectives is given by the Brauer tree T. We give an analogue of this result for orders, in showing that there is some R-order Λ associated with T in a natural way. Even though this turned out not to be needed to prove our main result, we include the construction of this order, since it should be of interest in itself.

When S denotes a k-algebra given by T, we let $\bar{S}_0$ be a simple S-module corresponding to an edge, one of whose vertices is a nonexceptional end point of the tree. Let

$$\bar{\mathscr{P}}_{\bar{S}0}\colon \cdots \to \overline{Q_{2e-1}} \to \cdots \to \overline{Q_1} \to \overline{Q_0} \to \overline{S_0} \to 0$$

be a minimal projective resolution of $\bar{S}_0$. As we have mentioned, the syzygy modules $S_0 = \bar{\Omega}_0, \bar{\Omega}_1, \ldots \bar{\Omega}_{2e-1}$ are all indecomposable and nonisomorphic. The edges to which the $\bar{Q}_i$ belong only depend on the edge of $\bar{S}_0$.

We recall that an R-order Λ is said to be a Bäckström order provided there is a hereditary R-order Λ with $\operatorname{rad}\Lambda = \operatorname{rad}\Gamma$. The representation theory of Bäckström orders is well understood [15], and for details on Bäckström orders we refer to [14].

We have the following main result of this section.

THEOREM 4.1. *With the above notation there exists a Bäckström R-order Λ satisfying*:

(i) $\Lambda/\mathfrak{p}\Lambda \simeq S$, *where S is given by the tree T.*

(ii) Λ *has exactly* $3e$ *nonisomorphic indecomposable R-free modules, which are the syzygies of one irreducible lattice.*

(iii) *If $M \in {}_\Lambda M^0$ is indecomposable, then so is $M/\mathfrak{p}M$. In particular, there exists $S_0 \in {}_\Lambda M^0$ with $S_0/\mathfrak{p}S_0$ and the minimal projective resolution of S_0 reduces to a minimal projective resolution of $\bar{S}_0$ modulo $\mathfrak{p}$.*

Proof. We first define a map

$$\mathscr{K}\colon \{\bar{\Omega}_i\}_{0 \le i < 2e} \to \{1, \ldots, e+1\}$$

where $\{1, \ldots, e+1\}$ represent the vertices of T. The minimal projective resolution $\bar{\mathscr{P}}_{\bar{S}_0}$ of $\bar{S}_0$ is constructed by walking around the Brauer tree. So as one walks from $\bar{Q}_{i-1}$ to $\bar{Q}_i$ one passes exactly one vertex which we define to be $\mathscr{K}(\bar{\Omega}_i)$. Then $\operatorname{card}(\mathscr{K}^{-1}(j)) = n(j)$ is the number of edges meeting in j. If j is not the exceptional vertex, we associate with j the following hereditary order

$$\Gamma_j = \begin{pmatrix} R & \mathfrak{p} & \cdots & & \mathfrak{p} \\ R & R & \mathfrak{p} & & \\ \vdots & & \ddots & & \vdots \\ & & & & \mathfrak{p} \\ R & R & & R & R \end{pmatrix}_{n(j)\times n(j)}$$

and we label the indecomposable Γ_j – lattices in the following way:

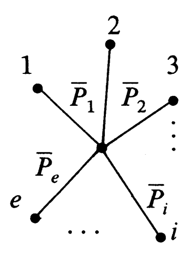
$$Q_{j,i} = \begin{pmatrix} \mathfrak{p} \\ \vdots \\ \mathfrak{p} \\ R \\ \vdots \\ R \end{pmatrix}_{n(j)\times 1} \begin{matrix} \left.\vphantom{\begin{matrix}\mathfrak{p}\\ \vdots\\ \mathfrak{p}\end{matrix}}\right\} i-1 \\ \\ \\ \\ \end{matrix} \quad ;1 \le i \le n(j).$$

Then $\operatorname{rad}_{\Gamma_j}(Q_{j,i}) = Q_{j,i+1}$, where $i+1$ is taken modulo $n(j)$. For the exceptional vertex j_0, let $\tilde{R}$ with radical $\tilde{\mathfrak{p}}$ be the totally ramified extension of R of degree m, let

$$\Gamma_{j_0} = \begin{pmatrix} \tilde{R} & \tilde{\mathfrak{p}} & & \cdots & \tilde{\mathfrak{p}} \\ \vdots & \tilde{R} & \tilde{\mathfrak{p}} & & \vdots \\ & & \ddots & \tilde{R} & \tilde{\mathfrak{p}} \\ \tilde{R} & & \cdots & & \tilde{R} \end{pmatrix}_{n(j_0)\times n(j_0)},$$

and label the indecomposable Γ_{j_0}-lattices as above. Let $\Gamma = \prod_{j=1}^{e+1}\Gamma_j$. We note that for each indecomposable projective Γ-module P we have $P/\operatorname{rad}_\Gamma P \simeq R/\mathfrak{p} = k$. Before we give the rather technical definition of our Bäckström order Γ, we illustrate the situation by means of two examples.

Example 1. *Let T be a star with the exceptional vertex in the centre*:



The exceptional vertex has multiplicity m and gets the number $e+1$, and i is the other end point of the edge corresponding to the indecomposable projective module $\overline{P}_i$, $1 \le i \le e$. Then we have

$$\Gamma = \prod_{i=1}^{e} R_i \times \begin{pmatrix} \tilde{R} & \tilde{\mathfrak{p}} & & \cdots & \tilde{\mathfrak{p}} \\ \vdots & \tilde{R} & & & \vdots \\ & & \ddots & \tilde{R} & \tilde{\mathfrak{p}} \\ \tilde{R} & & \cdots & & \tilde{R} \end{pmatrix}_{e\times e}, \quad R_i = R.$$

In $\Gamma/\operatorname{rad}\Gamma$ we consider the e-dimensional k-algebra

$$k_1 \times \cdots \times k_e, \quad k_i = k,$$

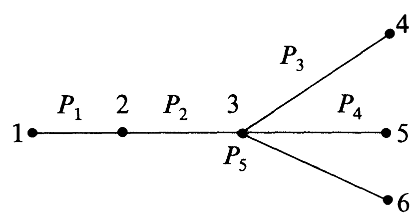
where k_j is diagonally embedded in

$$Q_{j,1}/\operatorname{rad} Q_{j,1} \oplus Q_{e+1,j}/\operatorname{rad} Q_{e+1,j}.$$

Then Λ is the pullback of the diagram

$$\begin{tikzcd} \Lambda \arrow[r] \arrow[d] & \Gamma \arrow[d] \\ k_1 \times \cdots \times k_e \arrow[r] & \prod_{j=1}^{e+1} \prod_{i=1}^{n(j)} Q_{j,i}/\operatorname{rad} Q_{j,i}. \end{tikzcd}$$

Example 2. [14]. The Mathiew group M_{11} at $p = 11$ has for the principal block the Brauer tree



where 5 is the exceptional vertex and has multiplicity 2.
Hence

$$\Gamma = R \times \begin{pmatrix} R & \mathfrak{p} \\ R & R \end{pmatrix} \times \begin{pmatrix} R & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ R & R & \mathfrak{p} & \mathfrak{p} \\ R & R & R & \mathfrak{p} \\ R & R & R & R \end{pmatrix} \times R \times \tilde{R} \times R.$$

We embed $k_1 \times \cdots \times k_5$ into $\Gamma/\operatorname{rad}\Gamma$ in the following way, denoting

$$Q_{i,j}/\operatorname{rad}_\Gamma Q_{i,j}$$

by $\overline{Q}_{i,j}$, where each map is a diagonal embedding:

$$k_1 \to \overline{Q}_{1,1} \oplus \overline{Q}_{2,1}, \quad k_2 \to \overline{Q}_{2,2} \oplus \overline{Q}_{3,1},$$
$$k_3 \to \overline{Q}_{3,2} \oplus \overline{Q}_{4,1}, \quad k_4 \to \overline{Q}_{3,3} \oplus \overline{Q}_{5,1}, \quad k_5 \to \overline{Q}_{3,4} \oplus \overline{Q}_{6,1}.$$

Again, Λ is the pullback in the diagram

$$\begin{tikzcd} \Lambda \arrow[rr] \arrow[d] & & \Gamma \arrow[d] \\ k_1 \times \cdots \times k_5 \arrow[rr] & & \Gamma/\mathrm{rad}\,\Gamma, \end{tikzcd}$$

where the embedding $k_1 \times \cdots \times k_5$ is induced from the above.

In both examples it is not hard to see that the claimed statements are true.

We now want to define Λ in general. We label the vertices and edges, with the associated projectives, as we meet them on our walk around the tree, starting with vertex 1 corresponding to S_0. We give an inductive definition of the embedding

$$\prod_{i=1}^{e} k_i \to \Gamma/\mathrm{rad}\,\Gamma, \quad k_i = k.$$

Corresponding to the edge 1 associated with $\overline{P}_1$ we have the diagonal embedding $k_1 \to Q_{1,1}/\mathrm{rad}_\Gamma Q_{1,1} \oplus Q_{2,1}/\mathrm{rad}\, Q_{2,1}$. Assume that we have embedded k_i, $i < i_0 \leq e$, into $\Gamma/\mathrm{rad}\,\Gamma$ as we followed the walk around the tree. Since T is a tree, the i_0-th edge meets the vertex $i_0 + 1$, and the other vertex of this edge is i_1 with $i_1 < i_0$. Assume we have already passed r edges meeting in the vertex i_1. Then we define the diagonal embedding

$$k_{i_0} \to Q_{i_1, r+1}/\mathrm{rad}\, Q_{i_1, k+1} \oplus Q_{i_0,1}.$$

We define Λ as the pullback of

$$\begin{tikzcd} \Lambda \arrow[rr] \arrow[d] & & \Gamma \arrow[d] \\ \prod_{i=1}^{e} k_i \arrow[rr] & & \Gamma/\mathrm{rad}\,\Gamma. \end{tikzcd}$$

We want to show that Λ satisfies the desired properties. Λ is a Bäckström order [15] with associated species a disjoint union of e copies of A_3, and so there are $3e$ indecomposable modules in ${}_\Lambda M^0$ which all occur as syzygy modules of any nonprojective indecomposable. We prove that $\Lambda/\mathfrak{p}\Lambda$ is given by T by induction on the number of edges of T. Let T be

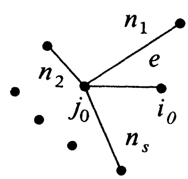
$$\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet$$

with the exceptional vertex of multiplicity $m(> 1)$. Then S is uniserial of length $m + 1$ over k. Moreover, if $\tilde{R}$ is a totally ramified extension of R of degree m, then Λ is the pullback of the diagram

$$\begin{array}{ccc} \Lambda & \rightarrow & R \times \tilde{R} \\ \downarrow & & \downarrow \\ k & \rightarrow & k \times k \end{array}$$

where $k \rightarrow k \times k$ denotes the diagonal embedding, and so $\Lambda/\mathfrak{p}\Lambda$ is uniserial of length $m + 1$ over k, that is $\Lambda/\mathfrak{p}\Lambda$ is given by T.

Now let e be the last edge we meet on our walk that has not been met before. Then one end point i_0 of e must be an end point of T. Let T' be the tree obtained from T by omitting e and i_0. Let m_{i_0} be the multiplicity of i_0. Let j_0 be the other end point of e, with multiplicity m_{j_0}. We then have the following picture:



Let $S_{T'}$ be an algebra of T' and $\Lambda_{T'}$ the constructed order of T'. Then passing from $S_{T'}$ to some S given by T means leaving the structure of the projective $\overline{P}_i$ for $i \neq n_j$ invariant and changing $\overline{P}_{n_j}$ by inserting m_{j_0} copies of the simple module S_e at the appropriate places in the composition series. We add a new indecomposable projective module $\overline{P}_e$. For the order we have

$$\Gamma_{j_0} = \left(\begin{array}{cccc|c} \hat{R} & \hat{\mathfrak{p}} & \cdots & \hat{\mathfrak{p}} & \hat{\mathfrak{p}} \\ \vdots & & & \vdots & \\ \hat{R} & & & \hat{R} & \hat{\mathfrak{p}} \\ \hline \hat{R} & \cdot & \cdots & & \hat{R} \end{array}\right)_{n_s+1)\times(n_s+1)},$$

where the framed region is Γ'_{j_0} corresponding to T'. Here $\hat{R} = R$, $\hat{\mathfrak{p}} = \mathfrak{p}$ if j_0 is not exceptional and $\hat{R} = \tilde{R}$, $\hat{\mathfrak{p}} = \tilde{\mathfrak{p}}$ if j_0 is exceptional.

Moreover, we have to add $\Gamma_{i_0} = \tilde{\tilde{R}}$, where $\tilde{\tilde{R}} = R$ if i_0 is not exceptional and $\tilde{\tilde{R}} = \tilde{R}$ if i_0 is exceptional. Now it is easily seen that $\Lambda/\mathfrak{p}\Lambda$ is given by T.

References

1. J. Alperin and G.J. Janusz, *Resolutions and periodicity*, Proc. Am. Math. Soc., vol. 37 (1973), pp. 403–406.
2. M. Auslander and I. Reiten, *Representation theory of artin algebras V: Methods for computing almost split sequences and irreducible morphisms*, Comm. Algebra, vol. 5 (1977), pp. 519–544.
3. ______, *Modules determined by their composition factors*, Illinois J. Math., vol. 29 (1985), pp. 280–301

4. D. Benson and R. Parker, *The Green ring of a finite group*, J. Algebra, vol. 87 (1984), pp. 290–331.
5. J.A. Green, *Walking around the Brauer Tree*, J. Austral. Math. Soc., vol. 27 (1974), pp. 197–213.
6. ______, *Vorlesungen über modulare Darstellungstheorie endlicher Gruppen*, Vorlesungen aus dem Math. Inst. Giessen, Heft 2, 1974.
7. P. Gabriel and C. Riedtmann, *Group representations without groups*, Comment. Math. Helv. vol. 54 (1979), pp. 240–287.
8. G.J. Janusz, *Indecomposable modules for finite groups*, Ann. of Math. (2), vol. 89 (1969), pp. 209–241.
9. H. Kupisch, *Unzerlegbare Moduln endlicher Gruppen mit zyklischer p-Sylow Gruppe*, Math. Zeitschr., vol. 108 (1969), pp. 77–104.
10. G. Michler, *Green correspondence between blocks with cyclic defect groups I*, J. Algebra, vol. 39 (1976), pp. 26–51.
11. I. Reiner, *Relations between integral and modular representations*, Mich. Math. J., vol. 13 (1966), pp. 372–375.
12. ______, *Integral representation algebras*, Trans. Amer. Math. Soc., vol. 124 (1966), pp. 111–121.
13. I. Reiten, *Almost split sequences for group algebras of finite representation type*, Trans. Amer. Math. Soc., vol. 233 (1977), pp. 125–136.
14. K. Roggenkamp, *Integral representations and structure of finite groups rings*, Presses de L'Université de Montréal, 1980, pp. 1–165.
15. K. Ringel and K. Roggenkamp, *Diagrammatic methods in the representation theory of orders*, J. Algebra, vol. 60 (1979), pp. 11–42.
16. P. Webb, *On the orthogonality coefficients for character tables of the Green ring of a finite group*, to appear.

Universität Stuttgart
Stuttgart, West Germany