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BOMBIERI'S NORM VERSUS MAHLER'S MEASURE

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Introduction

Factorization algorithms for polynomials with integer coefficients and one complex variable use an a priori bound on the size of the coefficients in any factor of P. The first bound of this type was given by Mignotte [5], using Mahler's measure. Then Beauzamy [2] and Beauzamy-Trevisan-Wang [4] gave sharper estimates, using Bombieri's norm. This leads to the natural question: for which polynomials is Bombieri's norm smaller than Mahler's measure? We give an answer here, in terms of the localization of the roots of P, more precisely a sufficient condition on the modulus of the roots, for a polynomial with complex coefficients and one complex variable to have its Bombieri's norm smaller than its Mahler's measure.

I. The results

Let

$$P(z) = a_n z^n + \cdots + a_0 = a_n \prod_{i=1}^{n} (z - \alpha_i)$$

be a polynomial of degree n, with complex coefficients and with complex zeros $(\alpha_i)_{1 \leq i \leq n}$. For our problem, we can obviously assume P to be monic, that is $a_n = 1$. Recall that, for a monic polynomial, Mahler's measure of P, denoted by $M(P)$, is

$$M(P) = \prod_{i=1}^{n} \max(1, |\alpha_i|)$$

and that Bombieri's norm, denoted by $[P]$, is

$$[P] = \left(\sum_{i=0}^{n} \frac{|a_i|^2}{\binom{n}{i}} \right)^{1/2}.$$

Let D be the open unit disk, $\overline{\mathrm{D}}$ its closure and C be the unit circle. A first and trivial observation is that if all roots of P lie in $\overline{\mathrm{D}}$, then $M(P) \leq [P]$. Indeed, in this case $M(P) = 1$ and $[P] \geq 1$. The same holds if all roots are outside D. So, if we want

$$[P] \leq M(P)$$

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this is possible only if some of the zeros are in $\overline{\mathrm{D}}$ and the others are outside $\overline{\mathrm{D}}$. Our main result is given by the following theorem:

THEOREM 1. *Let n, n_1, n_2, n_3 be positive integers such that $n = n_1 + n_2 + n_3$. Assume that P is a polynomial of degree n with n_1 roots, say $\alpha_1, \dots, \alpha_{n_1}$, inside the closed disk of center* 0 *and radius* $0 < \alpha < 1$, *n_2 roots, $\alpha_{n_1+1}, \dots, \alpha_{n_1+n_2}$ outside the closed disk of center* 0 *and radius* $\beta > 1$.

If the integers n, n_1, n_2, n_3 and the real numbers α, β satisfy the condition

$$(1+\alpha\beta)^{n_1}\left(1+\frac{1}{\beta^2}\right)^{n_2}(1+\beta)^{n_3}\left(\frac{n_1}{n}\frac{1+\alpha^2}{1+\alpha\beta}+\frac{n_2}{n}\frac{1+\alpha\beta}{1+\beta^2}+\frac{n_3}{n}\frac{1+\alpha}{1+\beta}\right)^{n_1}$$
$$\left(\frac{n_1}{n}\frac{1+\alpha}{1+\alpha\beta}+\frac{n_2}{n}\frac{1+\beta}{1+\beta^2}+\frac{n_3}{n}\frac{2}{1+\beta}\right)^{n_3} \leq 1$$

then $[P] \leq M(P)$.

We first recall main results about the Bombieri norm and its associated scalar product of polynomials in several variables in order to use their corollaries in the context of one complex variable polynomials. We then proceed to the proof of Theorem 1, after establishing an inequality independant of this context of polynomials. We eventually give a special case where Theorem 1 can be improved as well as examples showing to which extent it only gives a sufficient condition.

II. Bombieri norm and its associated scalar product

We recall in this section general results about the Bombieri norm, and its associated scalar product. For their proofs, see [3].

Let

$$P(x_1, \dots, x_N) = \sum_{|\alpha|=m} a_\alpha x_1^{\alpha_1} \cdots x_N^{\alpha_N} \tag{1}$$

be a homogeneous polynomial in N variables $x_1, \dots, x_N$, with complex coefficients and degree m. As usual, we write $\alpha = (\alpha_1, \dots, \alpha_N)$, $|\alpha| = \alpha_1 + \cdots + \alpha_N$.

For any $i_1, \dots, i_m$, $1 \leq i_1 \leq N, \dots, 1 \leq i_m \leq N$, we define

$$c_{i_1,\dots,i_m} = \frac{1}{m!}\frac{\partial^m P}{\partial x_{i_1}\cdots\partial x_{i_m}},$$

and by Taylor's formula, we have

$$P(x_1, \dots, x_N) = \sum_{i_1,\dots,i_m=1}^{N} c_{i_1,\dots,i_m} x_{i_1}\cdots x_{i_m}, \tag{2}$$

called the *symmetric form* of the polynomial.

For a polynomial P of degree m, Bombieri's norm is defined by

$$[P]_{(m)} = \left(\sum_{i_1,\dots,i_m=1}^{N} |c_{i_1,\dots,i_m}|^2 \right)^{1/2} .$$

If we start with any polynomial P given as in (1), it can be written in many ways in the form

$$P(x_1, \dots, x_N) = \sum_{i_1,\dots,i_m=1}^{N} b_{i_1,\dots,i_m} x_{i_1} \cdots x_{i_m}, \tag{3}$$

but the symmetric representation (2) has a particular property, given by the following proposition.

PROPOSITION 2. *Among all representations of P of the form* (3), *the symmetric one in* (2) *is the one for which the l_2-norm is minimal.*

There is a scalar product canonically associated with Bombieri's norm: if P, Q are two homogeneous polynomials with same degree m, written in symmetric form as

$$P(x_1, \dots, x_N) = \sum_{i_1,\dots,i_m=1}^{N} c_{i_1,\dots,i_m} x_{i_1} \cdots x_{i_m},$$

$$Q(x_1, \dots, x_N) = \sum_{i_1,\dots,i_m=1}^{N} d_{i_1,\dots,i_m} x_{i_1} \cdots x_{i_m},$$

then for the scalar product of P and Q we set

$$[P, Q]_{(m)} = \sum_{i_1,\dots,i_m=1}^{N} c_{i_1,\dots,i_m} \overline{d}_{i_1,\dots,i_m}.$$

In fact, in order to define the scalar product, only one of the polynomials needs to be written in symmetric form, according to the following proposition:

PROPOSITION 3. *Let $P = \sum_{i_1,\dots,i_m} c_{i_1,\dots,i_m} x_{i_1} \cdots x_{i_m}$ be written in symmetric form* (2), *and let*

$$Q = \sum_{j_1,\dots,j_m} d_{j_1,\dots,j_m} x_{j_1} \cdots x_{j_m}$$

be any homogeneous polynomial of degree m (*the d's need not be invariant under permutation of indices*). *Then*

$$[P, Q] = \sum_{i_1,\dots,i_m} c_{i_1,\dots,i_m} \overline{d}_{i_1,\dots,i_m}.$$

We now investigate a few special situations which will be useful for the further proofs.

PROPOSITION 4. *Let $P_1, \ldots, P_k$ be homogeneous polynomials in N variables $x_1, \ldots, x_N$, with degrees $m_1, \ldots, m_k$. Let $m = m_1 + \cdots + m_k$, and also let $q_1, \ldots, q_m$ be homogeneous polynomials of degree* 1. *Then*

$$[P_1 \cdots P_k, q_1 \cdots q_m] = \frac{1}{m!} \sum_{\sigma} [P_1, q_{\sigma(1)} \cdots q_{\sigma(m_1)}] \times \cdots \times [P_k, q_{\sigma(m-m_k+1)} \cdots q_{\sigma(m)}],$$

where σ runs over the set of all permutations of $\{1, \ldots m\}$.

COROLLARY 5. *Let $p_1, \ldots, p_m, q_1, \ldots, q_m$ be homogeneous polynomials of degree* 1, *with variables $x_1, \ldots, x_N$. Then*

$$[p_1 \cdots p_m, q_1 \cdots q_m] = \frac{1}{m!} \sum_{\sigma \in S_m} [p_1, q_{\sigma(1)}] \cdots [p_m, q_{\sigma(m)}],$$

where σ runs over the set S_m of all permutations of $\{1, \ldots, m\}$.

We now give an expression for the scalar product of two polynomials in one variable z, with same degree m. This expression uses the zeros of both polynomials, and is an obvious consequence of Corollary 5. We identify the one variable polynomial $z - a$ with the homogeneous two variables polynomial $z - az'$.

COROLLARY 6. *Let $P = (z - a_1) \cdots (z - a_m)$, $Q = (z - b_1) \cdots (z - b_m)$. Then*

$$[P, Q] = \frac{1}{m!} \sum_{\sigma \in S_m} (1 + a_1 \overline{b}_{\sigma(1)}) \cdots (1 + a_m \overline{b}_{\sigma(m)}),$$

where σ runs over the set S_m of all permutations of $\{1, \ldots, m\}$.

III. Inequalities

In the following lemma we propose a wide extension of the well-known result

$$\sum_{i+j=k} \binom{n}{i} \binom{m}{j} = \binom{n+m}{k}$$

where n, m, k are positive integers, that is the number of subsets of k elements out of a set of $n + m$ items, itself divided in two parts of respectively n and m elements. This equality can be further interpreted as the special case $A = B = 1$ in the inequality

$$\sum_{i+j=k} \binom{n}{i} \binom{m}{j} A^i B^j \leq \binom{n+m}{k} \left(\frac{nA + mB}{n+m} \right)^k$$

where A and B denote positive real numbers. In this paragraph we give an inequality of this type with six real numbers where the binomial coefficients above have been extended to generalized multinomial coefficients.

Let n, i, j be any positive integers such that $i \leq n$ and $j \leq n$. From now on we denote by $\binom{n}{i,j}$ the trinomial coefficient $\frac{n!}{i!j!(n-i-j)!}$ if $i+j \leq n$, else 0.

Lemma 7. Let n_1, n_2, n_3, n be positive integers such that $n_1+n_2+n_3 = n$, let k_1, k_2 be positive integers, and let A, B, C, D, E, F be positive real numbers such that $A \geq C \geq E$ and $B \geq D \geq F$. We have

$$\sum_{\substack{i_1+j_1+l_1=k_1\\ i_2+j_2+l_2=k_2}} \binom{n_1}{i_1, i_2}\binom{n_2}{j_1, j_2}\binom{n_3}{l_1, l_2} A^{i_1}B^{i_2}C^{j_1}D^{j_2}E^{l_1}F^{l_2}$$
$$\leq \binom{n}{k_1, k_2}\left(\frac{n_1A+n_2C+n_3E}{n}\right)^{k_1}\left(\frac{n_1B+n_2D+n_3F}{n}\right)^{k_2}.$$

Proof. The left-hand and right-hand sides of the inequality are the coefficients of $x^{k_1}y^{k_2}$ in the polynomials respectively $p(x, y)$ and $q(x, y)$ defined by

$$p(x, y) = (1+Ax+By)^{n_1}(1+Cx+Dy)^{n_2}(1+Ex+Fy)^{n_3},$$

$$q(x, y) = \left(1+\frac{n_1A+n_2C+n_3E}{n}x+\frac{n_1B+n_2D+n_3F}{n}y\right)^n.$$

Hence, it suffices to show that the coefficients of $q(x)-p(x)$ are all positive. If we now write $A = C+I$, $B = D+J$, $C = E+G$, $D = F+H$, the conditions become $G, H, I, J \geq 0$.

$$\begin{aligned}
q(x)-p(x) &= \left[(1+Ex+Fy)+\frac{n_1+n_2}{n}(Gx+Hy)+\frac{n_1}{n}(Ix+Jy)\right]^n\\
&\quad - [(1+Ex+Fy)+(Gx+Hy)+(Ix+Jy)]^{n_1}\\
&\quad \times [(1+Ex+Fy)+(Gx+Hy)]^{n_2}(1+Ex+Fy)^{n_3}\\
&= \sum_{k,i}\left[\binom{n}{k,i}\left(\frac{n_1+n_2}{n}\right)^i\left(\frac{n_1}{n}\right)^k - \binom{n_1}{k}\binom{n_1+n_2-k}{i}\right]\\
&\quad \times (1+Ex+Fy)^{n-k-i}(Gx+Hy)^i(Ix+Jy)^k
\end{aligned}$$

It now suffices to show that the coefficient

$$\binom{n}{k,i}\left(\frac{n_1+n_2}{n}\right)^i\left(\frac{n_1}{n}\right)^k - \binom{n_1}{k}\binom{n_1+n_2-k}{i}$$

is always nonnegative. In fact, a little factoring shows that it is equal to

$$\frac{n_1^k(n_1+n_2)^i}{k!i!} \times \left[\prod_{j=0}^{k+i-1}\left(1-\frac{j}{n}\right)-\left(1-\frac{k}{n_1+n_2}\right)^i\prod_{j=0}^{k-1}\left(1-\frac{j}{n_1}\right)\prod_{j=0}^{i-1}\left(1-\frac{j}{n_1+n_2-k}\right)\right].$$

We have $n > n_1$ and $n > n_1 + n_2 - k$ so for all j such that $0 \le j \le k + i - 1$,

$$1-\frac{j}{n} > 1-\frac{j}{n_1} \quad \text{and} \quad 1-\frac{j}{n} > 1-\frac{j}{n_1+n_2-k}.$$

Therefore

$$\begin{aligned}\prod_{j=0}^{k+i-1}\left(1-\frac{j}{n}\right) &> \prod_{j=0}^{k-1}\left(1-\frac{j}{n_1}\right)\prod_{j=0}^{i-1}\left(1-\frac{j}{n_1+n_2-k}\right)\\ &> \left(1-\frac{k}{n_1+n_2}\right)\prod_{j=0}^{k-1}\left(1-\frac{j}{n_1}\right)\prod_{j=0}^{i-1}\left(1-\frac{j}{n_1+n_2-k}\right)\end{aligned}$$

since $0 < 1 - \frac{k}{n_1+n_2} < 1$, and this ends the proof. □

IV. Proof of Theorem 1

In the context of polynomials with complex coefficients and one complex variable, we now use the preceeding results to establish the following lemmas leading to the proof of Theorem 1.

We are now in a position to compare the values of the Mahler measure of a polynomial in one variable P, $M(P)$, and its Bombieri norm, $[P]$. Let us just recall that for

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0 = \prod_{i=1}^{n}(z-\alpha_i),$$

$$M(P) = \prod_{i=1}^{n}\max(1, |\alpha_i|)$$

$$[P] = \left(\sum_{i=0}^{n}\frac{|a_i|^2}{\binom{n}{i}}\right)^{1/2}$$

where $a_n = 1$.

LEMMA 8. *Let n, n_1, n_2, n_3, be positive integers such that $n = n_1 + n_2 + n_3$. Assume that P is a polynomial of degree n with n_1 roots, say $\alpha_1, \ldots, \alpha_{n_1}$ inside the closed disk of center* 0 *and radius $\alpha \le 1$ and n_2 roots, $\alpha_{n_1+1}, \ldots, \alpha_{n_1+n_2}$ outside the closed disk of center* 0 *and radius $\beta \ge 1$. Then*

$$\frac{[P]}{M(P)} \le \frac{[(z+\alpha)^{n_1}(z+\beta)^{n_2}(z+1)^{n_3}]}{\beta^{n_2}}.$$

Geometrically speaking, Lemma 8 confirms the fact shown by Beaucoup [1] that $[P]$ is maximum when all roots of P lie on the same line, for instance here the real axis.

Proof of Lemma 8. First consider any polynomial P with k roots, say $\alpha_1, \ldots, \alpha_k$ inside the closed unit disk and the $n-k$ others outside. Then, by Corollary 6,

$$\left(\frac{[P]}{M(P)}\right)^2 = \frac{1}{n!}\sum_{\sigma\in S_n}\prod_{i=1}^{n} A_{i\sigma(i)}$$

where the complex numbers $A_{i\sigma(i)}$ may take the following values:

If $i \le k$ is such that $\sigma(i) \le k$ then $A_{i\sigma(i)} = 1 + \alpha_i\overline{\alpha}_{\sigma(i)}$.

If $i \le k$ is such that $\sigma(i) \ge k+1$ then $A_{i\sigma(i)} = \alpha_i + \frac{1}{\overline{\alpha}_{\sigma(i)}}$.

If $i \ge k+1$ is such that $\sigma(i) \le k$ then $A_{i\sigma(i)} = \overline{\alpha}_{\sigma(i)} + \frac{1}{\alpha_i}$.

If $i \ge k+1$ is such that $\sigma(i) \ge k+1$ then $A_{i\sigma(i)} = 1 + \frac{1}{\alpha_i\overline{\alpha}_{\sigma(i)}}$.

Note that all those numbers $A_{i\sigma(i)}$ have their moduli less than or equal to 2. We now look for a better majorization of $(\frac{[P]}{M(P)})^2$ when P satisfies the hypothesis of Lemma 8. For example, if $i \le n_1$ and $n_1 + 1 \le \sigma(i) \le n_1 + n_2$, then $|\alpha_i| \le \alpha \le 1$ and $|\bar{\alpha}_{\sigma(i)}| \ge \beta \ge 1$ then

$$|A_{i\sigma(i)}| = \left|\frac{1+\alpha_i\bar{\alpha}_{\sigma(i)}}{\bar{\alpha}_{\sigma(i)}}\right| = \left|\alpha_i + \frac{1}{\bar{\alpha}_{\sigma(i)}}\right| < |\alpha_i| + \frac{1}{|\bar{\alpha}_{\sigma(i)}|} < \alpha + \frac{1}{\beta}.$$

Similarly, we get the following majorizations on the $|A_{i\sigma(i)}|$:

If $i \le n_1$ and $\sigma(i) \le n_1$, $|A_{i\sigma(i)}| \le 1 + \alpha^2$.

If $i \le n_1$ and $\sigma(i) \ge n_1 + n_2 + 1$, $|A_{i\sigma(i)}| \le 1 + \alpha$.

If $n_1 + 1 \le i \le n_1 + n_2$ and $\sigma(i) \le n_1$, $|A_{i\sigma(i)}| \le \alpha + \frac{1}{\beta}$.

If $n_1 + 1 \le i \le n_1 + n_2$ and $n_1 + 1 \le \sigma(i) \le n_1 + n_2$, $|A_{i\sigma(i)}| \le 1 + \frac{1}{\beta^2}$.

If $n_1 + 1 \le i \le n_1 + n_2$ and $\sigma(i) \ge n_1 + n_2 + 1$, $|A_{i\sigma(i)}| \le 1 + \frac{1}{\beta}$.

If $i \geq n_1 + n_2 + 1$ and $\sigma(i) \leq n_1$, $|A_{i\sigma(i)}| \leq 1 + \alpha$.

If $i \geq n_1 + n_2 + 1$ and $n_1 + 1 \leq \sigma(i) \leq n_1 + n_2$, $|A_{i\sigma(i)}| \leq 1 + \frac{1}{\beta}$.

If $i \geq n_1 + n_2 + 1$ and $\sigma(i) \geq n_1 + n_2 + 1$, $|A_{i\sigma(i)}| \leq 2$.

This finishes the proof of Lemma 8.

Note that the above inequalities are all reached in the case of $P = (z+\alpha)^{n_1} (z+\beta)^{n_2} (z+1)^{n_3}$.

As a consequence of Lemma 8, we may assume from now on for the proof of theorem 1 that $P = (z+\alpha)^{n_1} (z+\beta)^{n_2} (z+1)^{n_3}$.

LEMMA 9. *We have*

$$\begin{aligned}
&\frac{[(z+\alpha)^{n_1}(z+\beta)^{n_2}(z+1)^{n_3}]^2}{\beta^{2n_2}} \\
&= \frac{1}{\binom{n}{n_1, n_3}}(1+\alpha\beta)^{n_1}\left(1+\frac{1}{\beta^2}\right)^{n_2}(1+\beta)^{n_3} \\
&\times \sum_{\substack{k_{11}+k_{21}+k_{31}=n_1 \\ k_{13}+k_{23}+k_{33}=n_3}} \binom{n_1}{k_{11}, k_{13}}\binom{n_3}{k_{31}, k_{33}}\binom{n_2}{k_{21}, k_{23}} \\
&\times \left(\frac{1+\alpha^2}{1+\alpha\beta}\right)^{k_{11}}\left(\frac{1+\alpha}{1+\alpha\beta}\right)^{k_{13}}\left(\frac{1+\alpha}{1+\beta}\right)^{k_{31}}\left(\frac{2}{1+\beta}\right)^{k_{33}} \\
&\times \left(\frac{1+\alpha\beta}{1+\beta^2}\right)^{k_{21}}\left(\frac{1+\beta}{1+\beta^2}\right)^{k_{23}}.
\end{aligned}$$

Proof of Lemma 9. Let $P = (z+\alpha)^{n_1}(z+\beta)^{n_2}(z+1)^{n_3} = q_1^{n_1}q_2^{n_2}q_3^{n_3}$ where q_1, q_2, q_3 denote respectively the factors of P, that is, $z+\alpha$, $z+\beta$, $z+1$. Using Proposition 4, we get

$$\begin{aligned}
[P]^2 = \frac{1}{n!}\sum_{\sigma\in S_n}&[q_1^{n_1}, q_{\sigma(1)}\cdots q_{\sigma(n_1)}]\,[q_2^{n_2}, q_{\sigma(n_1+1)}\cdots q_{\sigma(n_1+n_2)}] \\
&\times[q_3^{n_3}, q_{\sigma(n_1+n_2+1)}\cdots q_{\sigma(n)}].
\end{aligned}$$

For any fixed σ in S_n, let k_{11}, k_{12}, k_{13} denote the number of factors equal respectively to q_1, q_2, q_3 in the product $q_{\sigma(1)}\cdots q_{\sigma(n_1)}$. Similarly, define k_{21}, k_{22}, k_{23} for the product $q_{\sigma(n_1+1)}\cdots q_{\sigma(n_1+n_2)}$ as well as k_{31}, k_{32}, k_{33} for $q_{\sigma(n_1+n_2+1)}\cdots q_{\sigma(n)}$. Therefore the integers $(k_{ij})_{1\leq i,j\leq 3}$ satisfy the relations

$$n_i = \sum_{j=1}^{3} k_{ij} = \sum_{j=1}^{3} k_{ji} \quad \text{for} \quad i = 1, 2, 3. \tag{4}$$

Thus, we get

$$[P]^2 = \frac{1}{n!}\sum_{\sigma\in S_n}(1+\alpha^2)^{k_{11}}(1+\beta^2)^{k_{22}}2^{k_{33}}(1+\alpha\beta)^{k_{12}+k_{21}}(1+\alpha)^{k_{13}+k_{31}}(1+\beta)^{k_{23}+k_{32}}.$$

For any given integers $(k_{ij})_{1\le i,j\le 3}$ satisfying (4), let $C(k_{ij})$ be the number of permutations of S_n which map k_{11}, k_{12}, k_{13} (respectively k_{21}, k_{22}, k_{23}) indices in $\{1,\dots,n_1\}$ (respectively in $\{n_1+1,\dots,n_1+n_2\}$) onto k_{11} (respectively k_{21}) indices in $\{1,\dots,n_1\}$, k_{12} (respectively k_{22}) indices in $\{n_1+1,\dots,n_1+n_2\}$ and k_{13} (respectively k_{23}) indices in $\{n_1+n_2+1,\dots,n\}$. A routine computation of the coefficient $C(k_{ij})$ gives

$$C(k_{ij}) = n_1!\binom{n_1}{k_{11},k_{13}}n_2!\binom{n_2}{k_{21},k_{23}}n_3!\binom{n_3}{k_{31},k_{33}}.$$

Then

$$\begin{aligned}
[P]^2 &= \frac{1}{n!}\sum_{\substack{k_{11}+k_{21}+k_{31}=n_1\\ k_{13}+k_{23}+k_{33}=n_3}} C(k_{ij})\\
&\quad\times(1+\alpha^2)^{k_{11}}(1+\beta^2)^{n_2-k_{21}-k_{23}}2^{k_{33}}\\
&\quad\times(1+\alpha\beta)^{k_{21}+n_1-k_{11}-k_{13}}(1+\alpha)^{k_{13}+k_{31}}(1+\beta)^{k_{23}+n_3-k_{31}-k_{33}}\\
&= \frac{1}{n!}(1+\alpha\beta)^{n_1}(1+\beta^2)^{n_2}(1+\beta)^{n_3}\sum_{\substack{k_{11}+k_{21}+k_{31}=n_1\\ k_{13}+k_{23}+k_{33}=n_3}} C(k_{ij})\\
&\quad\times\left(\frac{1+\alpha^2}{1+\alpha\beta}\right)^{k_{11}}\left(\frac{1+\alpha}{1+\alpha\beta}\right)^{k_{13}}\left(\frac{1+\alpha}{1+\beta}\right)^{k_{31}}\left(\frac{2}{1+\beta}\right)^{k_{33}}\\
&\quad\times\left(\frac{1+\alpha\beta}{1+\beta^2}\right)^{k_{21}}\left(\frac{1+\beta}{1+\beta^2}\right)^{k_{23}}\\
&= \frac{1}{\binom{n}{n_1,n_3}}(1+\alpha\beta)^{n_1}(1+\beta^2)^{n_2}(1+\beta)^{n_3}\\
&\quad\sum_{\substack{k_{11}+k_{21}+k_{31}=n_1\\ k_{13}+k_{23}+k_{33}=n_3}}\binom{n_1}{k_{11},k_{13}}\binom{n_2}{k_{21},k_{23}}\binom{n_3}{k_{31},k_{33}}\\
&\quad\times\left(\frac{1+\alpha^2}{1+\alpha\beta}\right)^{k_{11}}\left(\frac{1+\alpha}{1+\alpha\beta}\right)^{k_{13}}\left(\frac{1+\alpha}{1+\beta}\right)^{k_{31}}\left(\frac{2}{1+\beta}\right)^{k_{33}}\\
&\quad\times\left(\frac{1+\alpha\beta}{1+\beta^2}\right)^{k_{21}}\left(\frac{1+\beta}{1+\beta^2}\right)^{k_{23}},
\end{aligned}$$

which is our claim.

Proof of Theorem 1. Apply Lemma 7 to Lemma 9 as well as Lemma 9 to Lemma 8 with $k_1 = n_1$, $k_2 = n_3$,

$$A = \frac{1+\alpha^2}{1+\alpha\beta}, B = \frac{1+\alpha}{1+\alpha\beta}, C = \frac{1+\alpha}{1+\beta}, D = \frac{2}{1+\beta}, E = \frac{1+\alpha\beta}{1+\beta^2}, F = \frac{1+\beta}{1+\beta^2}.$$

It is easily seen that $A \geq C \geq E > 0$ and $B \geq D \geq F > 0$.

Comment. One may be surprised that Lemma 7 requires six real numbers A, B, C, D, E, F in the given order. One might expect this lemma to hold for any positive real numbers but counterexamples show that an order between them has to be given. In these two independent results, the most surprising is that these restricted hypothesis of Lemma 7 happen to hold precisely in the context of Lemma 9. We do not believe this to be a coincidence but the true reason is still an open question to us.

V. Special cases and examples

Some special cases of the sufficient condition for $[P] \leq M(P)$ given in Theorem 1 are worth mentioning here as an illustration. First consider a polynomial P whose roots are well apart the unit circle, that is such that $n_3 = 0$. Then a sufficient condition on P is given by n_1, n_2, n, α, β such that

$$\left(1 + \frac{1}{\beta^2}\right)^{n_2} \left(\frac{n_1}{n}(1+\alpha^2) + \frac{n_2}{n}\frac{(1+\alpha\beta)^2}{1+\beta^2})\right)^{n_2} \leq 1.$$

Moreover, add the condition for n even that $n_1 = n_2 = \frac{n}{2}$; then Theorem 1 gives $\beta > \sqrt{\frac{1+\sqrt{5}}{2}}$ and $\frac{-\beta+\sqrt{2(\beta^4-\beta^2-1)}}{2\beta^2+1} < \alpha < 1$.

A great simplification of Theorem 1 can also be obtained in the case of a polynomial P with $n_3 = 0$, n_1 roots concentrated at the origin and n_2 other roots outside the unit circle. Setting $\alpha = 0$ in Theorem 1 gives $(1 + \frac{1}{\beta^2})^{n_2}(\frac{n_1}{n} + \frac{n_2}{n}\frac{1}{1+\beta^2})^{n_2} \leq 1$ but a better bound is given by $\beta > \sqrt{\frac{n}{n_1}}$. For a detailed proof of this result, ask the authors.

The table gives examples showing to what extent Theorem 1 only gives a sufficient condition.

We denote by RES the left-hand side of the inequality in Theorem 1.

n	$P(z)$	RES	$[P]$	$M(P)$	Comments
9	$(z-0,14)^4(z-2)^4(z+1)$	0, 979	13, 9	16	
10	$(z-0,14)^4(z-2)^4(z+1)^2$	$1,9 \geq 1$	19, 4	16	Th. 1 and claim unsat.
13	$(z-0,14)^6(z-2)^6(z+1)$	0, 6	43, 5	64	
14	$(z-0,14)^6(z-2)^6(z+1)^2$	$1,3 \geq 1$	60, 8	64	Th. 1 unsat., claim sat.
13	$(z-0,14)^7(z-2)^5(z+1)$	0, 6	20, 1	32	
14	$(z-0,14)^7(z-2)^5(z+1)^2$	$1,2 \geq 1$	27, 7	32	Th. unsat., claim sat.
12	$(z-0,14)^7(z-2)^5$	0, 341	14, 6	32	case $n_3=0, n_1 \neq n_2$
12	$(z-0,14)^6(z-2)^6$	0, 356	31, 3	64	case $n_3=0, n_1=n_2$
12	$z^6(z-2)^6$	0, 1	18, 7	64	case $\alpha=0, n_3=0, n_1=n_2$
12	$z^7(z-1,31)^5$	$1,1 \geq 1$	3, 4	3, 8	Th. 1 unsat., claim sat. $\sqrt{\frac{n}{n_1}}=\sqrt{\frac{12}{7}} \sim 1,309$

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