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JACOBI FORMS AND THE HEAT OPERATOR II

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1. Introduction

Classically, there are many interesting connections between differential operators and the theory of elliptic modular forms and many interesting results have been explored (see [5] and [6], for instance). Recently, many theorems about Jacobi forms have been studied. It turns out that Jacobi forms play important roles in the theory of numbers. Jacobi forms are connected with modular forms of half-integral weight as well as integral weight, Siegel forms and elliptic curves [4]. Despite the importance of the theory of Jacobi forms, it was only recently that Jacobi forms have been studied systematically by Eichler and Zagier [4].

The heat operator plays an important role in connecting Jacobi forms and elliptic modular forms. Recently, in [1], the bilinear differential operators, called Rankin-Cohen type bracket operators, on Jacobi forms were defined using the heat operator and it was shown how to construct Jacobi forms with Rankin-Cohen type bracket operators.

In this paper, we show how to construct more Jacobi forms using the heat operator. Originally, Eicher-Zagier have shown how to construct modular forms from Jacobi forms [4]. Our main result is a generalization of the result given in [4] to Jacobi forms. We also give examples which have expressions similar to those given in [3]. More precisely, Theorem 3.1 is a generalization of Theorem 3.3 in [4], replacing modular forms that occur there by Jacobi forms. Theorem 3.2 is an analog for Jacobi forms of the result given in [3] for modular forms.

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2. Definitions

First, we define Jacobi forms and the Rankin-Cohen type brackets using the heat operator. We follow the definitions and notations given in [4].

Let $\Gamma(1)$ be the modular group, the set of 2×2 matrices with integer entries and determinant 1. The following slash operators on functions $f\colon \mathcal{H} \times \mathcal{C} \to \mathcal{C}$ are given

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in [4]; for fixed integers k and m,

$$\left(f|_{k,m}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(\tau, z) := (c\tau + d)^{-k} e^{2\pi i m(\frac{-cz^2}{c\tau+d})} f\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right),$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$, and

$$(f|_m[\lambda, \nu])(\tau, z) := e^{2\pi i m(\lambda^2\tau + 2\lambda z)} f(\tau, z + \lambda\tau + \nu) \text{ for } [\lambda, \nu] \in Z^2.$$

From these definitions it is easy to check that, for $M, M' \in \Gamma(1)$, $X, X' \in Z^2$,

$$\begin{aligned}
(f|_{k,m}M)|_{k,m}M' &= f|_{k,m}(MM'), \\
(f|_m X)|_m X' &= f|_m(X + X'), \\
(f|_{k,m}M)|_m XM &= (f|_m X)|_{k,m}M.
\end{aligned}$$

These relations show that the slash operators define an action of the semi-direct product $\Gamma(1)^J := \Gamma(1) \propto Z^2 = \{(M, X) \mid M \in \Gamma(1), X \in Z^2\}$. $\Gamma(1)^J$ forms a group under the group law $(M, X)(M', X') = (MM', XM' + X')$. This is called the (full) Jacobi group.

Using these slash operators, we give the definition of Jacobi form [4].

DEFINITION 2.1. A *Jacobi form* of weight k and index m ($k, m \in Z^+$) on $\Gamma(1)$ is a holomorphic function $f\colon \mathcal{H} \times \mathcal{C} \to \mathcal{C}$ satisfying

$$\begin{aligned}
(f|_{k,m}M)(\tau, z) &= f(\tau, z) \ \text{ for } M \in \Gamma(1), \\
(f|_m X)(\tau, z) &= f(\tau, z) \ \text{ for } X \in Z^2
\end{aligned}$$

and such that it has a Fourier expansion of the form

$$f(\tau, z) = \sum_{\substack{n=1 \\ r \in Z,\, r^2 \leq 4nm}}^{\infty} c(n, r) q^n \zeta^r,$$

where $q = e^{2\pi i\tau}$ and $\zeta = e^{2\pi i z}$. If f satisfies the stronger condition on the Fourier expansion

$$f(\tau, z) = \sum_{\substack{n=1 \\ r \in Z,\, r^2 < 4nm}}^{\infty} c(n, r) q^n \zeta^r,$$

f is called a *Jacobi cusp form* of weight k and index m.

The vector space of all Jacobi forms of weight k and index m is denoted $J_{k,m}$. In particular, we denote the vector space of all Jacobi cusp forms of weight k and index m by $J_{k,m}^{\text{cusp}}$.

In [4], the heat operator $L_{(m)}$ on $J_{k,m}$ was studied. It plays an important role in connecting Jacobi forms and elliptic modular forms. This operator was used to construct the Rankin-Cohen type bracket on the space of Jacobi forms in [1]. We now define the heat operator.

DEFINITION 2.2. For integer m,

$$L_{(m)} = \frac{1}{(2\pi i)^2}\left(8\pi i m\frac{\partial}{\partial\tau} - \frac{\partial^2}{\partial z^2}\right)$$

This operator was used in [1] to define Rankin-Cohen type operators on the space of Jacobi forms.

3. Constructions and examples

The following theorem generalizes the idea of Eichler-Zagier given in [4], § I.3, pp. 28–35], replacing modular forms that occur there by Jacobi forms. As a result, we introduce $\tilde{f}$, a function of three variables, in place of the two variable $\tilde{f}$ occuring in [4], Theorem 3.3, p. 35. This leads us how to construct Jacobi forms using the heat operator.

THEOREM 3.1. *Let $\tilde{f}(\tau, z; X)$ be a formal power series in X,*

$$\tilde{f}(\tau, z; X) = \sum_{\nu=0}^{\infty} \chi_\nu(\tau, z)((2\pi i)^2 X)^\nu,$$

satisfying, for some k, m, the functional equation

$$\tilde{f}\left(M(\tau), \frac{z}{c\tau+d}; \frac{X}{(c\tau+d)^2}\right) = (c\tau+d)^k e^{2\pi i m\frac{cz^2}{c\tau+d}} e^{8\pi i m\frac{cX}{c\tau+d}} \tilde{f}(\tau, z; X) \qquad (1)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$. Furthermore, assume that coefficients $\chi_\nu(\tau, z)$ of $\tilde{f}(\tau, z; X)$, are holomorphic in $\mathcal{H} \times C$ and satisfy

$$\chi_\nu|_m[a, b] = \chi_\nu \text{ for all } [a, b] \in Z^2, \qquad (2)$$

with a Fourier expansion of the form

$$\chi_\nu(\tau, z) = \sum_{n\geq 1, r\in Z, r^2\leq 4mn} c(n, r)q^n\zeta^r \quad \text{with } q = e^{2\pi i\tau}, \zeta = e^{2\pi i z}. \qquad (3)$$

Then ξ_ν, defined as

$$\xi_\nu(\tau, z) = \sum_{j=0}^{\nu} \frac{(-1)^j L_{(m)}^j(\chi_{\nu-j})(\alpha+2\nu-j-2)!}{j!(\alpha+2\nu-2)!}, \qquad (4)$$

is a Jacobi form of weight $k+2\nu$ and index m. Here, $\alpha = k - \frac{1}{2}$, $x! = \Gamma(x+1)$.

REMARK 3.1. 1. Since $\tilde{f}$ is periodic in τ and z from the functional equation given in (1) and (2), it is clear that $\chi_\nu(\tau, z)$ has a Fourier expansion.
2. Recently, Theorem 3.1 has been applied to construct linearly independent Rankin-Cohen type brackets on the spaces of Jacobi forms [2].

If we invert the formula in (4), we get

$$\chi_\nu(\tau, z) = \sum_{j=o}^{\nu} \frac{L^j_{(m)}(\xi_{\nu-j})(\alpha + 2\nu - 2j - 1)!}{j!(\alpha + 2\nu - j - 1)!}.$$

By choosing $\xi_o = f, \ldots, \xi_\nu = 0$, where f is a Jacobi form, we get the following corollary.

COROLLARY 3.1. *Let $f(\tau, z)$ be a Jacobi form of weight k and index m. Then*

$$\tilde{f}(\tau, z; X) = \sum_{\nu=0}^{\infty} \frac{L^\nu_{(m)}(f(\tau, z))}{\nu!(\alpha + \nu - 1)!}((2\pi i)^2 X)^\nu$$

satisfies the functional equation

$$\tilde{f}\left(M(\tau), \frac{z}{c\tau + d}; \frac{X}{(c\tau + d)^2}\right) = (c\tau + d)^k e^{2\pi i m \frac{cz^2}{c\tau+d}} e^{8\pi i m \frac{cX}{c\tau+d}} \tilde{f}(\tau, z; X).$$

This function $\tilde{f}(\tau, z; X)$ in Cor.3.1 has been used to define the Rankin-Cohen type bracket on the space of Jacobi forms. The following corollary is an analog for Jacobi forms of that given by Kuznetsov-Cohen for modular forms (see [4], p. 35).

COROLLARY 3.2 [1]. *For each ν, define a bracket operator $[\ \ ]_\nu$ on $J_{k_1,m_1} \times J_{k_2,m_2}$ as*

$$[f_1, f_2]_\nu = \sum_{\ell=0}^{\nu} (-1)^\ell \binom{\alpha_1 + \nu - 1}{\nu - \ell} \binom{\alpha_2 + \nu - 1}{\ell} m_1^{\nu-\ell} m_2^\ell L^\ell_{(m_1)}(f_1) L^{\nu-\ell}_{(m_2)}(f_2),$$

with $f_i \in J_{k_i,m_i}$, $\alpha_i = k_i - \frac{1}{2}$ and $x! = \Gamma(x + 1)$. Then, this operator $[\ ]$ maps

$$[\ \ ]_\nu : J_{k_1,m_1} \times J_{k_2,m_2} \to J_{k_1+k_2+2\nu, m_1+m_2}$$

(for instance, $[f_1, f_2]_0 = f_1 f_2$, $[f_1, f_2]_1 = \alpha_1 m_1 f_1 L_{(m_2)}(f_2) - \alpha_2 m_2 L_{(m_1)}(f_1) f_2$, ... etc.).

Proof of Theorem 3.1. The proof is analogous to that given in [4], pp. 33–34. We introduce the operator $\tilde{L}_{k,m}$ defined by

$$\tilde{L}_{k,m} = \frac{2m}{\pi i}\frac{\partial}{\partial \tau} - \frac{1}{(2\pi i)^2}\frac{\partial^2}{\partial z^2} - \frac{(2k - 1)}{2(2\pi i)^2}\frac{\partial}{\partial X} - \frac{X}{(2\pi i)^2}\frac{\partial^2}{\partial X^2}.$$

Then, from direct computation using the functional equation (1), we get

$$(c\tau+d)^{-k-2}e^{2\pi im\frac{-cz^2}{c\tau+d}}e^{8\pi im\frac{-cX}{c\tau+d}}\tilde{L}_{k,m}(\tilde{f})\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d};\frac{X}{(c\tau+d)^2}\right)$$
$$=\tilde{L}_{k,m}((c\tau+d)^{-k}e^{2\pi im\frac{-cz^2}{c\tau+d}}e^{8\pi im\frac{-cX}{c\tau+d}}\tilde{f}\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d};\frac{X}{(c\tau+d)^2}\right). \tag{5}$$

So, by considering the space $\tilde{M}_{k,m}$ of functions $\tilde{f}$ satisfying the functional equation in (1), we see that, by (5), $\tilde{L}_{k,m}$ is a map from $\tilde{M}_{k,m}$ to $\tilde{M}_{k+2,m}$. Explicitly, we have

$$\tilde{L}_{k,m}:\sum_{\lambda\geq 0}\chi_\lambda((2\pi i)^2X)^\lambda\to\sum_{\lambda\geq 0}\{L_{(m)}(\chi_\lambda)-(\lambda+1)(\alpha+\lambda)\chi_{\lambda+1}\}((2\pi i)^2X)^\lambda,$$

with $\alpha=k-\frac{1}{2}$. Iterating this formula ν times, we find, by induction on ν, that the composite map

$$\tilde{M}_{k,m}\xrightarrow{\tilde{L}_{k,m}}\tilde{M}_{k+2,m}\xrightarrow{\tilde{L}_{k+2,m}}\tilde{M}_{k+4,m}\xrightarrow{\cdots}\cdots\xrightarrow{\tilde{L}_{k+2\nu-2,m}}\tilde{M}_{k+2\nu,m}$$

maps $\sum_{\lambda\geq 0}\chi_\lambda(\tau,z)((2\pi i)^2X)^\lambda$ to

$$\sum_{\lambda\geq 0}\sum_{j=0}^{\nu}\frac{(-1)^{j+\nu}L^j_{(m)}(\chi_{\lambda+\nu-j})\binom{\nu}{j}(\lambda+\nu-j)!(\lambda+\alpha+2\nu-j-2)!}{\lambda!(\lambda+\alpha+\nu-2)!}((2\pi i)^2X)^\lambda$$

and composing this with the map gives $\frac{(-1)^\nu(\alpha+2\nu-2)!\nu!}{(\alpha+\nu-2)!}\xi_\nu$, where

$$\xi_\nu(\tau,z)=\sum_{j=0}^{\nu}\frac{(-1)^jL^j_{(m)}(\chi_{\nu-j})(\alpha+2\nu-j-2)!}{j!(\alpha+2\nu-2)!}.$$

This means that from (5) we get

$$(\xi_\nu|_{k+2\nu,m}M)(\tau,z)=\xi_\nu(\tau,z)$$

for every $M\in\Gamma(1)$. ξ_ν also satisfies

$$(\xi_\nu|_m[a,b])(\tau,z)=\xi_\nu(\tau,z),$$

for every $[a,b]\in Z^2$. This is because of the condition on χ_ν in (2) and the fact that $L_{(m)}(\chi_\nu)|_m[a,b]=L_{(m)}(\chi_\nu|_m[a,b])$. From the condition on χ_ν in (3), it is easy to see that ξ_ν has an expansion of the form $\xi_\nu(\tau,z)=\sum_{n\geq 1,r^2\leq 4mn}a(n,r)q^n\zeta^r$.

We need the following well-known result to state the next theorem.

LEMMA 3.1 [3]. *For any $r \in R^+$, we have*

$$(1 - st + Nt^2)^{-r} = \frac{1}{(r-1)!}\sum_{n\geq 0} t^n \sum_{0\leq \ell\leq \frac{n}{2}} (-1)^\ell \frac{(n+r-\ell-1)!}{\ell!(n-2\ell)!} s^{n-2\ell} N^\ell$$

As an example of Theorem 3.1, we can construct Jacobi forms in the following way. This is an analog for Jacobi forms of a theorem given in [3].

THEOREM 3.2. *Let $f_i \in J_{k_i,m_i}$, $i = 1, 2$, with*

$$f_i(\tau, z) = \sum_{n\geq 1, r\in Z r^2\leq 4m_i n} a_i(n, r) q^n \zeta^r.$$

Then the coefficient of $t^{2\nu}$ in

$$\sum_{N,R\in Z} \sum_{\substack{s,r \\ R^2\leq 4m_2 s \\ r^2\leq 4m_1(N-s)}} \frac{a_1(N-s, r) a_2(s, R)}{(1 - 2\sqrt{m_1 m_2} rt + (4m_2 N - R^2)t^2)^{\alpha-1}} q^N \zeta^R$$

is a Jacobi form of weight $k_1 + k_2 + 2\nu$ and index m_2. Here, $\alpha = k_1 + k_2 - \frac{1}{2}$.

Proof of Theorem 3.2. For $f_i \in J_{k_i,m_i}$, $i = 1, 2$, with

$$f_i(\tau, z) = \sum_{\substack{r,n\in Z \\ r^2\leq 4m_i n}} a_i(n, r) q^n \zeta^r,$$

define

$$\tilde{f}(\tau, z; X) = f_1(\tau, 2\sqrt{\frac{m_2}{m_1}}\sqrt{X}) f_2(\tau, z). \tag{6}$$

Then, from the functional equations for f_i, $i = 1, 2$, $\tilde{f}$ satisfies

$$\tilde{f}\left(M\tau, \frac{z}{c\tau+d}; \frac{X}{(c\tau+d)^2}\right) = (c\tau+d)^{-k_1-k_2} e^{2\pi i m_2 \frac{-cz^2}{c\tau+d}} e^{8\pi i m_2 \frac{-cX}{c\tau+d}} \tilde{f}(\tau, z, X),$$

for every $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma(1)$. So, if we let

$$\tilde{f}(\tau, z; X) = \sum_{\nu\geq 0} \chi_\nu(\tau, z)((2\pi i)^2 X)^\nu,$$

then from (6) we have

$$\chi_\nu(\tau, z) = \frac{1}{(2\nu)!} \sum_{\substack{N,s,r,R\in Z \\ r^2\leq 4m_1(N-s) \\ R^2\leq 4sm_2}} (2\sqrt{m_1 m_2} r)^{2\nu} a_1(N-s, r) a_2(s, R) q^N \zeta^R. \tag{7}$$

Note that $\chi_\nu(\tau, z)$ satisfies properties (2) and (3) because of the functional equations for f_i, $i = 1, 2$, and $R^2 \leq 4m_2 s \leq 4m_2 N$.

Now, by substituting $\chi_\nu(\tau, z)$ from (7) into the formula for ξ_ν in (4), we get a Jacobi form of weight $k_1 + k_2 + 2\nu$ and index m_2:

$$\xi_\nu(\tau, z) = \sum_{\substack{N,s,r,R \in Z \\ r^2 \leq 4m_1(N-s) \\ R^2 \leq 4m_2 s}} \frac{a_1(N-s, r)a_2(s, R)}{(\alpha + 2\nu - 2)!}$$

$$\times \sum_{j=0}^{\nu} \frac{(-1)^j (4m_2N - R^2)^j (2\sqrt{m_1 m_2} r)^{2\nu - 2j} (\alpha + 2\nu - j - 2)!}{(2\nu - 2j)! j!} q^N \zeta^R,$$

with $\alpha = k_1 + k_2 - \frac{1}{2}$. Since, by Lemma 3.1, the coefficient of $t^{2\nu}$ in

$$\sum_{\substack{N,R,s,r \in Z \\ R^2 \leq 4m_2 s \\ r^2 \leq 4m_1(N-s)}} \frac{a_1(N-s, r)a_2(s, R)}{(1 - 2\sqrt{m_1 m_2} rt + (4m_2 N - R^2)t^2)^{\alpha - 1}} q^N \zeta^R$$

is

$$\frac{(\alpha + 2\nu - 2)!}{(\alpha - 2)!} \xi_\nu(\tau, z),$$

this concludes the proof of the theorem.

4. Conclusion

In this paper, we have shown how to use the heat operator to constuct more Jacobi forms. This is analogous to results given in [4]. Recently, Theorem 3.1 has been used to construct linearly independent bilinear differential operators, called Rankin-Cohen type brackets, on the spaces of Jacobi forms [2]. D. Zagier pointed out that Theorem 3.1 can be formulated in terms of Fourier-Jacobi expansions of Siegel forms of higher degree, so one can expect that a more general theory can be obtained in terms of Siegel forms. In a future work we shall pursue this more general theory from the point of view of Fourier-Jacobi expansions of Siegel forms of higher degree.

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