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M-IDEALS AND QUASI-TRIANGULAR ALGEBRAS

BY

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Introduction

Let $\mathscr{P}$ be a nest of projections on the Hilbert space H and let Alg $\mathscr{P}$ be the nest algebra consisting of those operators on H leaving invariant the ranges of all projections in $\mathscr{P}$. The quasi-triangular algebra associated with $\mathscr{P}$ is $\mathrm{QT}(\mathscr{P}) = \mathrm{Alg}\,\mathscr{P} + \mathscr{K}$, where $\mathscr{K}$ denotes the ideal of compact operators on H. It was shown in [9] that, if the nest $\mathscr{P}$ consists of a sequence of finite rank projections increasing strongly to the identity, then every bounded linear operator on H has a best approximant in $\mathrm{QT}(\mathscr{P})$. The methods used there were reminiscent of those used in [4] to establish best approximation of L^∞ functions by $H^\infty + C$ functions.

In this paper, we tackle the problem of the existence and uniqueness of best quasi-triangular approximants from a different angle employing the concept of M-ideals in a Banach space. These were first introduced in 1972 by Alfsen and Effros ([2]) and have been studied widely since. Much of this study has centered around determining those spaces X for which the compact operators on X form an M-ideal in the space of all bounded linear operators on X. See, for example, [5], [11], and [19].

In a somewhat different vein, Luecking [15] showed in 1980 that $(H^\infty + C)/H^\infty$ forms an M-ideal in L^∞/H^∞. In analogy with this function theoretic result, we will show that $(\mathrm{Alg}\,\mathscr{P} + \mathscr{K})/\mathrm{Alg}\,\mathscr{P}$ is an M-ideal in $\mathscr{L}(H)/\mathrm{Alg}\,\mathscr{P}$ for every nest $\mathscr{P}$.

Since Alfsen and Effros proved a result equivalent to the statement that if M is an M-ideal in X then every element of X has a best approximant in M [2, Cor. 5.6], the approximation properties have received considerable attention (cf. [14]). In 1975, Halmos, Scranton, and Ward [12] proved that best approximants in an M-ideal to a given element are not only not unique but abundant. In our context, we will be able to conclude the existence of many best quasi-triangular approximants to a given operator.

Among our tools will be an operator analogue of the abstract F. and M. Riesz Theorem. This is proven in Section 2 and serves to highlight some of the underlying parallels and interplay between operator theory and harmonic analysis.

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In the final section, some related results on quasi-triangular algebras and M-summands are presented.

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Since we arrived at these results, we have learned that there is some overlap with the work of Kenneth R. Davidson and Stephen C. Power who have proved some results on M-ideals in a C^*-algebra setting [6]. For instance, they prove that if J is a closed ideal in a C^*-algebra $\mathscr{A}$ and if S is a subalgebra of $\mathscr{A}$ such that $S \cap J$ contains a bounded approximate identity for J, then $(S + J)/S$ is an M-ideal in $\mathscr{A}/S$.

1. Preliminaries

In what follows, $\mathscr{L}(H)$ will denote the algebra of bounded linear operators on the separable, infinite dimensional Hilbert space H with $\mathscr{K}(H)$, or simply $\mathscr{K}$, denoting the ideal of compact operators in $\mathscr{L}(H)$. All subspaces of H are assumed to be closed and all projections are self-adjoint. For a projection P, let $P^{\perp} = 1 - P$.

A nest is a family of projections which is linearly ordered by range inclusion, contains 0 and 1, and is closed in the strong operator topology (SOT). If $\mathscr{P}$ is a nest and $P \in \mathscr{P}$, define $P_- = \sup\{E \in \mathscr{P}\colon E < P\}$. Since $\mathscr{P}$ is (SOT) closed, $P_- \in \mathscr{P}$. Whenever $P_- < P$, the projection $(P - P_-)$ is called an atom of the nest. $\mathscr{P}$ is said to be continuous if it has no atoms and purely atomic if its atoms sum strongly to the identity, that is $\Sigma_{P \in \mathscr{P}}(P - P_-) = 1$.

For a nest $\mathscr{P}$, the associated nest algebra is

$$\operatorname{Alg} \mathscr{P} = \{T \in \mathscr{L}(H)\colon P^{\perp} TP = 0, \text{ all } P \in \mathscr{P}\}.$$

In [3], Arveson established the following distance formula for an arbitrary nest algebra:

$$(1.1) \qquad d(T, \operatorname{Alg} \mathscr{P}) = \sup\{\|P^{\perp} TP\|\colon P \in \mathscr{P}\}, \quad T \in \mathscr{L}(H).$$

That for every T in $\mathscr{L}(H)$ there is a closest element of $\operatorname{Alg} \mathscr{P}$ is a consequence of the fact that every nest algebra is closed in the weak operator topology (WOT) and a standard argument involving the compactness, in the (WOT), of the closed unit ball in $\mathscr{L}(H)$.

The quasi-triangular algebra associated with a nest $\mathscr{P}$ is defined by $\mathrm{QT}(\mathscr{P}) = \operatorname{Alg} \mathscr{P} + \mathscr{K}(H)$. It was shown in [8] that $\mathrm{QT}(\mathscr{P})$ is a norm closed subalgebra of $\mathscr{L}(H)$.

Let X be a Banach space and X^* its dual space. For a subset J of X, the annihilator of J in X^* will be denoted by $J^{\perp}$. Similarly, if N is a subset of X^* then the preannihilator of N in X is denoted ${}^{\perp}N$. The metric complement of a closed subspace J of X is denoted by J^0 and is defined by

$$J^0 = \{x \in X: \|x\| = d(x, J)\},$$

where $d(x, J)$ denotes the distance of the element x from the subspace J.

A closed subspace N of X^* is said to be an L-summand of X^* if there is a projection E of X^* onto N such that $\|f\| = \|Ef\| + \|f - Ef\|$ for all $f \in X^*$. A subspace J of X is said to be an M-ideal in X if its annihilator $J^\perp$ is an L-summand of X^*. The subspace J is said to be an M-summand of X if there is a subspace J' such that $J \cap J' = \{0\}$, $J + J' = X$, and

$$\|x + y\| = \max\{\|x\|, \|y\|\}$$

whenever $x \in J$ and $y \in J'$. Every M-summand is an M-ideal but not conversely.

We shall need the following important result on best approximation in M-ideals due to Holmes, Scranton, and Ward.

THEOREM 1.2 [12]. *Let J be an M-ideal in X. For each $x \in X \setminus J$, the set $\mathscr{S}(x) = \{y \in J: \|x - y\| = d(x, J)\}$ algebraically spans J.*

A class of operators on Hilbert space which will appear in the sequel is the trace class, denoted (tc). □ $T \in$ (tc) provided both $T \in \mathscr{K}(H)$ and the eigenvalues of $(T^*T)^{1/2}$ are summable. If $T \in$ (tc), then the number

$$\operatorname{tr}(T) = \sum_j \langle Te_j, e_j \rangle$$

defines the trace of T, where $\{e_j\}$ is an orthonormal basis for H. This series converges absolutely and its sum is independent of the choice of orthonormal basis.

$\mathscr{L}(H)$ may be identified with the dual space of (tc) and every trace class operator T induces a bounded linear functional on $\mathscr{L}(H)$, namely $\phi_T(A) = \operatorname{tr}(AT)$ for all $A \in \mathscr{L}(H)$.

We shall need the following results, the first due to Fall, Arveson, and Muhly and the second a decomposition theorem due to Dixmier.

THEOREM 1.3 [8, APPENDIX]. *If* Alg $\mathscr{P}$ *is a nest algebra and* $T \in$ (tc) *then* ϕ_T *annihilates* Alg $\mathscr{P}$ *if and only if* $(1 - P_-)TP = 0$ *for every* $P \in \mathscr{P}$.

THEOREM 1.4 [7], [18]. *Every element ϕ of the dual space $\mathscr{L}(H)^*$ can be represented uniquely by $\phi = \phi_0 + \phi_T$, where $\phi_0 \in \mathscr{K}(H)^\perp$ and ϕ_T is induced by the operator $T \in$ (tc). Moreover, $\|\phi\| = \|\phi_0\| + \|\phi_T\|$.*

This last theorem appeared in 1950 and constitutes the first proof that $\mathcal{K}(H)$ is an M-ideal in $\mathcal{L}(H)$, though Dixmier, of course, did not use that terminology.

2. Nest algebras and M-ideals

Throughout this section let $\mathcal{P}$ be a nest of projections and let $\mathcal{A} = \operatorname{Alg} \mathcal{P}$ be the associated nest algebra.

PROPOSITION 2.1. *Suppose* $\phi \in \mathcal{L}(H)^*$ *and has the decomposition* $\phi = \phi_0 + \phi_T$, *as in Theorem* 1.4. *If* $\phi \in \mathcal{A}^{\perp}$, *then* $\phi_0 \in \mathcal{A}^{\perp}$ *and* $\phi_T \in \mathcal{A}^{\perp}$ *as well.*

Proof. Clearly, it is enough to show that ϕ_T annihilates $\mathcal{A}$ if ϕ does. Suppose, to the contrary, that $\phi_T \notin \mathcal{A}^{\perp}$. It follows from Theorem 1.3 that $(1 - P_-)\, TP \neq 0$ for some $P \in \mathcal{P}$. Let

$$A = [(1 - P_-)TP]^* = PT^*(1 - P_-).$$

If $E \in \mathcal{P}$, then either $E < P$ in which case $E \leq P_-$ and $(1 - P_-)E = 0$ or $E \geq P$ in which case $(1 - E)P = 0$. In any event, we have

$$(1 - E)AE = (1 - E)PT^*(1 - P_-)E = 0$$

which implies that $A \in \mathcal{A}$.

To calculate $\operatorname{tr}(AT)$, let $\{e_j\}$ be an orthonormal basis for the range of P and let $\{f_k\}$ be an orthonormal basis for the range of $P^{\perp}$. We have

$$\begin{aligned}
\operatorname{tr}(AT) &= \sum_j \langle ATe_j, e_j\rangle + \sum_k \langle ATf_k, f_k\rangle \\
&= \sum_j \langle PT^*(1 - P_-)Te_j, e_j\rangle + \sum_k \langle PT^*(1 - P_-)Tf_k, f_k\rangle \\
&= \sum_j \langle PT^*(1 - P_-)Te_j, e_j\rangle \quad (\text{since } Pf_k = 0 \quad \text{for all } k) \\
&= \sum_j \|(1 - P_-)TPe_j\|^2
\end{aligned}$$

Since $(1 - P_-)TP \neq 0$ but $(1 - P_-)TPf_k = 0$ for all k, it follows that

$$(1 - P_-)TPe_j \neq 0 \quad \text{for some } j$$

and, hence, that $\operatorname{tr}(AT) \neq 0$.

Since $T \in (\mathrm{tc})$, we also have $A \in (\mathrm{tc})$. In particular, $A \in \mathcal{K}(H)$ so that $\phi_0(A) = 0$. Therefore, $\phi(A) = \phi_0(A) + \phi_T(A) = \operatorname{tr}(AT) \neq 0$ contradicting the assumption that $\phi \in \mathcal{A}^{\perp}$. ■

The preceding proposition may be viewed as an operator analogue of the abstract F. and M. Riesz Theorem with the nest algebra Alg $\mathscr{P}$ playing the role of the space of bounded analytic functions H^∞.

By the standard theory of commutative C^* algebras, the space L^∞ of bounded measurable functions on the unit circle may be identified with the space of continuous functions on its maximal ideal space $\mathscr{M}$. The dual space $(L^\infty)^*$ may therefore be identified with the space of regular Borel measures on $\mathscr{M}$. Among these measures is a lifting of Lebesgue measure, say m, defined by

$$\int_{\mathscr{M}} \hat{f}\, dm = \frac{1}{2\pi} \int_0^{2\pi} f\, dt$$

where $\hat{f}$ is the continuous function on $\mathscr{M}$ identified with the L^∞ function f. Every regular Borel measure μ on $\mathscr{M}$ has a decomposition $\mu = \mu_a + \mu_s$, where μ_a is absolutely continuous and μ_s is singular with respect to m. Since μ_a can be associated with a function in $L^1(\mathscr{M})$ and $\|\mu_a + \mu_s\| = \|\mu_a\| + \|\mu_s\|$, this decomposition parallels the decomposition $\phi = \phi_0 + \phi_T$ for $\phi \in \mathscr{L}(H)^*$ with μ_a and ϕ_T analogous to one another. The fact that $\mathscr{L}(H)$ is a second dual space (of $\mathscr{K}(H)$) while L^∞ is not is responsible for some major flaws in the analogy.

The abstract version of the F. and M. Riesz Theorem (cf. [1], [10, II, Theorem 7.6]) says that if μ annihilates the space H_0^∞ of bounded analytic functions of mean value 0, then μ_a also annihilates H_0^∞ and μ_s annihilates H^∞.

The F. and M. Riesz Theorem provides the key ingredient in Luecking's proof [15] that $(H^\infty + C)/H^\infty$ is an M-ideal in L^∞/H^∞. Proposition 2.1 will do the same in the proof of our main result which we now give.

THEOREM 2.2. *If $\mathscr{P}$ is a nest of projections and $\mathscr{A} =$ Alg $\mathscr{P}$, then $(\mathscr{A} + \mathscr{K}(H))/\mathscr{A}$ is an M-ideal in $\mathscr{L}(H)/\mathscr{A}$.*

Proof. We must show that $((\mathscr{A} + \mathscr{K})/\mathscr{A})^\perp$ is an L-summand in $(\mathscr{L}(H)/\mathscr{A})^*$. For this, we first make the following standard identifications:

$$(\mathscr{L}(H)/\mathscr{A})^* \cong \mathscr{A}^\perp; \quad ((\mathscr{A} + \mathscr{K})/\mathscr{A})^\perp \cong (\mathscr{A} + \mathscr{K})^\perp.$$

Note that we need the fact that $\mathscr{A} + \mathscr{K}$ is norm closed. To prove the theorem we will show that $(\mathscr{A} + \mathscr{K})^\perp$ is an L-summand in $\mathscr{A}^\perp$.

From Theorem 1.4, every ϕ in $\mathscr{L}(H)^*$, and hence every ϕ in $\mathscr{A}^\perp$, can be written as $\phi = \phi_0 + \phi_T$ where $\phi_0 \in \mathscr{K}^\perp$ and ϕ_T is induced by the operator $T \in$ (tc). Also, $\|\phi\| = \|\phi_0\| + \|\phi_T\|$. Define $E\phi = \phi_0$ for every $\phi \in \mathscr{L}(H)^*$. Then $\|\phi\| = \|E\phi\| + \|\phi - E\phi\|$ and it only remains to show that the image of $\mathscr{A}^\perp$ under E is exactly $(\mathscr{A} + \mathscr{K})^\perp$. For this, suppose

$$\phi = \phi_0 + \phi_T \in \mathscr{A}^\perp.$$

Proposition 2.1 implies that $\phi_0 \in \mathscr{A}^\perp$ as well. Since $\phi_0 \in \mathscr{K}^\perp$ by definition and since $\mathscr{A}^\perp \cap \mathscr{K}^\perp = (\mathscr{A} + \mathscr{K})^\perp$, it follows that $E\phi \in (\mathscr{A} + \mathscr{K})^\perp$ whenever $\phi \in \mathscr{A}^\perp$.

To see that E maps $\mathscr{A}^\perp$ onto $(\mathscr{A} + \mathscr{K})^\perp$, suppose that

$$\phi = \phi_0 + \phi_T \in (\mathscr{A} + \mathscr{K})^\perp .$$

Thus, ϕ and ϕ_0 both annihilate $\mathscr{K}$ which implies that ϕ_T does too. But (tc) $\cong \mathscr{K}(H)^*$ so ϕ_T is the zero functional on $\mathscr{K}$ if and only if $T = 0$. Therefore $\phi = \phi_0 = E\phi$ and E maps $\mathscr{A}^\perp$ onto $(\mathscr{A} + \mathscr{K})^\perp$. This also shows that E is a projection. The proof of the theorem is complete. ∎

Before proceeding, note that each equivalence class in $(\mathscr{A} + \mathscr{K})/\mathscr{A}$ can be represented by a compact operator and that the norm in $\mathscr{L}(H)/\mathscr{A}$ of an equivalence class $T + \mathscr{A}$ is just $d(T, \mathscr{A})$. Also, note that

$$d(T + \mathscr{A}, (\mathscr{A} + \mathscr{K})/\mathscr{A}) = d(T, \mathscr{A} + \mathscr{K}).$$

For each operator T in $\mathscr{L}(H)$ define the set

$$\mathscr{S}(T) = \{K + \mathscr{A}\colon K \in \mathscr{K} \text{ and } d(T - K\mathscr{A}) = d(T, \mathscr{A} + \mathscr{K})\}.$$

This is the set of equivalence classes in $(\mathscr{A} + \mathscr{K})/\mathscr{A}$ which are best approximants to $T + \mathscr{A}$. By Theorems 1.2 and 2.2, the set $\mathscr{S}(T)$ algebraically spans $(\mathscr{A} + \mathscr{K})/\mathscr{A}$ whenever $T \notin \mathscr{A} + \mathscr{K}$. To obtain two different best approximants in $\mathscr{A} + \mathscr{K}$ to T, select two compact operators K and K_1 representing different equivalence classes in $\mathscr{S}(T)$. As we have just noted, there are many possible choices for K and K_1. Then find operators A and A_1 in $\mathscr{A}$ such that

$$\|T - K - A\| = d(T - K, \mathscr{A}) \quad \text{and} \quad \|T - K_1 - A_1\| = d(T - K_1, \mathscr{A}).$$

By the choice of K and K_1, we see that $A + K$ and $A_1 + K_1$ are two different best approximants in $\mathscr{A} + \mathscr{K}$ to T. We have thus proved the following.

COROLLARY 2.3. *For every operator $T \in \mathscr{L}(H)$ there exists an operator $B \in \mathscr{A} + \mathscr{K}$ such that $\|T - B\| = d(T, \mathscr{A} + \mathscr{K})$. Furthermore, if $T \notin \mathscr{A} + \mathscr{K}$, then the operator B is not unique.*

3. Metric complements and quasi-triangularity

Let $\mathscr{P}$ be any nest, let $\mathscr{A} = \operatorname{Alg}\mathscr{P}$, and let $M = (\mathscr{A} + \mathscr{K})/\mathscr{A}$. M is a closed subspace of $\mathscr{L}(H)/\mathscr{A}$ and its metric complement is given by

$$M^0 = \{T + \mathscr{A} \in \mathscr{L}(H)/\mathscr{A}\colon d(T, \mathscr{A}) = d(T, \mathscr{A} + \mathscr{K})\}.$$

We will show that M is not an M-summand of $\mathscr{L}(H)/\mathscr{A}$.

Our first result is an application of Arveson's distance formula (1.1).

PROPOSITION 3.1. *M^0 is nowhere dense in $\mathscr{L}(H)/\mathscr{A}$.*

Proof. Since M^0 is a closed subset of $\mathscr{L}(H)/\mathscr{A}$, it is enough to show that it contains no open balls. For this, notice first that if $T + \mathscr{A} \in M^0$ and λ is a scalar then

$$d(\lambda T, \mathscr{A}) = |\lambda|\, d(T, \mathscr{A}) = |\lambda|\, d(T, \mathscr{A} + \mathscr{K}) = d(\lambda T, \mathscr{A} + \mathscr{K})$$

so that $\lambda T + \mathscr{A} \in M^0$. Thus, by taking K to be any compact operator such that $d(K, \mathscr{A}) = 1$, it is easy to see that M^0 contains no open ball in $\mathscr{L}(H)/\mathscr{A}$ centered at the origin.

It remains to show that M^0 contains no other open balls. Suppose $T + \mathscr{A} \in M^0$ and choose $\delta > 0$ such that $0 < \delta < d(T, \mathscr{A})$. By Arveson's distance formula (1.1) for nest algebras, $d(T, \mathscr{A}) = \sup\{\|P^{\perp}TP\|\colon P \in \mathscr{P}\}$. Choose $E \in \mathscr{P}$ such that $\|E^{\perp}TE\| > d(T, \mathscr{A}) - \delta/8$. Then select a unit vector x in H such that $Ex = x$ and $\|E^{\perp}TEx\| \geq d(T, \mathscr{A}) - \delta/4$. Set $r = \|E^{\perp}TEx\|$ and define the rank one operator $S = (\delta/2r)(E^{\perp}TEx \otimes x)$. That is,

$$Sy = \left(\frac{\delta\langle y, x\rangle}{2r}\right) E^{\perp}TEx \quad \text{for all } y \in H.$$

Note that

$$\|S\| = \left(\frac{\delta}{2r}\right)\|E^{\perp}TEx\| \cdot \|x\| = \frac{\delta}{2}.$$

For any $P \in \mathscr{P}$, we have $\|P^{\perp}SP\| \leq \|S\| = \delta/2$. On the other hand,

$$\|E^{\perp}SE\| = (\delta/2r)\|E^{\perp}TEx\| \cdot \|Ex\| = \delta/2.$$

Applying Arveson's formula (1.1) we see that $d((T + S) - T, \mathscr{A}) = d(S, \mathscr{A}) = \delta/2$. This shows that the coset $T + S + \mathscr{A}$ is within distance $\delta/2$ of $T + \mathscr{A}$ in $\mathscr{L}(H)/\mathscr{A}$. The proof will be complete once we show that $T + S + \mathscr{A} \notin M^0$.

Again by the distance formula (1.1),

$$\begin{aligned} d(T + S, \mathscr{A}) &= \sup\{\|P^{\perp}(T + S)P\colon P \in \mathscr{P}\} \\ &\geq \|E^{\perp}(T + S)E\| \\ &\geq \|E^{\perp}(T + S)Ex\| \\ &= (1 + \delta/2r)\|E^{\perp}TEx\| \\ &= r + \delta/2 \\ &\geq d(T, \mathscr{A}) - \delta/4 + \delta/2 \\ &> d(T, \mathscr{A}) \\ &= d(T + S - S, \mathscr{A}) \\ &= d(T + S, \mathscr{A} + S) \\ &\geq d(T + S, \mathscr{A} + \mathscr{K}) \quad \text{since } S \in \mathscr{K}. \end{aligned}$$

Hence, $d(T+S, \mathscr{A}+\mathscr{K}) < d(T+S, \mathscr{A})$ which proves that $T+S+\mathscr{A} \notin M^0$. The proof of the proposition is complete. ■

For each operator T in $\mathscr{L}(H)$, denote by $\mathscr{S}(T)$ the set of best approximants to $T+\mathscr{A}$ in $(\mathscr{A}+\mathscr{K})/\mathscr{A}$. Thus,

$$\mathscr{S}(T) = \{K+\mathscr{A}: K \in \mathscr{K} \text{ and } d(T-K, \mathscr{A}) = d(T, \mathscr{A}+\mathscr{K})\}.$$

Note that if $T+\mathscr{A} = S+\mathscr{A}$ then $\mathscr{S}(T) = \mathscr{S}(S)$ and that if $K+\mathscr{A} \in \mathscr{S}(T)$ then

$$d(T-K, \mathscr{A}) = d(T, \mathscr{A}+\mathscr{K}) = d(T-K, \mathscr{A}+\mathscr{K})$$

so that $T-K+\mathscr{A} \in M^0$.

PROPOSITION 3.2. *For every $T \in \mathscr{L}(H)$, the set $\mathscr{S}(T)$ has no interior points relative to $(\mathscr{A}+\mathscr{K})/\mathscr{A}$.*

Proof. If $T \in \mathscr{A}+\mathscr{K}$, then $\mathscr{S}(T)$ is a singleton and the result is clear. Assume, then, that $T \notin \mathscr{A}+\mathscr{K}$ and suppose that $\mathscr{S}(T)$ has an interior point $K+\mathscr{A}$ relative to $(\mathscr{A}+\mathscr{K})/\mathscr{A}$. This implies that $K+S+\mathscr{A} \in \mathscr{S}(T)$ whenever $S \in \mathscr{A}+\mathscr{K}$ and $d(S, \mathscr{A})$ is sufficiently small. This, in turn, implies that $(T-K)-S+\mathscr{A} \in M^0$ for all such S. This last statement, however, contradicts the previous proposition. ■

The last two results would not be very interesting, of course, if M^0 consisted only of the zero coset $(0+\mathscr{A})$ and if $\mathscr{S}(T)$ were empty. Since $\mathscr{A}$ is a nest algebra, however, we have already seen that $\mathscr{S}(T)$ is quite large whenever $T \notin \mathscr{A}+\mathscr{K}$ and hence that M^0 is also a large set. The last proposition assures us, in this case, that the elements of $\mathscr{S}(T)$ do not algebraically span $(\mathscr{A}+\mathscr{K})/\mathscr{A}$ for a trivial reason.

In [12], Holmes, Scranton, and Ward state, without proof, that the metric complements of M-summands have non-empty interior. Below, we provide a proof of this claim.

PROPOSITION 3.3 [12]. *Let X be a Banach space and let $J \subset X$ be an M-summand. Then the metric complement J^0 has non-empty interior.*

Proof. We may decompose X as $X = J + J'$ where $J \cap J' = \{0\}$ and

$$\|e+f\| = \max\{\|e\|, \|f\|\}$$

whenever $e \in J$ and $f \in J'$. Let $x_0 = e_0 + f_0$ with $e_0 \in J$ and $f_0 \in J'$. Then

$$\begin{aligned} d(x_0, J) &= \inf\{\|(e_0 - e) + f_0\| : e \in J\} \\ &= \inf\{\max\{\|e_0 - e\|, \|f_0\|\} : e \in J\} = \|f_0\|. \end{aligned}$$

Thus,

$$J^0 = \{e + f \in J + J' : \|f\| \geq \|e\|\}.$$

Now, choose $x_0 = e_0 + f_0$ such that $\|f_0\| - \|e_0\| > \delta > 0$. Suppose $x \in X$ satisfies $x = e + f$ and $\|x - x_0\| < \delta/2$. Then

$$\|x - x_0\| = \max\{\|e - e_0\|, \|f - f_0\|\} < \delta/2.$$

We have

$$\begin{aligned}\|e\| &\leq \|e_0\| + \|e - e_0\| \\ &< \|f_0\| - \delta + \delta/2 = \|f_0\| - \delta/2 \leq \|f_0\| - \|f - f_0\| \leq \|f\|.\end{aligned}$$

Hence, $x \in J^0$ which shows that J^0 contains an open ball. This proves the proposition. ∎

COROLLARY 3.4. *Let $\mathscr{A} = \operatorname{Alg} \mathscr{P}$ be a nest algebra. Then $(\mathscr{A} + \mathscr{K})/\mathscr{A}$ is not an M-summand in $\mathscr{L}(H)/\mathscr{A}$.*

Proof. The result follows immediately from Propositions 3.1 and 3.3. ∎

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