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STRICT SINGULARITY OF INCLUSION OPERATORS BETWEEN γ-SPACES AND MEYER-KÖNIG ZELLER TYPE THEOREMS

BY

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Introduction

Let E, F be BK-spaces containing Φ and having $F \subset E$. Then $F < E$ means that whenever G is a BK-space containing Φ and satisfying $E = F + G$, then $E = G$ holds. It is known for instance that $c_0 < l^\infty$, (cf. [10]), and that $l^p < l^q$ for $1 \leq p < q \leq \infty$ (cf. [9]). Also $l < E$ is known to be valid for every BK-space E into which l is weakly compactly included (cf. [9]). In a certain sense $F < E$ indicates that F is a small subspace of E.

In [11], Snyder has shown that $F < E$ is valid if and only if the inclusion operator $F \to E$ is strictly cosingular in the sense of Pelczynski (also called a Pelczynski operator). This clearly throws new light on the above examples and, moreover, provides various other situations $F < E$ (see [11]). Strict cosingularity of an operator has a dual description. In our situation it tells that the inclusion operator i: $F \to E$ is strictly cosingular if and only if its adjoint, the restriction operator i': $E' \to F'$ is strictly singular in the weak star sense. In [11], Snyder proved that the latter is equivalent to strict singularity of i' with respect to the dual norms in the case where F is a separable space.

From the point of view of sequence space theory it seems desirable to express the relation $F < E$ in terms of β- or γ-duality rather than abstract topological duality. In [9], Snyder has pursued this program, discussing a property of the inclusion operator $E^\gamma \to F^\gamma$, called the Meyer-König Zeller property (MKZ property for short), which in many cases gives rise to a dual version of $F < E$. Closing the circle in [11], Snyder introduced an abstract version of the MKZ property in Banach spaces and proved that the restriction operator i': $E' \to F'$ has this abstract MKZ property precisely when it is strictly singular.

The abstract dual description of $F < E$ being complete in the case where F is separable, this is far from being true in the case of concrete duality. The result in [9] expressing $F < E$ in terms of MKZ for $E^\gamma \to F^\gamma$ requires that both E, F are BK-AD-spaces and that, in addition, the closure E_0' of Φ in E'

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has finite codimension in E'. In the present paper we keep to the concrete approach in the spirit of [9]. Using a different technique allows us to avoid the AD condition on the space F and to relax the assumption on E_0'.

Our investigation shows that it is useful to discuss another property of the inclusion $E^\gamma \to F^\gamma$, also introduced in [9], which is called the gliding humps property. We clarify the interrelation of the two properties, proving that they are equivalent in many cases. Also we prove that $E^\gamma \to F^\gamma$ has MKZ if and only if the inclusion is strictly singular with respect to the norm topologies. This is a parallel to the result of Snyder in [11]. However, strict cosingularity of $F \to E$ and strict singularity of $E^\gamma \to F^\gamma$ are no longer related by means of natural duality, so that a direct proof for the mentioned equivalence has to be given.

1. Definitions and preliminaries

In general our terminology follows the monograph [12]. We use two notations different from [12]. Firstly, the sections of a sequence x are noted $P_n x$, $n = 1, 2, \ldots$ Secondly, given a BK-space X containing Φ, we denote by X_{AB} the sectionally bounded part of X, i.e., the linear subspace of X consisting of all sequences $x \in X$ for which $\{P_n x \colon n \in \mathbf{N}\}$ is bounded in X. Notice that X_{AB} is a BK-space when endowed with the norm

$$\|x\|_{AB} = \|x\| + \sup_{n \in \mathbf{N}} \|P_n x\|,$$

where $\| \quad \|$ denotes the norm on X.

In the following we recall some notions of particular interest in our present investigation. A sequence (z^n) of vectors $z^n \in \Phi$, $z^n \neq o$, is called a block sequence if there exists a strictly increasing sequence (k_n) of indices such that $z_k^n = o$ for k not in the interval $(k_{n-1}, k_n]$.

Let $\zeta = (z^n)$ be a block sequence. We denote by $c_0(\zeta)$ the sequence space

$$c_0(\zeta) = \left\{ \sum_{n=1}^{\infty} \lambda_n z^n \colon (\lambda_n) \in c_0 \right\},$$

where summation is understood in the coordinatewise sense. $c_0(\zeta)$ is a BK-space with the norm

$$\|z\|_\infty = \left\| \sum_{n=1}^{\infty} \lambda_n z^n \right\|_\infty = \sup_{n \in \mathbf{N}} |\lambda_n|.$$

In the same sense we use the notations $l^\infty(\zeta)$ and $\Phi(\zeta)$.

Let X, Y be sequence spaces having $\Phi \subset X \subset Y$. Suppose X is a BK-space. The inclusion $X \to Y$ is said to have the gliding humps property if, given any block sequence $\zeta = (z^n)$ with $\|z^n\|_X = 1$, $n \in \mathbf{N}$, there exists a sequence $(\lambda_n) \in l^\infty$ such that $\sum_{n=1}^\infty \lambda_n z^n \in Y \setminus X$, i.e.,

$$l^\infty(\zeta) \cap Y \not\subset X \quad (\text{cf. } [9]).$$

Let X be a BK-space containing Φ. X is called null for block sequences if, given any block sequence $\zeta = (z^n)$, the relation $c_0(\zeta) \subset X$ implies $z^n \to o$ $(n \to \infty)$ in X (cf. [4]).

The following result has been proved in [4, Theorem 1]. We shall need it again in the present paper, and shall therefore give an alternative proof.

LEMMA 1. *Let E be a* BK-*space containing Φ. Suppose $X = E^\gamma$ is separable. Then X is null for block sequences.*

Proof. Every BK-space which is a γ-space may be written as the dual of a BK-AK-space. Indeed, let F be the closure of Φ in $E^{\gamma\gamma}$. Then F is BK-AK, so $F' = F^\beta = F^\gamma$. It remains to prove that $F^\gamma = E^\gamma$. It follows from [1, Prop. 1, (iii)] that $E^\gamma = B_0^*(E)^\gamma$, where $B_0^*(E)$ denotes the linear hull of $bv_0 \cdot E$. Clearly $B_0^*(E) \subset E^{\gamma\gamma}$. But actually the elements of $B_0^*(E)$ have AK in $E^{\gamma\gamma}$ (see [1]). So $B_0^*(E) \subset F$ which gives $F^\gamma = E^\gamma$.

Suppose now we had $c_0(\zeta) \subset E^\gamma = X$ for a block sequence $\zeta = (z^n)$ with $z^n \nrightarrow o$. Passing to a subsequence if necessary, we may assume $\|z^n\|_X \geq \eta > o$ for all n. But now the inclusion $c_0(\zeta) \to X$ turns out to be an embedding. Indeed, for $z = \sum_{n=1}^\infty \lambda_n z^n \in c_0(\zeta)$ we find

$$|\lambda_n| \leq \eta^{-1}\|\lambda_n z^n\|_X = \eta^{-1}\|P_{k_n} z - P_{k_{n-1}} z\|_X \leq 2\eta^{-1}\|z\|_X$$

in view of the monotonity of the norm $\| \quad \|_X$ (cf. [12]). This proves that $c_0(\zeta)$ is a closed subspace of X. Since $c_0(\zeta) \cong c_0$, this is a contradiction since no separable dual space may contain a copy of c_0 (see [3]), and since $X = F'$ by the above. This ends the proof. □

There is a surprising interrelation between the concept of nullity for block sequences and the gliding humps property.

LEMMA 2. *Let E be a* BK-*space containing Φ and let $X = E^\gamma$. Then the following are equivalent:*

(1) *There exists a sequence space Y such that $X \to Y$ has the gliding humps property;*
(2) *$X \to \omega$ has the gliding humps property;*
(3) *X is null for block sequences.*

Proof. Clearly (1) and (2) are equivalent. Assume (2). Let $\zeta = (z^n)$ be a block sequence having $c_0(\zeta) \subset X$. Suppose we had $z^n \nrightarrow o$. Selecting a subsequence if necessary we assume $\|z^n\|_X \geq \eta > o$ for all n. Let $\xi = (v^n)$ be the normalized sequence, i.e., $v^n = (1/\|z^n\|_X) \cdot z^n$. Then $c_0(\xi) \subset c_0(\zeta)$. But notice that $l^\infty(\xi) \subset c_0(\xi)^{\gamma\gamma} \subset X^{\gamma\gamma} = X$, the first inclusion being established in analogy with the classical fact $l^\infty \subset c_0^{\gamma\gamma}$. Since $\|v^n\|_X = 1$, this contradicts the fact that $X \to \omega$ has the gliding humps property. So our assumption must be incorrect, i.e., (2) implies (3). The reverse implication being clear, the proof of the Lemma is complete. □

Remarks. (1) Lemmas 1 and 2 are not valid for β-spaces $X = E^\beta$. This may be seen by taking $E = bv$, $X = cs$. Clearly X is separable, but it is not null for block sequences. Take $z^n = \delta^{2n} - \delta^{2n-1}$, then $c_0(z^n) \subset cs$, but $\|z^n\|_{cs} = 1$. Also the inclusion $cs \to \omega$ has the gliding humps property, so that Lemma 2 is not true for $X = cs$.

(2) Notice that $X \to \omega$ has the gliding humps property if and only if $l^\infty(\zeta) \subset X$ for a block sequence $\zeta = (z^n)$ implies $z^n \to o$ in X.

2. Meyer-König Zeller property

Let X, Y be FK-spaces having $\Phi \subset X \subset Y$. The inclusion $X \to Y$ is said to have the Meyer-König Zeller property (MKZ property for short) if for any FK-space F, the relation $Y \cap F \subset X$ implies that $X \cap F$ is closed in F. This notion has been introduced in [9] as Meyer-König Wilansky Zeller property (MKWZ for short), and later on in [11] has been renamed as presented here. We refer to [9, 11] for various examples concerning MKZ. Let us just mention that the choice of the name comes from the fact that Meyer-König and Zeller proved that $c_0 \to l^\infty$ has MKZ (see [9, 11]).

Our first result concerning the MKZ property indicates a close relation to the gliding humps property discussed in the previous section.

LEMMA 3. *Let E be a* BK*-space containing Φ, and let $X = E^\gamma$. Let Y be any FK-space containing X. Then the following statements are equivalent*:

(1) *$X \to Y$ has the gliding humps property*;

(2) *X is null for block sequences and $X \to Y$ has MKZ*;

(3) *X is null for block sequences and, given any BK-space F with $Y \cap F = X$, X is closed in F_{AB}.*

Proof. (1) implies (2) by Lemma 2 and [9, Theorem 1]. Notice that the result in [9] is stated under more restrictive assumptions, but that the part of the proof needed here does not actually use these. We prove that (2) implies (3). So let F be an BK-space satisfying $Y \cap F = X$. Then (2) implies that X is closed in F. But X is an AB-space, so is contained in F_{AB}. Clearly this implies that X is closed in F_{AB}, so (3) is proved. Finally assume (3). We prove that $X \to Y$ has the gliding humps property. Assume the contrary and find a block sequence $\zeta = (z^n)$ having $\|z^n\|_X = 1$ and $l^\infty(\zeta) \cap Y \subset X$. Let $F = c_0(\zeta) + X$. Then

$$Y \cap F = (Y \cap c_0(\zeta)) + (Y \cap X) = X.$$

So (3) implies that X is closed in F_{AB}. But notice that F itself has AB. Indeed, this follows when we check that every $z = \sum_{n=1}^{\infty} \lambda_n z^n \in c_0(\zeta)$ has bounded sections in F. So let $k \in \mathbf{N}$, $k_{j-1} < k \leq k_j$, where (k_j) is the sequence of indices corresponding with the block sequence (z^n). Then we have

$$P_k z = \sum_{i=1}^{j-1} \lambda_i z^i + \lambda_j P_k z^j,$$

and by the definition of the norm $\| \quad \|_F$ on F (cf. [9, §2]) this implies

$$\begin{aligned} \|P_k z\|_F &\leq \left\| \sum_{i=1}^{j-1} \lambda_i z^i \right\|_\infty + \|\lambda_j P_k z^j\|_X \\ &\leq \|z\|_\infty + |\lambda_j| \cdot \|P_k z^j\|_X \\ &\leq 2\|z\|_\infty, \end{aligned}$$

where the last inequality uses the monotonity of the norm $\| \quad \|_X$ and the fact that $\|z^j\|_X = 1$. This proves in fact that z has bounded sections in F. So we have proved that X is closed in F.

We claim that, on the other hand, X is dense in F. Again this results when we prove that every $z = \sum \lambda_n z^n \in c_0(\zeta)$ can be approximated in F by vectors from ϕ. Since $P_{k_j} z \to z (j \to \infty)$ in $c_0(\zeta)$, and since the inclusion $c_0(\zeta) \to F$ is continuous, this is clear.

We have proved $X = F$, so $c_0(\zeta) \subset X$. This, however, contradicts the fact that X is null for block sequences. This ends our proof. □

Remark. Lemma 3 generalizes [9, Theorem 1], since the assumptions made for the space U in [9] imply that the latter is null for block sequences.

We end this paragraph with a criterion for the presence of the gliding humps property for an inclusion $X \to Y$. This involves another definition which is a variation of a concept introduced in [8].

Let Y be a BK-space containing Φ. Then Y is said to have the strong gliding humps property if, given any block sequence $\zeta = (z^n)$ having $\|z^n\|_Y = 1$, there exists a strictly increasing sequence (r_j) of indices such that $l^\infty(\xi) \subset Y$, where $\xi = (z^{r_n})$. For examples concerning the gliding humps property for spaces we refer to [8].

LEMMA 4. *Let X, Y be BK-spaces having $\Phi \subset X \subset Y$. Suppose $X \to \omega$ has the gliding humps property, and Y has the strong gliding humps property in the above sense. Then $X \to Y$ has the gliding humps property.*

Proof. Let $\zeta = (z^n)$ be a block sequence satisfying $\|z^n\|_X = 1$, $n \in \mathbf{N}$. First consider the case where $\|z^{n_k}\|_Y \to o$ $(k \to \infty)$ for some sequence (n_k) of indices. Then we may select a subsequence (m_k) having

$$\sum_{k=1}^{\infty} \|z^{m_k}\|_Y < \infty.$$

Setting $\xi = (z^{m_k})$, this clearly implies $l^\infty(\xi) \subset Y$.

Next consider the case where $\|z^n\|_Y \geq \eta > o$ for all n. Let $(v^n) = \lambda$ be the normalized block sequence, i.e., $v^n = (1/\|z^n\|_Y) \cdot z^n$. Then the strong gliding humps property for the space Y provides (m_k) so that

$$l^\infty(\kappa) \subset Y, \qquad \kappa = (v^{m_k}).$$

But notice that $l^\infty(\xi) \subset l^\infty(\kappa)$, where $\xi = (z^{m_k})$.

In both cases we have $l^\infty(\xi) \subset Y$, where $\xi = (\mathrm{z}^{\mathrm{m}_k})$. Now we apply the fact that $X \to \omega$ has the gliding humps property. This gives $l^\infty(\xi) \not\subset X$. Therefore $X \to Y$ has the gliding humps property. □

Remark. Notice that $X \to \omega$ has the gliding humps property whenever X is a separable BK-AB-space. Consequently, for every space X of this kind included in l^∞, $X \to l^\infty$ has the gliding humps property and hence MKZ. Indeed, we have to use the fact that l^∞ has the strong gliding humps property.

3. Main theorem

In this section we obtain our main result which derives the validity of $F < E$ from the fact that $E^\gamma \to F^\gamma$ has MKZ or rather has the gliding humps property.

Before starting we need a definition. Given a BK-space E with topological dual E', we note E_0' the norm closed linear hull of the projection functionals $x \to x_n$, $n \in \mathbf{N}$, in E'. Clearly if E' turns out to be a sequence space, then E_0' is just the closure of Φ in E'.

THEOREM 1. *Let E be a* BK-AD*-space such that E_0' is complemented in E'. Let F be any FK-space satisfying $\Phi \subset F \subset E$. Let $X = E^\gamma$, $Y = F^\gamma$, and suppose $X \to Y$ has the gliding humps property. Then $F < E$.*

Proof. Let G be an FK-space containing Φ and having $E = F + G$. We have to prove $E = G$. This requires six steps.

(I) We claim that the weak topologies $\sigma(\Phi, G)$ and $\sigma(\Phi, E)$ have the same null sequences. Since every $\sigma(\Phi, E)$-null sequence is clearly null in $\sigma(\Phi, G)$, we are left to prove the reverse implication. So let (y^n) be a null sequence in $\sigma(\Phi, G)$. It suffices to prove that (y^n) is bounded with respect to the norm $\| \quad \|_X$. Indeed, suppose this has been established, $\|y^n\|_X \le M$, say. Fix $x \in E$ and some $\varepsilon > o$. As E is an AD-space, there exists $\bar{x} \in G$ having $\|x - \bar{x}\|_E \le \varepsilon/2M$. But now we have

$$\begin{aligned} |\langle x, y^n\rangle| &\le \|x - \bar{x}\|_E \cdot \|y^n\|_X + |\langle \bar{x}, y^n\rangle| \\ &\le \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon, \end{aligned}$$

for $n \ge n(\varepsilon)$. So (y^n) is in fact null for $\sigma(\Phi, E)$.

(II) We prove that (y^n) is bounded for $\| \quad \|_X$. Assume the contrary, $\|y^n\|_X \ge 2^n$, say. Since G contains Φ, the sequence (y^n) is coordinatewise null. This permits us to find strictly increasing sequences (n_i), (k_i) of integers satisfying:

(a) $\|y^{n_i} - P_{k_{i-1}} y^{n_i}\|_X \ge 2^i$;

(b) y^{n_i} has length $\le k_i$, $i = 1, 2, \ldots$

Now let us define $v^i = y^{n_i} - P_{k_{i-1}} y^{n_i}$, $\alpha_i = 1/\|v^i\|_X$, $z^i = \alpha_i v^i$. Then $\zeta = (z^n)$ is a block sequence having $\|z^n\|_X = 1$.

(III) We claim that $l^\infty(\zeta) \subset G^\gamma$. Indeed, let $z = \sum \lambda_n z^n \in l^\infty(\zeta)$ be fixed. Let $x \in G$. For $k \in \mathbf{N}$ choose j satisfying $k_{j-1} < k \le k_j$. Then we have

$$\begin{aligned} \sum_{i=1}^{k} x_i z_i &= \sum_{i=1}^{j-1} \lambda_i \alpha_i \sum_{r=k_{i-1}+1}^{k_i} x_r v_r^i + \lambda_j \alpha_j \sum_{r=k_{j-1}+1}^{k} x_r v_r^j \\ &= \sum_{i=1}^{j-1} \lambda_i \alpha_i \langle x, v^i\rangle + \lambda_j \langle P_k x - P_{k_{j-1}} x, \alpha_j v^j\rangle. \end{aligned}$$

Here the first term on the right hand side converges ($k \to \infty$, $k_{j-1} < k \le k_j$) in view of $(\alpha_j) \in l$ and $\langle x, v^i\rangle \to o$ $(i \to \infty)$. The second term remains bounded ($k \to \infty$, $k_{j-1} < k \le k_j$) as a consequence of $(\lambda_j) \in l^\infty$ and the estimate

$$\left|\langle P_k x - P_{k_{j-1}}x, \alpha_j v^j\rangle\right| \le \|P_k x - P_{k_{j-1}}x\|_{X^\gamma}\cdot\|\alpha_j v^j\|_X$$
$$\le 2\|x\|_{X^\gamma}.$$

Here we use the monotonity of the norm $\|\quad\|_{X^\gamma}$ on $X^\gamma = E^{\gamma\gamma}$.

(IV) Using the fact that $X \to Y$ has the gliding humps property, we find some $\lambda \in l^\infty$ such that $z = \Sigma\lambda_n z^n \in Y\setminus X$. But $z \in G^\gamma$ by III. This contradicts the equality $E^\gamma = F^\gamma \cap G^\gamma$. So the sequence (y^n) must be bounded, and this proves that $\sigma(\Phi, G)$ and $\sigma(\Phi, E)$ have the same null sequences in Φ.

(V) We end the proof of the theorem by showing that G is a barrelled subspace of E. The closed graph theorem will then imply $E = G$.

Let U be a barrel in G. We have to prove that U is actually a neighbourhood of o. Now by assumption E_0' is complemented in E', i.e., there exists a closed linear subspace Q of E' such that $E' = E_0' \oplus Q$. Let $M = (E_0')^\perp$ be the annihilator of E_0' calculated in the dual pairing $\langle E'', E'\rangle$. Then $M \cap E = \{o\}$ since E_0' separates the points of E. Notice that $M \cong Q'$, where Q' is the dual of Q with the dual norm. Let B be the polar of the unit ball in Q calculated in the dual pairing $\langle M, Q\rangle$. Then B is $\sigma(M, Q) = \sigma(Q', Q)$-compact, so B is compact with respect to the topology $\sigma(E'', E')$ when regarded as a subset of E''. This follows from $\sigma(E'', E')|M = \sigma(M, Q)$.

Let $V = U + B$. Then V spans $G + M$, therefore V^0, calculated in the pairing $\langle E'', E'\rangle$, is $\sigma(E', G + M)$-bounded. We prove that V^0 is actually bounded in the dual norm on E'.

Let (y^n) be any sequence chosen from V^0. Using the decomposition of E' we find sequences (p^n) in E_0', (q^n) in Q having $y^n = p^n + q^n$. Let $\psi \in M$ be fixed. Then we have

$$\langle \psi, q^n\rangle = \langle \psi, q^n\rangle + \langle \psi, p^n\rangle = \langle \psi, y^n\rangle = O(1),$$

which proves that (q^n) is $\sigma(Q, M) = \sigma(Q, Q')$ bounded. This implies the norm boundedness of (q^n). It remains to prove that (p^n) is bounded in norm.

Using the definition of E_0' we find vectors $r^n \in \Phi$ such that

$$\|p^n - r^n\| \le 1,\ n \in \mathbf{N},$$

where $\|\quad\|$ denotes the dual norm and r^n is identified with an element of E'. So the sequence (r^n) must be bounded with respect to $\sigma(E', G)$, since this is true for (p^n). Regarding (r^n) as a sequence in Φ, this means that it is bounded for $\sigma(\Phi, G)$. But now part I of our proof comes into action. Since the topologies $\sigma(\Phi, G)$ and $\sigma(\Phi, E)$ have the same null sequences, they also have the same bounded sequences, thus (r^n) is bounded in $\sigma(\Phi, E)$, so is bounded in $\sigma(E', E)$ via identification. Hence (r^n) is norm bounded, and thus so is (p^n).

(VI) We have proved that V^0 is norm bounded, so V^{00} is a neighbourhood of o in E''. Thus $V^{00} \cap G$ is a norm neighbourhood of o in G. Clearly V^{00} is the $\sigma(E'', E')$-closure of V. Since B is $\sigma(E'', E')$-compact, the latter is $\bar{V} = \bar{U} + B$. Thus we find

$$V^{00} \cap G = \bar{V} \cap G = \bar{U} \cap G = U,$$

proving that U is a norm neighbourhood of o in G. Here the last equality follows from the fact that U is weakly closed in G. This ends the proof of our theorem. □

4. Consequences

In this section we obtain two consequences of our main result. The first consequence below generalizes [9, Theorem 2].

COROLLARY 1. *Let E be a* BK-AK-*space such that $E_0' = \bar{\Phi}$ is separably complemented in E'. Then the following statements are equivalent for any* FK-*space F having $\Phi \subset F \subset E$:*

(1) $F < E$;

(2) $E^\gamma \to F^\gamma$ *has the* MKZ *property.*

Proof. (2) implies (1) by Theorem 1 when we observe that $E^\gamma = E'$, being separable, is null for block sequences, so that $E^\gamma \to F^\gamma$ has the gliding humps property by Lemma 3.

We prove that (1) implies (2). We check that condition (3) from Lemma 3 is fulfilled. So let X be an FK-space satisfying $X \cap F^\gamma = E^\gamma$. We have to prove that E^γ is closed in X_{AB}. Actually we shall prove $E^\gamma = X_{\mathrm{AB}}$.

Observe that $X_{\mathrm{AB}} \cap F^\gamma = E^\gamma$ must hold, since a vector $x \in X$ belonging to E^γ has bounded sections in E^γ, so its sections must be bounded in X as well. Now using [2, Satz 2.3(c)] we have $X_{\mathrm{AB}}^\gamma + F^{\gamma\gamma} = E^{\gamma\gamma}$. So

$$X_{\mathrm{AB}}^{\gamma\gamma} \cap F^\gamma = E^\gamma,$$

dualizing once more. Now observe that $X_{\mathrm{AB}}^{\gamma\gamma} = (X_{\mathrm{AB}}^\gamma \cdot bv_0)^\gamma$ holds by [1, Prop. 1 (iii)]. Since $X_{\mathrm{AB}}^\gamma \subset E^{\gamma\gamma}$, we deduce

$$X_{\mathrm{AB}}^\gamma \cdot bv_0 \subset E^{\gamma\gamma} \cdot bv_0 = E,$$

where the last equality comes from the fact that E has AK. So we have shown that

$$G = X_{AB}^{\gamma} \cdot bv_0 + F$$

is a dense FK-subspace of E satisfying $G^\gamma = E^\gamma$, hence $G^\beta = E^\beta$. Now using the fact that E has the Wilansky property (cf. [4, Prop. 2] or [5, Theorem 2]), we deduce $E = G$.

Applying $F < E$ now, we derive $E = X_{AB}^{\gamma} \cdot bv_0$, giving $E^\gamma = X_{AB}^{\gamma\gamma}$, so actually $E^\gamma = X_{AB}$. This ends the proof. □

Remark. Notice that the result in [9] uses a similar reasoning. Also the equality $E = G$ has to be derived from the weaker $E^\gamma = G^\gamma$. The author, however, does not have the full strength of the converse theorem from [4], [5], so has to use the converse theorem [12, 8.6.1]. The latter, however, relies on the notion of f-duality, so the space G has to be an AK-space, and this is possible to arrange only when F is assumed to be AK.

Our next result gives a condition for $l < E$. The reader might compare this with the results in [7].

COROLLARY 2. *Let E be a* BK-AD-*space containing l. Suppose E_0' is complemented in E' and E^γ is null for block sequences. Then $l < E$.*

Proof. This follows from Theorem 1 when we prove that $E^\gamma \to l^\infty$ has the gliding humps property. But this follows from Lemma 4, since the inclusion $E^\gamma \to \omega$ has the gliding humps property by assumption, while l^∞ has the strong gliding humps property. □

5. Strict singularity

We have already mentioned in the introduction that Snyder [11] has introduced an abstract version of the MKZ property for the restriction operator $E' \to F'$ which turns out to be equivalent to strict singularity of this operator at least when the space F is assumed separable. This naturally raises the question of whether in the case of the concrete γ-duality, the MKZ property of $E^\gamma \to F^\gamma$ may be expressed equivalently by the strict singularity of this inclusion. The purpose of this paragraph is to settle this question in the positive.

First we have to recall the notion of strict singularity. A linear and continuous operator T: $X \to Y$ between Banach spaces X, Y is called strictly singular if the closed linear subspaces S of X for which $T|S$ is a homeomorphism with closed range are the finite dimensional ones (if there are any). In the special case of an inclusion operator i: $X \to Y$ with X, Y

BK-spaces, strict singularity of i means that every subspace S of X which is closed in Y must be finite dimensional.

Our investigation starts with the interrelation of the gliding humps property and the notion of strict singularity.

PROPOSITION 1. *Let X, Y be* BK-*spaces having $\Phi \subset X \subset Y$. Suppose Y has AB. If $X \to \omega$ has the gliding humps property and $X \to Y$ is strictly singular, then $X \to Y$ has the gliding humps property.*

Proof. Let $\zeta = (z^n)$ be a block sequence satisfying $\|z^n\|_X = 1$, $n \in \mathbf{N}$. Suppose we had $l^\infty(\zeta) \cap Y \subset X$. Then by the strict singularity of the inclusion, $l^\infty(\zeta) \cap Y$ would not be closed in Y since it is infinite dimensional.

We claim the existence of a sequence (n_k) of indices such that $\|z^{n_k}\|_Y \to o$ $(k \to \infty)$. For otherwise we would have $\|z^n\|_Y \geq \eta > o$ for all n. This, however, implies that $l^\infty(\zeta) \cap Y$ is closed in Y. Indeed, denoting by $\| \quad \|$ the norm on $l^\infty(\zeta) \cap Y$ and fixing $z = \sum \lambda_n z^n$ within, we have

$$|\lambda_n| \leq \eta^{-1}\|\lambda_n z^n\|_Y = \eta^{-1}\|P_{k_n} z - P_{k_{n-1}} z\|_Y$$
$$\leq 2\vartheta\eta^{-1}\|z\|_Y,$$

where ϑ is a constant such that $\|P_k x\|_Y \leq \vartheta\|x\|_Y$ for all $x \in Y$ and all k. This gives $\|z\|_\infty \leq 2\vartheta\eta^{-1}\|z\|_Y$, so the norms $\| \quad \|_Y$ and $\| \quad \|$ are equivalent on $l^\infty(\zeta) \cap Y$. This contradiction proves our claim.

We choose a subsequence (m_k) of (n_k) having $\sum_{k=1}^\infty \|z^{m_k}\|_Y < \infty$. Then clearly $l^\infty(\xi) \subset Y$, where $\xi = (z^{m_k})$. Applying the fact that $X \to \omega$ has the gliding humps property shows that, on the other hand, $l^\infty(\xi) \not\subset X$. So $X \to Y$ has the gliding humps property. □

Replacing the AB-condition on the space Y by the stronger fact that $Y = F^\gamma$ for a BK-space F, we can show that strict singularity of $X \to Y$ alone implies MKZ. Before proving this, we need the following result.

LEMMA 5. *Let F be a* BK-*space containing Φ, and let $Y = F^\gamma$. Then Φ is norming when considered as a subspace of Y', i.e.,*

$$\|y\|_Y = \sup\{|\langle x, y\rangle| : x \in \Phi,\ \|x\|_{Y'} = 1\}$$

is satisfied.

Proof. The proof of Lemma 1 shows that Y may be represented as the dual $E' = E^\gamma$ of a BK-AK-space E. Then Φ clearly is a norming subspace of $E'' = Y'$ when $Y = E'$ is endowed with the dual norm corresponding with a fixed norm on E. But $\| \quad \|_Y$ is equivalent with the mentioned dual norm, and this implies that Φ is also norming for $\| \quad \|_Y$. $\square$

PROPOSITION 2. *Let F be a BK-space containing Φ and let $Y = F^\gamma$. Let X be any BK-space having $\Phi \subset X \subset Y$. Suppose $X \to Y$ is strictly singular. Then it has the MKZ property.*

Proof. Let G be a BK-space satisfying $G \cap Y \subset X$. We have to prove that $G \cap Y$ is closed in G. Assume the contrary. Consequently, there exists a sequence (y^n) of vectors in $G \cap Y$ such that

$$\|y^n\|_Y = 1, \quad \|y^n\|_G \le 2^{-n}.$$

Since G is a K-space, the sequence (y^n) is certainly coordinatewise null, i.e., $y^n \to o$ in $\sigma(Y, \Phi)$.

Applying [6, Prop.] and Lemma 5, we deduce that there exists a subsequence (y^{n_k}) of (y^n) which is basic with respect to the norm $\| \quad \|_Y$ on Y.

Let L denote the closed linear span of (y^{n_k}) in Y. We prove $L \subset X$, and this contradicts the strict singularity of $X \to Y$, since L is clearly infinite dimensional.

Let $y = \sum \rho_k y^{n_k} \in L$, the series converging in norm $\| \quad \|_Y$. Then we find

$$|\rho_k| = \|\rho_k y^{n_k}\|_Y = \left\| \sum_{i=1}^{k} \rho_i y^{n_i} - \sum_{i=1}^{k-1} \rho_i y^{n_i} \right\|_Y \to o \quad (k \to \infty)$$

in view of the convergence of the series. So $(\rho_k) \in c_0$, and this proves $y \in G$, having regard of $\|y^{n_k}\|_G \le 2^{-n_k}$. Thus $y \in X$. $\square$

What about the reverse implication? We obtain a positive answer in the case where both, X and Y, are γ-spaces.

PROPOSITION 3. *Let E, F be* BK-*spaces containing Φ, and let $X = E^\gamma$, $Y = F^\gamma$ satisfy $X \subset Y$. Let E_0, F_0 be the unique* BK-AK-*spaces having $E_0' = E_0^\gamma = X$, $F_0' = F_0^\gamma = Y$. Then the following statements are equivalent*:

(1) *$X \to Y$ has the* MKZ *property*;
(2) *$X \to Y$ is strictly singular with respect to the norm topology*;
(3) *$X \to Y$ is strictly singular with respect to the weak star topology $\sigma(Y, F_0) = \sigma(F_0', F_0)$*;
(4) *$F_0 < E_0$*;
(5) *$F_0 \to E_0$ is strictly cosingular.*

Proof. Recall the proof of Lemma 1, which tells that E_0 is just the norm closure of Φ in $X^\gamma = E^{\gamma\gamma}$, and similarly for F_0. Clearly $F_0 \subset E_0$, and the inclusion $F_0 \to E_0$ is dense. But now Proposition 3 is a reformulation of [11, Theorem 4.7]. Indeed, the inclusion operator $X \to Y$ is just the restriction operator $E_0' \to F_0'$, since F_0 is dense in E_0. Also the abstract MKZ property for the restriction operator in the sense of [11] is the same as the concrete MKZ property for $X \to Y$. Finally observe that the result in [11] is applicable since F_0 is a separable space. □

Remark. Let us mention another consequence of the fact that γ-spaces may be represented as duals of BK-AK-spaces. Suppose X, Y are BK γ-spaces with X closed in Y. Then $X = Y$. Indeed, writing $X = E^\gamma = E'$, $Y = F^\gamma = F'$ for BK-AK-spaces E, F gives $F \subset E$. But the norms of E, F are equivalent, so $E = F$ since F is dense in E. This may as well be seen by writing X, Y as f-duals and applying [12, 8.6.1].

Corollary 3 (Compare [9, Theorem 2]). *Let E be a* BK-AK-*space, F a* BK-AD-*space, and suppose $F \subset E$. Let $E^\gamma \to F^\gamma$ have the* MKZ *property. Then $F < E$. The converse is true when F has* AK.

Proof. MKZ for $E^\gamma \to F^\gamma$ implies $F_0 < E_0$ by Proposition 3. But here we have $F \subset F_0 \subset E_0 = E$, and this immediately gives $F < E_0 = E$. So the first part of the statement follows. The second part of the statement is clear from Proposition 3. □

We conclude with the following useful combination of Corollary 1 and Proposition 3.

Theorem 2. *Let E be a* BK-AK-*space such that E_0' is separably complemented in E', and let F be any* BK-*space having $\Phi \subset F \subset E$. Then the following statements are equivalent*:

(1) *$F \to E$ is strictly cosingular*;

(2) *$E^\gamma \to F^\gamma$ is strictly singular.*

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