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CHARACTERIZATIONS WITHOUT CHARACTERS

BY

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1. Introduction

The current program of revision of the classification of the finite simple groups has led us to approach old characterization problems via new avenues. The direction of the new approaches has made it natural to look for new ways to complete the characterizations, more consistent with the new avenues of approach.

For example, the group A_7 was characterized by Michio Suzuki in 1959 in a paper [10] which made extensive and detailed use of character theory. This characterization was invoked by Gorenstein and Walter to complete their "Dihedral Paper" [9] and, again, by Bender in his revision of the Gorenstein-Walter theorem [5]. However Bender invokes Suzuki at a point where both the centralizer of an involution and the group order are known. At this point a far more elementary and character-free argument is available and this is provided in Sections 3 and 5 of this paper. In thinking about this we realized that combining some of Bender's arguments with our own affords an almost character-free proof of the following case of the Dihedral Theorem.

THEOREM 1.1. *Let G be a finite simple group with a dihedral Sylow 2-subgroup. Suppose that the centralizer H of an involution of G is* **properly** *contained in a subgroup $\tilde{H}$ of G with $F^*(\tilde{H}) = F(\tilde{H})$. Then G is isomorphic to either A_5, $PSL(3,2)$, $PSL(2,9)$ or A_7.*

The lion's share of the proof of (1.1) is in Bender [5]. After some preliminaries in Sections 2, 3, 4 and 5, we outline Bender's reductions (with some improvements) and the completion of the proof of (1.1) in Section 6.

The remainder of the paper is devoted to a character-free proof of Brauer's well-known result [7]:

THEOREM 1.2. *Let G be a finite simple group with an involution $t \in G$ such that $H = C_G(t) \cong GL(2,3)$. Then G is isomorphic either to M_{11} or to $PSL(3,3)$.*

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This result is invoked in the classification of groups with a semi-dihedral or wreathed Sylow 2-subgroup. It is interesting to note that, just as in the dihedral case, if the centralizer of an involution is not maximal in such a group G, then it may be proven without character theory that G is isomorphic to $PSL(3, q)$ for some odd q. The proof is lengthy and will appear in Part III of the revision work of Gorenstein, Lyons and Solomon. If the centralizer is maximal, then character theory seems to be essential to pin down the structure of the core of the centralizer H, even in the case $H/O(H) \cong GL(2,3)$.

2. Counting arguments

We shall frequently have occasion to count involutions in cosets of a subgroup M. These methods probably go back to Burnside. They are well presented by Bender in [4].

Notation. (1) For M a subgroup of G, let $\mathscr{M}_n = \mathscr{M}_n(M)$ denote the set of all right cosets of M distinct from M and containing exactly n involutions. Let

(2) $$b_n = b_n(M) = |\mathscr{M}_n(M)|.$$

The philosophy is that frequently we can determine b_n for all $n \geq 2$. If G has one class of involutions and if $|M| > |H|$ where H is the centralizer of an involution t in G, then we might expect each coset of M to contain at least two involutions. Comparing $|G:M|$ with $|G:H| = |t^G|$ often confirms this and permits us to determine $|G|$. The following remark is useful.

LEMMA 2.1. *Let M be a subgroup of G. Then M acts on $\mathscr{M}_n(M)$ by right multiplication.*

Proof. The map $x \mapsto m^{-1}xm$ defines a bijection between the involutions of Mt and those of Mtm for any $t \in G$ and $m \in M$.

3. A Construction for $PSL(2,9)$

PROPOSITION 3.1. *Let H be a group generated by subgroups B and $\langle \tau \rangle$ satisfying:*

(1) $B = N_H(P) = P\langle u \rangle = \langle c, c^u \rangle \langle u \rangle$

where

(2) $P \cong \mathbf{Z}_3 \times \mathbf{Z}_3$, $\langle u \rangle \cong \mathbf{Z}_4$ *and* u^2 *inverts* P;

(3) τ *is an involution of* $H - B$ *such that*

$(*)$ $u^\tau = u^{-1}$,

$(**)$ $(\tau c)^3 = 1$,

$(***)$ $(u\tau^c)^3 = 1$.

Then $H \cong PSL(2,9)$.

Proof. Since $u \in B \cap B^\tau$ and $P \neq P^\tau$, we must have $B \cap B^\tau = \langle u \rangle$. Thus

$$|B\tau B| = |B|\,|B : B \cap B^\tau| = 9|B|.$$

Set $H_0 = B \cup B\tau B$. Then

$$|H_0| = 10|B| = 2^3 \cdot 3^2 \cdot 5 = |PSL(2,9)|.$$

We wish to show that H_0 is a subgroup of H (and hence equal to H). For this it suffices to show that

$$\tau b \tau \in B \cup B\tau B \quad \text{for all } b \in B.$$

By $(*)$, we may assume $b \in P^\#$. As each element of $P^\#$ is $\langle u \rangle$-conjugate either to c or to cc^u, it remains to show:
(a) $\tau c \tau \in B\tau B$, and
(b) $\tau cc^u \tau \in B\tau B$.
By $(**)$,

$$\tau c \tau = c^{-1}\tau c^{-1} \in B\tau B.$$

So (a) holds. For (b), set $t = u^2$ and note that

$$\tau cu^{-1}cu\tau = \tau cutcu\tau = \tau cuc^{-1}tu\tau = \tau cuc^{-1}u^{-1}\tau = \tau cuc^{-1}\tau u.$$

From $(***)$,

$$(uc^{-1}\tau c)(uc^{-1}\tau c) = c^{-1}\tau cu^{-1}.$$

So

$$(uc^{-1})(\tau cuc^{-1}\tau)c = c^{-1}\tau cu^{-1}.$$

and

$$\tau cuc^{-1}\tau = (cu^{-1}c^{-1})\tau(cu^{-1}c^{-1}) \in B\tau B.$$

Thus $\tau cuc^{-1}\tau u \in B\tau B$, proving (b).

It follows that $H_0 = H$ is a group of order 360 with a uniquely determined multiplication table. We conclude that $H \cong PSL(2,9)$.

4. A characterization of $PSL(3,2)$ and $PSL(2,9)$

In [4], Bender provides an elementary counting argument which establishes the following result.

LEMMA 4.1. *Let G be a finite simple group with an involution t such that $H = C_G(t) \cong D_8$. Then $H \subseteq \tilde{H} \subseteq G$ with $\tilde{H} \cong S_4$ and either*

(1) $|G| = 2^3 \cdot 3 \cdot 7$, *or*

(2) $|G| = 2^3 \cdot 3^2 \cdot 5$ *and* $B = \langle c, c^u\rangle\langle u\rangle \subseteq G$ *with* $c \in \tilde{H}$, $\langle c, c^u\rangle \cong \mathbf{Z}_3 \times \mathbf{Z}_3$, $\langle u\rangle \cong \mathbf{Z}_4$ *and* $u^2 = s \in H$ *inverting* $\langle c, c^u\rangle$.

We now show the following result.

PROPOSITION 4.2. *Let G be a finite simple group with an involution t such that $H = C_G(t) \cong D_8$. Then either $G \cong PSL(3,2)$ or $G \cong PSL(2,9)$.*

LEMMA 4.3. *In case* (1) *of* (4.1), $G \cong PSL(3,2)$.

Proof. As G has two conjugacy classes of 4-groups, we may define a geometry Γ with points $\mathscr{P} = U^G$ and lines $\mathscr{L} = V^G$ where U and V are representatives of these classes. A point U^g is incident with a line V^{g_1} if $U^g V^{g_1}$ is a 2-group. Now $|\mathscr{P}| = 7 = |\mathscr{L}|$. Each point (line) is incident with exactly 3 lines (points).

As $\tilde{H}$ acts on its 7 right cosets with orbits of length $|\tilde{H} : \tilde{H} \cap \tilde{H}^g|$ and as $O_2(\tilde{H}) \not\subseteq \tilde{H}^g$ for any $g \notin \tilde{H}$, we easily see that $\tilde{H}$ has only one non-trivial orbit and so $|\tilde{H} \cap \tilde{H}^g| = 4$ for all $g \in G - \tilde{H}$. If $\tilde{H} \cap \tilde{H}^g$ is cyclic, then $N_G(\tilde{H} \cap \tilde{H}^g) \subseteq \tilde{H} \cap \tilde{H}^g$, which is not the case. Hence $\tilde{H} \cap \tilde{H}^g \in V^G$. Geometrically this says that any two points lie on one and only one line.

Hence Γ is a projective plane of order 2. It is easy to see that Γ is unique and

$$G \subseteq \operatorname{Aut}\Gamma = PSL(3,2).$$

(For example, see [1, 2.26].) Hence $G \cong PSL(3,2)$.

LEMMA 4.4. *In case* (2) *of* (4.1), $G \cong PSL(2,9)$.

Proof. As $[s,t] = 1$, t inverts u and $(tc)^3 = 1$. As $t^c \in O_2(\tilde{H})$, we see that both u and t^c normalize $\langle s,t\rangle$ and so lie in $N_G(\langle s,t\rangle) \cong S_4$. It follows easily that $(ut^c)^3 = 1$. Thus by (3.1), $G \cong PSL(2,9)$.

This completes the proof of Proposition 4.2.

5. A characterization of A_7

PROPOSITION 5.1. *Let G be a finite simple group with a dihedral Sylow 2-subgroup. Let t be an involution in G and $H = C_G(t)$. Suppose H satisfies the following*:

(a) $O(H) \simeq Z_3$ *and* $H/O(H) \simeq D_8$.

(b) $H \subseteq \tilde{H}$, *a maximal subgroup of G with* $O(\tilde{H}) = O(H)$ *and* $\tilde{H}/O(\tilde{H}) \cong S_4$ *and* $C_{\tilde{H}}(O(\tilde{H})) = O^2(\tilde{H})$.

(c) $|G| = 2^3 \cdot 3^2 \cdot 5 \cdot 7$.

Then $G \cong A_7$.

Notation. $W = O_2(\tilde{H})$. V is a 4-subgroup of $\tilde{H}$ with $W \neq V$ and $t \in V$.

We proceed via a series of lemmas.

LEMMA 5.2. *$N_G(V) = \langle t, c, \tau \rangle \cong S_4$ with $V = \langle t, t^c \rangle$, $[t, \tau] = 1$ and*

$$\left\langle c, \tau | c^3 = \tau^2 = (c\tau)^2 = 1 \right\rangle \cong S_3.$$

Proof. As $C_G(V) \subseteq C_G(t)$, we have $C_G(V) = V$ and $V \notin W^G$. Let $V = \langle t, t_1 \rangle$. Then as $t_1 \in t^G$, extremal conjugation implies that there exists $c \in G$ with $t_1^c = t$ and

$$V^c \subseteq \langle W, V \rangle \in Syl_2(H).$$

Thus $V^c = V$. As t_1 was arbitrary, we conclude that $N_G(V)$ is transitive on $V^\#$ and so

$$N_G(V) = \langle t, c, \tau \rangle \cong S_4.$$

LEMMA 5.3. *Let $c \in P \in Syl_3(G)$. Then $N_G(\langle c \rangle) = P\langle \tau \rangle$ and $N_G(P) = P\langle u \rangle$ with $u^2 = \tau$, $P \cong \mathbf{Z}_3 \times \mathbf{Z}_3$, $C_P(\tau) = 1$.*

Proof. As $\tilde{H}$ is maximal in G, we have $\tilde{H} = N_G(O(\tilde{H}))$ and $O^2(\tilde{H}) = C_G(O(\tilde{H}))$. If $P_1 \in Syl_3(\tilde{H})$, it follows that $C_G(P_1) = P_1$ and $|N_{\tilde{H}}(P_1)| = 18$. By Sylow we conclude that $|N_G(P)| = 36$. As $C_G(P) = P$, it follows that $4 \mid |\mathrm{Aut}\, P|$ and so $P \cong \mathbf{Z}_3 \times \mathbf{Z}_3$.

Suppose $|C_G(c)|$ is even. Since τ inverts c, we see τ centralizes some involution u in $C_G(c)$. Now $u \in C_G(\tau) \subseteq \tilde{H} = N_G(W)$. But $\langle W, c \rangle = \langle \tau, t, c \rangle = N_G(V)$ and so $u \in N_G(N_G(V)) = N_G(V)$, a contradiction. So $|C_G(c)|$ is odd. Now P is inverted by an involution s in $N_G(\langle c \rangle)$ and it follows that s inverts $C_G(c)$. Then $C_G(c) \subseteq C_G(P) = P$ and $N_G(\langle c \rangle) = P\langle \tau \rangle$.

If $x \in P^\#$ commutes with an involution, then $x^G \cap O(\tilde{H}) \neq \phi$. However for $x_1 \in O(\tilde{H})^\#$, x_1 is not centralized by any involution in $N_{\tilde{H}}(P_1) = N_G(\langle x_1 \rangle) \cap N_G(P_1)$. It follows that $N_G(P) = P\langle u \rangle$ with $u^2 = \tau$, $C_P(\tau) = 1$.

LEMMA 5.4. *The following relations hold*:

$(*)$ $u^t = u^{-1}$,

$(**)$ $(tc)^3 = 1$,

$(***)$ $(ut^c)^3 = 1$.

Proof. We may assume $W = \langle t, \tau \rangle \leq \tilde{H}$. As $C_G(\tau) \subseteq \tilde{H}$, we have $u \in \tilde{H}$. As $[\tau, u] = 1$, we have $t^u = \tau t$, whence $u^t = u\tau = u^{-1}$, proving $(*)$.

As $C_G(t) \subseteq \tilde{H}$, we have $t^c \in \tilde{H}$. Since $\langle t, \tau, u \rangle$ and $\langle \tau, t, t^c \rangle$ are distinct Sylow 2-subgroups of $\tilde{H}$, we have $(ut^c)^3 \in \langle t, \tau \rangle$.

Suppose ut^c has order 6. Then by the structure of $\tilde{H}$, we have $ut^c \in C_G(W)$. In particular, $[\tau, ut^c] = 1$. Then $[\tau, t^c] = 1$, contrary to the structure of $N_G(V)$. Thus $(ut^c)^3 = 1$, proving $(***)$.

Finally as $O^2(N_G(V)) = \langle t, c \rangle \cong A_4$, it is clear that $(tc)^3 = 1$, proving $(**)$.

DEFINITION. Let $B = N_G(P)$ and $G_0 = B \cup BtB$.

LEMMA 5.5. $G_0 \cong A_6$.

Proof. This is immediate from Proposition 3.1 and Lemmas 5.3 and 5.4.

Proof of Proposition 5.1. As G is simple and has a subgroup of index 7, G is isomorphic to a subgroup of S_7 of order $2^3 \cdot 3^2 \cdot 5 \cdot 7$. Thus $G \cong A_7$.

6. Bender's reductions

In this section we complete the proof of Theorem 1.1.

THEOREM 1.1. *Let G be a finite simple group with a dihedral Sylow 2-subgroup. Suppose that the centralizer H of an involution t in G is properly contained in a subgroup $\tilde{H}$ of G with $F^*(\tilde{H}) = F(\tilde{H})$. Then G is isomorphic to either A_5, $PSL(3,2)$, $PSL(2,9)$ or A_7.*

We proceed by induction to reach the hypotheses of Proposition 4.2 or 5.1. Our reductions are a simplified version of Bender's arguments in [3] and [5]. We shall not repeat all of Bender's arguments but rather we shall indicate how to extract from Bender's work an *almost* character-free proof of Theorem 1.1.

Let G be a minimal counterexample to Theorem 1.1 and let $\tilde{H}$ be maximal subject to $H \subseteq \tilde{H}$ and $F^*(\tilde{H}) = F(\tilde{H})$. We proceed via a sequence of lemmas.

LEMMA 6.1. *Suppose $\tilde{H} \subseteq M$, a maximal subgroup of G, with either*
(a) *$M = N_G(S)$ where $S \in Syl_2(G)$, $|S| = 4$, and $|M : C_G(S)| = 3$, or*
(b) *$M = O(M)E(M)$ with $[O(M), E(M)] = 1$ and $E(M)/Z(E(M)) \cong A_5$, $PSL(3,2)$, $PSL(2,9)$ or A_7.*

Then $G \cong A_5$.

Proof. In either case, M is a strongly embedded subgroup of G and the character-free proof in Bender [3] shows that case (a) holds and

(i) $|G| = 12u(4u+1)$, where $C_G(S) = U \times S$ and $|U| = u$;

(ii) G is 2-transitive on the cosets of M; and

(iii) $|G|_3 = 3$ and if $\langle w \rangle \in Syl_3(G)$, then $N_G(\langle w \rangle) \cong S_3$.

Now a bit of character theory completes the proof. Let χ be the non-principal irreducible constituent of 1_M^G. As $|C_G(w)| = 3$, the orthogonality relations [8, (2.14)] yield an irreducible character, ψ, of G such that a portion of the character table of G is

	1	t^G	w^G
1_G	1	1	1
χ	$4u$	0	1
ψ	$4u+1$	1	-1

Let

$$\mathscr{A} = \{(t_1, w_1) \in t^G \times w^G : t_1 w_1 = w\}$$

and let $a = |\mathscr{A}|$. Then by [8, (2.15)], we have

$$a = \frac{12u(4u+1)}{4u \cdot 3}\left(1 + \frac{1 \cdot (-1)^2}{4u+1}\right) = 4u + 2.$$

On the other hand, if $(t_1, w_1) \in \mathscr{A}$, then $\langle t_1, w_1 \rangle \cong A_4$ and $w \in \langle t_1, w_1 \rangle$. By Sylow, $N_G(\langle w \rangle)$ transitively permutes the subgroups A of G with $w \in A$ and $A \cong A_4$. Hence there are only two such subgroups and an easy count yields $a = 6$. Thus $u = 1$ and $|G| = 60$, whence $G \cong A_5$ as claimed.

LEMMA 6.2. (a) *$\tilde{H}$ is a maximal subgroup of G.*

(b) $O(H) = O(\tilde{H})$.

(c) $O_2(\tilde{H}) = V \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

(d) $\tilde{H}/O(\tilde{H}) \cong S_4$.

Proof. Suppose that $\tilde{H}$ is properly contained in M, a maximal subgroup of G. By the maximal choice of $\tilde{H}$, we have $E = E(M) \neq 1$. As E is a quasi-simple group with a dihedral Sylow 2-subgroup, E has one class of involutions, t^E. Thus $M = EH$ and so $H \cap E$ is properly contained in $\tilde{H} \cap E$. Moreover, as $\tilde{H} \cap E \trianglelefteq \tilde{H}$, we have that $E(\tilde{H} \cap E) = 1$. Thus by induction, $E/Z(E) \cong A_5$, $PSL(3,2)$, $PSL(2,9)$ or A_7. Moreover $O_2(\tilde{H} \cap E) \neq 1$ and so $O_2(\tilde{H}) \neq 1$. Thus $|G|_2 \leq 8$. If E contains a Sylow 2-subgroup of G, then $M = O(M)E(M)$ with $[O(M), E(M)] = 1$ and (6.1) yields a contradiction. Otherwise $E \cong A_5$ and $S \cong D_8$ with $S \in Syl_2(H)$. In this case the second paragraph of Bender's proof of Theorem 2.6 in [5] yields a contradiction, proving (a).

If $O_2(\tilde{H}) \neq 1$, then as $\tilde{H} \neq H$, we must have $O_2(\tilde{H}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\tilde{H}/C_{\tilde{H}}(O_2(\tilde{H})) \cong A_3$ or S_3. In the former case, $G \cong A_5$ by (6.1). In the latter case, we are done.

Thus we may assume that $\tilde{H}$ is maximal in G and that $F^*(\tilde{H}) = O(F(\tilde{H}))$. But then the remainder of the proof of (2.6) in [5] yields a final contradiction.

If $O(H) = 1$, then Proposition 4.2 yields $G \simeq PSL(3,2)$ or $PSL(2,9)$ and henceforth we may assume that $O(H) \neq 1$.

Notation. (1) $U = O(H)$.

(2) For s an involution of H, $I_U(s)$ denotes the set of elements of U inverted by s.

(3) S is a Sylow 2-subgroup of H.

LEMMA 6.3. *Suppose that M is a maximal subgroup of G with $N_G(X) \subseteq M$ for some $X \subseteq F(U)$. Suppose that $S \subseteq M \neq \tilde{H}$. Then either*

(1) *$|[M,t]|$ is relatively prime to $|F(\tilde{H})|$ and $[S,U] \nsubseteq F(U)$ or*

(2) *$t \in E(M)$.*

In particular, if $I_U(s)$ is a Hall subgroup of $F(U)$ for each involution $s \in S$, then $t \in E(M)$.

Proof. The first part is Bender's Lemma 2.7 in [5]. If $I_U(s)$ is a Hall subgroup of $F(U)$ for each involution s in S, then $[S,U] \subseteq F(U)$ and so (1) does not hold. Thus $t \in E(M)$ in this case.

PROPOSITION 6.4. *One of the following holds:*

(1) *For each involution $s \in H$, $I_U(s)$ is a Hall subgroup of $F(U)$; and for every subgroup $X \neq 1$ of $F(U)$, $N_G(X) \subseteq \tilde{H}$.*

(2) *G has a maximal subgroup M with $t \in E(M)$ and with $N_G(X) \subseteq M$ for some $X \subseteq F(U)$.*

Proof. This is Theorem 2.10 of [5] and follows from (6.3).

In Bender's Section 3 of [5], case (1) of (6.4) is treated. As we have $O_2(\tilde{H}) \cong Z_2 \times Z_2$, we only require the elementary counting argument which begins with the sentence "Thus $V = O_2(\tilde{H})$ is of type (2,2)." The conclusion he reaches corresponds to the hypotheses of Proposition 5.1. Thus $G \simeq A_7$ in this case.

Thus it remains to derive a contradiction in case (2) of (6.4).

LEMMA 6.5. *Let M be a maximal subgroup of G with $O_2(H) \subseteq M$ and with $t \in E(M)$. Then $O_2(H) \subseteq E(M)$.*

Proof. Suppose not. Let $V = \langle t, v\rangle = O_2(\tilde{H})$ and choose M with $V \not\subseteq E(M)$. Then M has exactly 2 classes of involutions with representatives t and v. Let

$$X = \{(x_1, x_2) | x_1, x_2 \text{ are } M\text{-conjugates of } t \text{ and } v \text{ respectively}\}.$$

If $(x_1, x_2) \in X$ then $(x_1x_2)^i = \rho$ for some positive integer i, where ρ is a conjugate of t or v. Let $a(\rho)$ be the number of elements (x_1, x_2) in X with $(x_1x_2)^i = \rho$. Then

$$|X| = |M : C_M(t)|a(t) + |M : C_M(v)|a(v).$$

Let $C_M(t) = (\langle t, v\rangle \times W)\langle s\rangle$, where $\langle t, v\rangle\langle s\rangle \cong D_8$ and $W \subseteq U$. Let $|W| = w$ and $k = |W : C_W(s)|$. Since v and vt are the only conjugates of v in $C_M(t)$ and all involutions in $C_M(t)$ not in $\langle v, t\rangle$ are M-conjugates of t we have $a(t) = 4k$. As $[V, U] = 1$ and as $v \in t^G$, we have that $U = O(C_G(v))$. Hence $C_M(v) = \langle t, v\rangle \times W$ and so $a(v) = 1$. Thus

$$|X| = \frac{|M|^2}{|C_M(t)||C_M(v)|} = \frac{|M|}{|C_M(t)|}4k + \frac{|M|}{|C_M(v)|}.$$

We obtain $|M| = 8w(1 + 2k)$.

Now let $Y = \{(y_1, y_2)|y_1, y_2 \text{ are } M\text{-conjugates of } t\}$. If s_1 is an involution in $C_M(t)$ with $s_1 \notin \langle t, v\rangle$ then $(s_1, s_1tw) \in Y$ for all $w \in W$ with $s_1ws_1 = w^{-1}$. The number of such pairs is $2k^2$. Of course $(t, t) \in Y$. Hence

$$|Y| \geq \frac{|M|}{|C_M(t)|} + \frac{|M|}{|C_M(t)|}2k^2.$$

We obtain

$$\frac{|M|^2}{|C_M(t)|^2} \geq \frac{|M|}{|C_M(t)|}(1 + 2k^2),$$

or

$$|M| \geq |C_M(t)|(1 + 2k^2).$$

Since $|M| = 8w(1 + 2k)$ and $|(C_M(t)| = 8w$, we get $k \leq 1$, which contradicts the simplicity of $E/Z(E)$.

LEMMA 6.6. *Case* (2) *of* (6.4) *does not occur.*

Proof. Choose a maximal subgroup M of G with $t \in E(M)$ and with $N_G(X) \subseteq M$ for some $X \subseteq F(U)$. Since $[U, V] = 1$, we have that $V \subseteq M$ and so, by (6.5), $V \subseteq E = E(M)$. Since $\bar{E} = E/Z(E)$ is a simple group with a dihedral Sylow 2-subgroup, $\bar{E}$ has one class of involutions. Hence

$$N_{\bar{E}}(\bar{V})/C_E(\bar{V}) \cong S_4.$$

Since $[\bar{V}, O(C_{\bar{E}}(\bar{t})] = 1$, $C_{\bar{E}}(\bar{t})$ is not a maximal subgroup of $\bar{E}$. Since G is a minimal counterexample, we have $\bar{E} \cong A_5$, $PSL(2,9)$, $PSL(3,2)$ or A_7. Also $M = N_G(E)$.

Let $F = F(U) \cap O(M)$. As E has one class of involutions, we have $M = E(H \cap M)$ and so $F \trianglelefteq M$. Thus $F \neq F(U) \cap M$ and so $E/Z(E) \cong A_7$.

If $D \neq 1$ and $D \subseteq F$, then by (6.3), we have $E \subseteq E(N_G(D))$ and so $E = E(N_G(D))$. If $h \in \tilde{H}$ with $D = F \cap F^h \neq 1$, then $E = E^{h^{-1}}$ and so $h \in M$. But as $\tilde{H} \not\subseteq M$, we must have

$$N_{\tilde{H}}(F(U) \cap M) \not\subseteq M.$$

Choose $h \in N_{\tilde{H}}(F(U) \cap M)$ with $h \notin M$. Then $F \cap F^h = 1$ and so $F \cong F(U) \cap M/F \cong Z_3$. As $F_{3'}(U) \subseteq C_G(F) \subseteq M$, we have $F(U)$ is a 3-group.

Let $K = [S, U \cap M]$. Then $K \cong Z_3$ and $F \times K = F(U) \cap M$. Let $s \in S$ of order 2 with $[K, s] = K$. Then $F = C_{F(U)}(s) \cap M$. As $N_G(F) \subseteq M$, this forces $F = C_{F(U)}(s)$.

If $F(U) \subseteq M$, then $F(U) = F \times K = U$ and so $\tilde{H} \subseteq M$, which is not the case. As $Z(F(U)) \subseteq F \times K$ and as K is $\langle s \rangle$-invariant; this forces $E \cong A_7$. Now $K = O(\tilde{H} \cap E)$ and there exists $e \in N_E(KK^e)$ with $K^e \subseteq \tilde{H}$ but $K^e \not\subseteq O(\tilde{H})$.

We turn to the structure of the 3-group $P = F(U)$. Since $F(U) \cap M = F \times K$, we have $N_P(F) = C_P(F) = F \times K$. Let R be a normal subgroup of P with $|P/R| \geq 9$. If $F \subseteq R$, then $|F^P| < |R|$ and so $|N_P(F)| > |P:R| \geq 9$, contrary to $|N_P(F)| = 9$. So it follows that $F \not\subseteq R$. In particular $F \not\subseteq \Phi(P)$. Let $\bar{P} = P/\Phi(P)$. Then $\bar{P} = C_{\bar{P}}(s) \times [\bar{P}, s]$ and so s inverts a normal abelian complement W to F in P. Moreover either W is characteristic in F or P is extraspecial of order 3^3, since $|A| \leq 9$ for any abelian subgroup of P not contained in W.

If $W \trianglelefteq \tilde{H}$, then as $K^e \subseteq \tilde{H}$ and s inverts K^e, we see that $[W, K^e] = 1 = [F, K^e]$ and so $P \subseteq \tilde{H}^e = N_G(K^e)$. As $[K, V^e] \neq 1$, we have $K \not\subseteq O(\tilde{H}^e)$. But $K \subseteq P' \subseteq O(\tilde{H}^e)$, a contradiction. Thus $W \ntrianglelefteq \tilde{H}$ and $|P| = 3^3$.

As $P = F(U)$, it follows that $P = U$. As $PK^e \in Syl_3(\tilde{H})$ and as $K = Z(PK^e) \trianglelefteq \tilde{H}$, we have $PK^e \in Syl_3(G)$. Let $V_0 = F \times K \times K^e$ and consider $N = N_G(V_0)$. Then $V_0 = C_G(V_0)$ and as F is not G-conjugate to K, we infer from $|GL(3,3)|$ that $|N/V_0|$ divides 24. Now $N_M(V_0)/V_0 \cong S_3$ and, as $e \in N$, we have $K \ntrianglelefteq N$ and so $PK^e \ntrianglelefteq N$. Thus $N/V_0 \cong S_4$. Now e^2 inverts KK^e. Thus $e^2 \in N_G(K) = \tilde{H}$ and so $e^2 \in N_G(PK^e)$. But $e^2 \in O^2(N)$ and $N_{O^2(N)}(PK^e) = PK^e$, a final contradiction.

7. A characterization of M_{11} and $PSL(3,3)$

In this section, we begin the proof of Theorem 1.2. For the remainder of the paper, G is a finite simple group, t is an involution in G and $H = C_G(t)$ is isomorphic to $GL(2,3)$.

We proceed via a sequence of lemmas.

LEMMA 7.1. *G has one class of involutions. If E is a 4-subgroup of G, then $N_G(E) \cong S_4$.*

Proof. H has one class of 4-groups and one class of non-central involutions. The result follows from the Thompson transfer lemma (See [2, 37.4]) and extremal conjugation (See [2, p. 207, exercise 1]).

LEMMA 7.2. *Suppose that $c \in H$ of order 3 and $c \in B \subseteq G$ with $B \cong S_4$ and with $N_B(\langle c\rangle) \subseteq H$. Then $\langle B, t\rangle \cong S_5$.*

Proof. We wish to show that $\langle B, t\rangle = B \cup BtB$. Let $N_B(\langle c\rangle) = \langle c, t_1\rangle$ and let $\langle \tau\rangle = C_{O_2(B)}(t_1)$. Then $\langle \tau, t_1, t\rangle \subseteq C_G(t_1)$ and so either $(\tau t)^2 \in \langle t_1\rangle$ or $(\tau t)^3 \in \langle t_1\rangle$. In the former case $t\tau t \in \tau\langle t_1\rangle \subseteq B$. In the latter case $t\tau t \in \tau\langle t_1\rangle t\tau \subseteq BtB$. In any case, this and $[\langle c, t_1\rangle, t] = 1$ shows that $B \cup BtB$ is a group. As $t \notin N_G(B)$, we have $B \cap B^t = \langle c, t_1\rangle$ and so

$$|B \cup BtB| = |B| + |B|\,|B : B \cap B^t| = 5|B| = 120.$$

It follows trivially that $\langle B, t\rangle \cong S_5$.

For the next two lemmas we assume that $|G|_3 = 3$. Let $c \in H$ of order 3 with

$$N_H(\langle c\rangle) = \langle c, t, t_1\rangle, \quad t_1^2 = 1.$$

LEMMA 7.3. (a) $N_G(\langle c\rangle) = N_H(\langle c\rangle) \cong D_{12}$.
(b) *There exists $S \subseteq G$ with $S \cong S_5$.*

Proof. As $N_H(\langle c\rangle) \cong D_{12}$, we have $N_G(\langle c\rangle) = O(N_G(\langle c\rangle))\langle t, t_1\rangle$. As $|G|_3 = 3$, we have $O(N_G(\langle c\rangle)) = \langle c\rangle$, proving (a). Now (b) follows from Lemmas 7.1 and 7.2.

LEMMA 7.4. *Let $y \in S$ of order 5. Then $C_G(y) = \langle y\rangle$.*

Proof. Let A be a $\{2,3\}'$-subgroup of G containing y and maximal subject to being inverted by an involution τ. If X is a non-identity subgroup of A, then $X \le A\langle \tau\rangle$ and so by maximality, $A = C_G(X)$. Let $M = N_G(A)$.

As $A = C_G(A)$ and τ inverts A, we have $M = AN_{C_G(\tau)}(A)$. As $A = C_G(y)$ and $N_S(\langle y \rangle)$ contains a $\mathbf{Z}_4$, we conclude that $N_{C_G(\tau)}(A)$ is isomorphic to one of $\mathbf{Z}_4, \mathbf{Q}_8, \mathbf{Z}_8$ or $SL(2,3)$.

Let $|A| = k$. As M contains $C_G^*(x)$ for all $x \in O(M)^\#$, no involution outside M inverts any $x \in O(M)^\#$. Thus if σ is an involution of $G - M$ and if $M\sigma$ contains more than one involution, then σ centralizes exactly one M-conjugate of τ, of which there are k. Thus

$$|G:H| = 13k + b_1 \quad \text{in all cases.}$$

We easily compute

$$\begin{array}{rlll} \text{Case } \mathbf{Z}_4 \text{ or } \mathbf{Z}_8. & b_4 = k, & b_2 = 4k, & b_n = 0, \text{ for } n = 3 \text{ or } n \geq 5. \\ \text{Case } \mathbf{Q}_8. & b_4 = 3k, & & b_n = 0, \text{ for } n \geq 2,\ n \neq 4. \\ \text{Case } SL(2,3). & b_{12} = k, & & b_n = 0, \text{ for } n \geq 2,\ n \neq 12. \end{array}$$

Now we use the equations

$$|G:H| = 13k + b_1,$$

$$|G:M| = 1 + \sum_{n \geq 0} b_n,$$

$$|G:H| = \frac{k}{r}|G:M|$$

where $r = |C_G(\tau) : N_{C_G(\tau)}(A)|$ to get

$$13k + b_1 = \frac{k}{r}[1 + b_0 + b_1 + sk]$$

with $s = 5$, 3 or 1 depending on the case. Thus

$$13k - \frac{k}{r} - \frac{s}{r}k^2 = \frac{k}{r}b_0 + \left(\frac{k}{r} - 1\right)b_1.$$

If $k \geq 12$, then the right hand side is non-negative and as $s/r \geq \frac{5}{12}$, we get $13k - \frac{5}{12}k^2 > 0$ or $k \leq 31$.

As $k \equiv 0 \pmod 5$ and $k \equiv 1 \pmod 4$, we get $k = 5$ or 25. If $k = 5$, we are done; so assume $k = 25$. The only cases are:

$$\mathbf{Z}_4 \text{ Case.} \quad 30 \cdot 25 = 25b_0 + 13b_1,$$

$$\mathbf{Q}_8 \text{ Case.} \quad 2 \cdot 25 = 25b_0 + 19b_1.$$

By Lemma 2.1, M acts on $\mathscr{M}_i$ of cardinality b_i and if $Mg \in \mathscr{M}_i$, the M-orbit of Mg has length

$$|M : M \cap M^g|.$$

Now if $a \in A^\# \cap M^g$, then $A^g \subseteq C_G(a) = A$ and so $g \in M$. Thus

$$|M : M \cap M^g| \equiv 0(\bmod 25) \quad \text{for all } g \in G - M.$$

Thus $b_0 \equiv b_1 \equiv 0(\bmod 25)$. This easily rules out both cases.

LEMMA 7.5. $|G|_3 > 3$.

Proof. If not, then by Lemmas 7.3 and 7.4, there exists $S \subseteq G$ with $S \cong S_5$ and with $C_G^*(x) \subseteq S$ for all $x \in S$ of order 3, 5 or 6. It follows easily that if $g \in G - S$ and g inverts some elements of $S^\#$, then g centralizes one and only one involution of S. Also we see that if g inverts $w \in S$ of order 4, then $g \in S$. So $b_n = 0$ for $n \geq 3$. Also $b_2 = 4 \times 25 = 100$. Thus

$$|G : S| = 1 + b_0 + b_1 + 100,$$
$$|G : H| = 25 + b_1 + 200.$$

As $|G : H| = \frac{120}{48}|G : S|$ we obtain

$$b_1 + 225 = \tfrac{5}{2}(b_0 + b_1 + 101),$$

an obvious contradiction.

LEMMA 7.6. *Let $c \in H$ of order 3, with normalizer $\langle c, t, t_1 \rangle$. A Sylow 3-subgroup of $C_G(c)$ is either elementary of order 9 or extraspecial of order 27 and exponent 3.*

Proof. Let $K = C_G(c)$. As $\langle t, t_1 \rangle$ normalizes $O(K)$, we see that $O(K)$ is a 3-group of order 9 or 27 and $K = O(K)\langle t \rangle$. Moreover $O(K)$ has exponent 3, as it is so generated and has class at most 2.

Suppose $O(K)$ is elementary of order 27. Let $N = N_G(O(K))$, $\overline{N} = N/O(K)$. As $C_{\overline{N}}(\bar{\tau}) = \langle \bar{t}, \bar{t}_1 \rangle$ for all $\bar{\tau} \in \langle \bar{t}, \bar{t}_1 \rangle^\#$, we see that $\overline{N}$ is isomorphic to a subgroup of A_5. As $5 \mid |GL(3,3)|$, either $\overline{N} = \langle \bar{t}, \bar{t}_1 \rangle$ or $\overline{N} \cong A_4$. The former possibility violates Burnside's Lemma. Thus $\overline{N} \cong A_4$ and, as $O(K)$ is characteristic in a Sylow 3-subgroup of N, we have that N contains a Sylow 3-subgroup of G. Moreover N controls fusion in $O(N)$.

Let $w \in N - N'$ of order 3. We shall argue that $w \in G - G'$, giving a contradiction. Let

$$P \in Syl_3(N), Z = Z(P).$$

First note that $C_G(Z)$ has odd order and so, easily, $C_G(Z) = O_{3',3}(C_G(Z))$ and

$$N_G(Z) = O_{3'}(C_G(Z))N_G(P) = O_{3'}(C_G(Z))P.$$

In particular, Z is neither inverted nor centralized by an involution. If $w \in N - N'$ with $w^G \cap O(N) \neq \phi$, then by Alperin's Theorem [2, Section 38], $w^{N_1} \cap O(N) \neq \phi$ where N_1 is the normalizer either of $\langle w, Z \rangle$ or of an extraspecial subgroup of P. In the latter case, $N_1 \subseteq N_G(Z)$, which is impossible. In the former case, $N_1/C_G(\langle w, \rangle)$ is isomorphic to a subgroup of $GL(2,3)$ of order divisible by 3 but without a normal 3-subgroup. But then N_1 contains an involution inverting $\langle w, Z \rangle$, a contradiction.

Thus $w^G \cap O(N) = \phi$ for all $w \in N - N'$. Every element of $N - N'$ of order 3 is N-conjugate to w or w^{-1}. As $C_P(w) = \langle w, Z \rangle$, we see that $N_G(\langle w \rangle)$ has odd order and so w is not G-conjugate to w^{-1}. Now by the Thompson transfer lemma [2, 37.4)], $w \notin G'$, a contradiction.

LEMMA 7.7. *Let $c \in P \in Syl_3(G)$. Either*

(1) *$P \cong \mathbf{Z}_3 \times \mathbf{Z}_3$ and $B = N_G(P) = PQ$, Q quasi-dihedral of order* 16, *or*

(2) *$|P| = 3^3$ and there are two elementary subgroups, E_1 and E_2, of P with $N_G(E_i)/E_i \cong GL(2,3)$. Also $E_2 \notin E_1^G$.*

Proof. Let $P \in Syl_3(C_G(c))$ and let $B = N_G(P)$. Suppose $|P| = 9$. As $|C_B(c)| = 18$, we have $|B/P| \leq 16$. If $|B/P| = 12$, then $O(B)$ is extraspecial and we may assume $Z(O(B)) = C_{O(B)}(tt_1)$. But then $\langle c \rangle$ and $Z(O(B))$ are G-conjugate, contrary to assumption. Thus $P \in Syl_3(G)$ and B is transitive on its involutions which do not invert P. Hence $B = PQ$ with Q quasi-dihedral.

Suppose $|P| = 27$. Clearly $P \in Syl_3(G)$. Let $E_1 = \langle c, c_1 \rangle$ where $\langle c_1 \rangle = C_P(t_1)$. Then $N_G(E_1)$ is transitive on the subgroups of E_1 of order 4 and so $N_G(E_1)/E_1 \cong GL(2,3)$. Ditto for $\langle c, c_2 \rangle = E_2$, where $\langle c_2 \rangle = C_P(tt_1)$. Where $E_2^G = E_1^G$, we would have $E_2 \in E_1^{N_G(P)}$. But $N_G(P) = N_G(Z(P)) = P\langle t, t_1 \rangle$.

8. The identification of M_{11}

Assume that $|P| = 9$.

LEMMA 8.1. $|G| = 2^4 \cdot 3^2 \cdot 5 \cdot 11 = 11 \cdot 10 \cdot 9 \cdot 8.$

Proof. As $B \supseteq C_G^*(x)$ for all $x \in P^\#$, if σ is an involution of $G - B$ and $B\sigma$ contains more than one involution, then σ centralizes a unique involution j of B. If j is 2-central in B, then $B\sigma$ contains 4 involutions and there are two such cosets for each 2-central j. If j is not 2-central in B, then $B\sigma$

contains 2 involutions and there are 3 such cosets for each non-2-central j. Hence $b_2 = 3 \times 12 = 36$, $b_4 = 2 \times 9 = 18$ and $b_n = 0$ for $n = 3$ or $n \geq 5$.
Thus

$$|G:B| = 1 + b_0 + b_1 + 36 + 18 = b_0 + b_1 + 55.$$

Also

$$|G:H| = |z^G| = 21 + b_1 + 2b_2 + 4b_4 = b_1 + 165.$$

As $|G:H| = 3|G:M|$, we conclude that $b_0 = b_1 = 0$ and $|G:M| = 55$.

Notation. $\langle c, t, t_1\rangle = N_H(\langle c\rangle)$ with $C_P(tt_1) = 1$.

LEMMA 8.2. *There exists a 4-subgroup $U = \langle \tau, \tau^c\rangle$ in G with $N_G(U) = U\langle c, t_0\rangle$ where $t_0 = tt_1$.*

Proof. By (7.1), the normalizer of any 4-subgroup of G is S_4. By inspection in $N_G(\langle c, c_1\rangle)$, G has two classes of S_3 subgroups: $\langle c, t_1\rangle^G$ and $\langle c, tt_1\rangle^G$. Thus we need to prove that there is no S_4-subgroup $B = \langle \tau, \tau^c\rangle\langle c, t_1\rangle$. Suppose there is. By (7.2), $S = \langle B, t\rangle \cong S_5$ and clearly $t \notin S'$ and $t_1 \notin S'$. Hence $tt_1 \in S'$. Let $u \in C_G(tt_1)$ with $u^2 = tt_1$ and $u^{t_1} = u^{-1}$. Then $D = \langle u, t_1\rangle$ is the unique D_8 subgroup of G with $Z(D) = \langle tt_1\rangle$ and $t_1 \in D$. Hence $D = C_S(tt_1)$. Also as tt_1 is central in a Sylow 2-subgroup of $N_G(\langle c, c_1\rangle)$ containing t_1, we have $u \in N_G(\langle c, c_1\rangle)$. But then $\langle c, c^u\rangle = \langle c, c_1\rangle \subseteq S$, which is absurd.

Notation. (1) $\langle c, c_1\rangle \in Syl_3(G)$. t_0 inverts $\langle c, c_1\rangle$.
(2) $N_G(U) = U\langle c, t_0\rangle$, $U \cong \mathbf{Z}_2 \times \mathbf{Z}_2$.
(3) $\langle\tau\rangle = C_U(t_0)$; $u \in C_G(t_0)$ with $u^2 = t_0$, $u^\tau = u^{-1}$.
(4) $u_1 \in C_G(t_0)$ with $\langle u, u_1\rangle = O_2(C_G(t_0))$.
(5) $B_1 = \langle c, c_1\rangle\langle u\rangle = \langle c, c^u\rangle\langle u\rangle$.
(6) $G_1 = B_1 \cup B_1\tau B_1$.
(7) $G_0 = \langle G_1, u_1\rangle$.

LEMMA 8.3. $G_1 \cong PSL(2,9)$ *and* $G_0 \cong M_{10}$.

Proof. Suppose $G_1 \cong PSL(2,9)$. As $u_1 \in O_2(C_G(t_0))$ and as $N_G(\langle c, c_1\rangle)$ contains a Sylow 2-subgroup of $C_G(t_0)$, we see that $u_1 \in N_G(B_1)$. As $[u_1, \tau] \in \langle u\rangle \subseteq B$, we have $u_1 \in N_G(G_1)$. Thus $|G_0 : G_1| = 2$ and so $G_0 \cong M_{10}$.

By (3.1), to show $G_1 \cong PSL(2,9)$, it suffices to verify $(*)$, $(**)$ and $(***)$. Now $(*)$ holds by choice of u and $(**)$ holds since $\langle\tau, c\rangle \cong A_4$ and $\tau c \notin O_2(\langle\tau, c\rangle)$. Finally $u \in N_G(\langle t_0, \tau\rangle)$ by choice of u. Also $\tau^c \in N_G(\langle t_0, \tau\rangle)$ by the structure of $N_G(U)$. As $N_G(\langle t_0, \tau\rangle) \cong S_4$ and as u and τ^c

centralize different involutions of $\langle t_0, \tau \rangle$, we have $(u\tau^c)^3 = 1$, i.e., $(***)$ holds.

COROLLARY 8.4. $G \cong M_{11}$.

Proof. Lemma XII (3.2) of Blackburn-Huppert [4]. Note that they denote by $M(9)$ the group we call M_{10}.

9. The identification of $PSL(3, 3)$

Assume that $|P| = 27$.

LEMMA 9.1. *Let $M = N_G(E_i)$ for $i = 1$ or 2. Then for all $g \in G - M$,*
(a) $|M \cap M^g| = 36$,
(b) $O_3(M \cap M^g) \in E_{3-i}^G$,
(c) $|G : M| = 13$.

Proof. By the structure of $N_G(P)$, E_1 and E_2 are the only E_9 subgroups of P inverted by an involution. Hence each Sylow 3-subgroup of G lies in a unique conjugate of M. In particular, if $g \in G - M$, then $3 \mid |M : \cap M^g|$. Suppose $4 \mid |M : M \cap M^g|$. Then we may assume

$$t \in Z^*(M) \cap Z^*(M^g)$$

and so there exists $m \in M$ with $t^{g^{-1}m} = t$. But then $g^{-1}m \in H \subseteq M$ and so $g \in M$ contrary to assumption. Thus $12 \mid |M : M \cap M^g|$ for all $g \in G - M$.

As M contains $C_G^*(x)$ for all $x \in O_{3,2}(M)$, we have $b_n = 0$ for $n > 6$. Now

$$|G : H| = 9|G : M| = 9(1 + b_0 + b_1 + \cdots + b_6)$$

and

$$|G : H| = 45 + b_1 + 2b_2 + \cdots + 6b_6.$$

Thus

$$9b_0 + 8b_1 + \cdot + 3b_6 = 36.$$

By the above, $b_n \equiv 0 (\mathrm{mod}\ 12)$. Hence $b_6 = 12$ and $b_n = 0$ for $n \neq 6$. We infer that $|G : M| = 13$ and that $|M : M \cap M^g| = 12$ for all $g \in G - M$, proving (a) and (c).

As $|M \cap M^g| = 36$, we see that $O_3(M \cap M^g)$ is a 9-group inverted by an involution, hence is in E_1^G or E_2^G. Were $O_3(M \cap M^g) \in E_i^G$, we would have $O_3(M) = O_3(M \cap M^g) = O_3(M^g)$ and $g \in M$, contrary to assumption.

DEFINITION. We define a geometry Γ with points $\mathscr{P} = E_1^G$, lines $\mathscr{L} = E_2^G$ and incident pairs $(E_1^g, E_2^{g'})$ if and only if $E_1^g E_2^{g'} \in Syl_3(G)$.

LEMMA 9.2. *Γ is a projective plane of order* 3.

Proof. By 9.1 (b), any two "points" are normalized by (i.e. incident with) a unique line and any two "lines" are incident with a unique point. Clearly each line has 4 points and each point lies on 4 lines. There are 13 points in all.

COROLLARY 9.3. $G \cong PSL(3,3)$.

Proof. It is an easy game to check that there is a unique affine plane of order 3 and hence a unique projective plane Γ of order 3. G is isomorphic to a subgroup of Aut $\Gamma = PGL(3,3)$ (see e.g. [1, 2.26]). Hence $G \cong PSL(3,3)$.

This completes the proof of Theorem 1.2.

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