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DVORETZKY'S THEOREM FOR QUASI-NORMED SPACES

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1. Introduction

Recall the definition of a quasi-norm $\|\cdot\|$ defined on a vector space X. This is a map $X \to R$ so that

$$\|x\| > 0 \quad x \neq 0, \tag{1.1}$$

$$\|tx\| = |t|\,\|x\|, \quad t \in R, x \in X, \tag{1.2}$$

$$\|x + y\| \leq C \max\{\|x\|, \|y\|\}, \quad x, y \in X, \tag{1.3}$$

where $C = C_X$ is a constant. $C_X \geq 2$ with equality iff $\|\cdot\|$ is a norm.

By Theorem 1.2 [KPR], if p is chosen so that $2^{1/p} = C_X$ then the formula

$$\|x\|_p =: \inf\left\{\left(\sum_{i=1}^{n} \|x_i\|^p\right)^{1/p}; x = \sum x_i\right\}$$

defines another quasi-norm $\|\cdot\|_p$ which satisfies

$$4^{-1/p}\|x\| \leq \|x\|_p \leq \|x\|, \quad x \in X, \tag{1.4}$$

$$\|x + y\|_p^p \leq \|x\|_p^p + \|y\|_p^p, \quad x, y \in X, \tag{1.5}$$

(1.4) says the two norms are equivalent and (1.5) implies that $\|\cdot\|_p$ is a continuous function since

$$\left|\|x\|_p^p - \|y\|_p^p\right| \leq \|x - y\|_p^p, \quad x, y \in X. \tag{1.6}$$

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A quasi-norm $\|\cdot\|_p$ that satisfies (1.1), (1.2), and (1.5) is called a p-norm and $(X, \|\cdot\|_p)$ is called a p-normed space. Of course, (1.5) implies that (1.3) holds with $C = 2^{1/p}$.

We shall denote by S^{n-1} the Euclidean sphere of R^n, $|x| = 1$, where $|\cdot|$ is the Euclidean norm. Let ρ be the geodesic metric on S^{n-1}, and given a subset $A \subset S^{n-1}$, its ε-neighborhood A_ε is the set $\{x \in S^{n-1}; \rho(x, A) \leq \varepsilon\}$. Given $f \in C(S^{n-1})$, the median M_f is a number M such that $\mu(f \geq M) \geq 1/2$ and $\mu(f \leq M) \geq 1/2$, here μ denotes the rotation invariant probability measure on the sphere.

Levy's classical isoperimetric inequality [L] states:

(1.7) For any closed subset $A \subset S^{n-1}$ and $0 < \varepsilon < \pi$, if $C \subset S^{n-1}$ is a cap with $\mu(C) = \mu(A)$, then $\mu(A_\varepsilon) \geq \mu(C_\varepsilon)$.

It was also observed by Levy [L] that (1.7) implies the following:

(1.8) Let $f \in C(S^{n-1})$ and $A =: f^{-1}(M_f)$.

Then

$$\mu(A_\varepsilon) \geq 1 - \sqrt{\pi/2}\, \exp(-(n-2)\varepsilon^2/2).$$

This is because (1.7) implies that the measure of an ε-neighborhood of $A = f^{-1}(M_f)$ is minimal when A is an equator, and the R.H.S. of (1.8) is a lower estimate for $\mu(A_\varepsilon)$ when A is an equator. Using (1.8) Milman gave in [M] a new and simpler proof of Dvoretzky's theorem [Dv1] which gave also an estimate for the dimensions of ε-Euclidean sections in convex bodies, and in particular, in Banach spaces. In its original form, Dvoretzky's theorem states:

(1.9) Given an integer $n \geq 1$, and $\varepsilon > 0$, there exists an integer $N = N(\varepsilon, n)$ so that for any N-dimensional Banach space X there is a one-to-one operator $T\colon l_2^n \to X$ such that

$$1 - \varepsilon \leq \|Tx\| \leq 1 + \varepsilon, \quad x \in S^{n-1}.$$

The application of (1.8) in Milman's proof yielded the estimate

$$N(\varepsilon, n) = c_1 \exp\bigl(c_2 n \varepsilon^{-2} \log(2 + 1/\varepsilon)\bigr),$$

where c_1, c_2 are absolute constants.

Other proofs of Dvoretzky's theorem, also in the Banach space setting, were given in [F], [FLM], [LM], [Pi1], [S]; in addition, the three recent books [MS], [T], and [Pi2] contain proofs of the theorem and many diverse applications as well.

In [G1] we showed that the log term in Milman's estimate may be eliminated. To prove this we used instead of 1.8 the following result of [G1].

(1.10) Let $X_{i,j}$, $Y_{i,j}$, $1 \le i \le n$, $1 \le j \le m$, be two doubly indexed centered Gaussian processes which satisfy the following inequalities for all $1 \le i$, $i' \le n$, $1 \le j$, $j' \le m$:
(i) $\mathbf{E}(X_{i,j} - X_{i,j'})^2 \le \mathbf{E}(Y_{i,j} - Y_{i,j'})^2$ if $j \neq j'$,
(ii) $\mathbf{E}(X_{i,j} - X_{i',j'})^2 \ge \mathbf{E}(Y_{i,j} - Y_{i',j'})^2$ if $i \neq i'$.
Then, $\mathbf{E} \min_i \max_j X_{i,j} \le \mathbf{E} \min_i \max_j Y_{i,j}$.

(A simpler proof of (1.10) and generalizations were later given in [G2]).

In comparison to the extensive literature on the applications of Dvoretzky's theorem in the Banach space setting, the quasi-normed space received very little attention. Kalton proved it (unpublished [K]) in the case when X is an infinite-dimensional quasi-normed space; i.e., $N = \infty$, $2 \le n < \infty$, and $\varepsilon = \varepsilon(C_X)$ is not arbitrary but depends on the value of C_X. The fact that ε cannot be taken arbitrarily close to 0 is clear because the general quasi-norm need not be a continuous function. The finite dimensional setting, dim $X = N$ was proved by Dilworth in [Di], who followed the [FLM] approach, and the dependence of N on ε is missing there.

The new ingredient in this presentation is the combination of the "classical" Milman approach based on (1.8) and the modern approach based on (1.10). This combination is necessary to improve the estimates on $N(\varepsilon, n)$ for quasi-normed spaces. On the other hand, if in §2 one uses only Levy's inequality instead of (1.10), then the resulting argument is a cleaned-up version of the classical proof of Dvoretzky's theorem which works even in the p-normed setting.

2. Dvoretzky's Theorem

The main result in this paper is:

THEOREM 2.1. *Let $n \ge 1$, $0 < \varepsilon < 1$, $0 < p \le 1$ be such that $l =: [c\varepsilon^{2/p} \ln(1 + n)]$ is greater than 1. If $X_p = (R^n, \|\cdot\|_p)$ is a p-normed space, then there is an l-dimensional subspace E of R^n and a linear operator T on R^n so that*

$$1 - \varepsilon \le \|Tx\|_p^p \le 1 + \varepsilon, \quad x \in E \cap S^{n-1} \tag{2.1}$$

(*where c denotes an absolute positive constant*).

Proof. Without loss of generality we may assume that the ellipsoid of maximal volume contained in the unit ball B_p of X_p is the standard ball of

$l_2^n = (R^n, |\cdot|)$. It was observed by Dvoretzky [Dv2], and also used in [Di], that the Dvoretzky-Roger's lemma [DR] holds in this case; that is, there exists an orthonormal basis $\{e_i\}_{i=1}^n$ and contact points $y_i \varepsilon \partial B_p \cap S^{n-1}$ (B_p is the unit ball of X_p) so that for all $i = 1, 2, \ldots, n$,

$$y_i = \sum_{j=1}^{i} \omega_{i,j} e_j, \tag{2.2}$$

$$\sum_{j=1}^{i-1} \omega_{i,j}^2 = 1 - \omega_{i,i}^2 \leq \frac{i-1}{n}. \tag{2.3}$$

Let

$$f(x) =: \left\| \sum_{i=1}^{m} \xi_i y_i \right\|_p^p$$

be defined for points $x = \sum_{i=1}^n \xi_i e_i \in S^{n-1}$, where $m = [\sqrt{2n}]$.

By (1.6), if $y = \sum_{i=1}^n \eta_i e_i$ then

$$|f(x) - f(y)| \leq f(x-y) \leq \left| \sum_{1}^{m} (\xi_i - \eta_i) y_i \right|^p. \tag{2.4}$$

But,

$$\begin{aligned}
\left| \sum_1^m a_i y_i \right|^2 &= \left| \sum_{i=1}^m a_i \omega_{i,i} e_i + \sum_{i=1}^m a_i \sum_{j=1}^{i-1} \omega_{i,j} e_j \right|^2 \\
&\leq 2 \left\{ \sum_{i=1}^m a_i^2 \omega_{i,i}^2 + \sum_{j=1}^{m-1} \left(\sum_{i=j+1}^m a_i \omega_{i,j} \right)^2 \right\} \\
&\leq 2 \left\{ \sum_{i=1}^m a_i^2 + \sum_{j=1}^{m-1} \left(\sum_{i=j+1}^m a_i^2 \right) \left(\sum_{i=j+1}^m \omega_{i,j}^2 \right) \right\} \\
&\leq 2 \left(\sum_{i=1}^m a_i^2 \right) \left(1 + \sum_{i=2}^m \sum_{j=1}^{i-1} \omega_{i,j}^2 \right) \\
&\leq 2 \left(\sum_{i=1}^m a_i^2 \right) \left(1 + \sum_{i=2}^m \frac{i-1}{n} \right) \leq 4 \sum_{i=1}^m a_i^2.
\end{aligned}$$

Hence

$$|f(x)-f(y)| \le 2^p\left(\sum_{i=1}^{m}(\xi_i-\eta_i)^2\right)^{p/2} \le 2^p|x-y|^p;$$

this implies that the modulus of continuity of f satisfies

$$\omega_f(\delta) =: \max\{|f(x)-f(y)|;\rho(x,y)\le\delta\} \le (2\delta)^p, \quad \delta>0. \tag{2.5}$$

Hence, letting $A=f^{-1}(M_f)$, it follows from (2.5) and (1.8) that

$$\begin{aligned}
|\mu(f)-M_f| &\le \mu\big(|f-M_f|\big) = \int_0^\infty \mu\big(|f-M_f|\ge t\big)\,dt\\
&= \int_0^\infty \mu\big(|f-M_f|\ge (2\delta)^p\big)\,d\big((2\delta)^p\big)\\
&\le \int_0^\infty \mu\big(|f-M_f|\ge \omega_f(\delta)\big)\,d\big((2\delta)^p\big)\\
&\le \int_0^\infty \mu\big(S^{n-1}\setminus A_\delta\big)\,d\big((2\delta)^p\big)\\
&\le \sqrt{\pi/2}\int_0^\infty \exp(-(n-2)\delta^2/2)\,d\big((2\delta)^p\big)\\
&= \big(\sqrt{\pi/2}\,\big)2^{3p/2}\Gamma\Big(\frac{p+2}{2}\Big)(n-2)^{-p/2}.
\end{aligned} \tag{2.6}$$

Using the fact that $\|y_i\|_p=1$, we obtain from (1.5) and a standard estimate

$$\begin{aligned}
\mu(f) &= \int_{S^{n-1}} Av_{\pm}\left\|\sum_{i=1}^{m}\pm\,\xi_i y_i\right\|_p^p d\mu \ge 2^{p-1}\int_{S^{n-1}}\max_{1\le i\le m}|\xi_i|^p\,d\mu\\
&\ge c_1\left(\frac{\ln m}{n}\right)^{p/2},
\end{aligned} \tag{2.7}$$

where c_1 is a constant (see e.g. Lemma 5.7 [MS]).

We will now prove that when $n\ge n(\delta)$ $(0<\delta<\pi/2)$ there is a subspace E of dimension $l\ge c_2 n\delta^2$ such that for all $f\in C(S^{n-1})$,

$$|f(x)-M_f| \le \omega_f(c_3\delta) \le (2c_3\delta)^p, \quad x\in S^{n-1}\cap E. \tag{2.8}$$

(2.8) will complete the proof because (2.6), (2.7), (2.8), and the estimate for l, imply that for all $x \in S^{n-1} \cap E$ we have

$$
\begin{aligned}
\left|\frac{f(x)}{\mu(f)} - 1\right| &\le \left\{(2c_3\delta)^p + |M_f - \mu(f)|\right\}/\mu(f) \\
&\le c\left\{\left(\frac{l}{\ln(1+n)}\right)^{p/2} + O\left((\ln(1+n))^{-p/2}\right)\right\} \qquad (2.9)\\
&\le c\left(\frac{l}{\ln(1+n)}\right)^{p/2} \quad (\text{since } l \ge 1),
\end{aligned}
$$

and this is smaller than ε provided l satisfies the assumptions of the theorem and δ is chosen so that $n\delta^2 \sim l$.

To make the paper self-contained, we give the essential ingredients of the proof of (2.8) taken from [G3] and rewrite it as a lemma. A weaker version of this lemma follows from Levy's inequality alone [M], [MS].

LEMMA 2.2. *Given $\pi/2 > \delta > 0$, there is a function $n(\delta)$, so that for every $f \in C(S^{n-1})$, $n \ge n(\delta)$, there is a subspace E of dimension $l \ge -1 + [n\delta^2/4]$ such that*

$$|f(x) - M_f| \le \omega_f(2\delta), \quad x \in S^{n-1} \cap E.$$

Proof. Let $A = f^{-1}(M_f)$, $1 \le k < n$. Given any $x \in (A_{2\delta})^c =: S^{n-1} \setminus A_{2\delta}$ and $y \in S^{k-1}$, we define the two Gaussian processes

$$X_{x,y} =: \sum_{i=1}^{n} h_i\xi_i + \sum_{j=1}^{k} g_j\eta_j \quad \left(x = \sum_{i=1}^{n} \xi_i e_i,\ y = \sum_{j=1}^{k} \eta_j e_j\right)$$

and

$$Y_{x,y} =: \sum_{i=1}^{n}\sum_{j=1}^{k} g_{i,j}\xi_i\eta_j = \langle G(x), y\rangle$$

where $g_j, h_i, g_{i,j}$ are independent $N(0,1)$ Gaussian variables and $G =: (g_{i,j})$ is the induced Gaussian operator from R^n to R^k. It is obvious that

$$\mathbf{E}(X_{x,y} - X_{x',y'})^2 - \mathbf{E}(Y_{x,y} - Y_{x',y'})^2 = 2(1 - \langle x, x'\rangle)(1 - \langle y, y'\rangle) \ge 0$$

with equality to zero if $x = x'$. By (1.10) and since $h = (h_i)_1^n$ is a symmetric

Gaussian vector

$$
\begin{aligned}
\mathbf{E}\Big(\min_{x\in(A_{2\delta})^c}|G(x)|\Big) &= \mathbf{E}\Big(\min_{x\in(A_{2\delta})^c}\ \max_{y\in S^{k-1}} Y_{x,y}\Big)\\
&\geq \Big(\min_x \max_y X_{x,y}\Big)\\
&= \mathbf{E}\sqrt{\sum_{j=1}^k g_j^2} - \mathbf{E}\max_{x\in(A_{2\delta})^c}\langle x,h\rangle \qquad (2.10)\\
&= a_k - a_n\int_{S^{n-1}}\max_{x\in(A_{2\delta})^c}\langle x,u\rangle\mu(du)
\end{aligned}
$$

where

$$
a_k =: \mathbf{E}\sqrt{\sum_{j=1}^k g_j^2} < \Bigg(\mathbf{E}\Big(\sum_{j=1}^k g_j^2\Big)\Bigg)^{1/2} = \sqrt{k}\,.
$$

Note that by evaluating a_k in terms of the Γ function, we obtain $a_k a_{k+1} = k$, therefore

$$
a_k > \frac{k}{\sqrt{k+1}} > \sqrt{k-1}\,.
$$

We now estimate the integral of (2.10). Since $u \in (A_\delta)^c$ implies $\max_{x\in(A_{2\delta})^c}\langle x,u\rangle \leq 1$, and if $u\in A_\delta$, then $\max_{x\in(A_{2\delta})^c}\langle x,u\rangle \leq \cos\delta$, it follows that if $0<\delta\leq\pi/2$,

$$
\begin{aligned}
\int_{S^{n-1}}\max_{x\in(A_{2\delta})^c}\langle x,u\rangle\mu(du) &= \int_{(A_\delta)^c} + \int_{A_\delta}\\
&\leq 1-\mu(A_\delta)(1-\cos\delta)\\
&\leq 1-\frac{\delta^2}{\pi}\mu(A_\delta).
\end{aligned}
$$

By Levy's inequality (1.8), $\mu(A_\delta)\geq 1-\sqrt{\pi/2}\exp(-(n-2)\delta^2/2)$, hence

$$
\begin{aligned}
\int_{S^{n-1}}\max_{x\in(A_{2\delta})^c}\langle x,u\rangle\mu(du) &\leq 1-\frac{\delta^2}{\pi}\Big(1-\sqrt{\pi/2}\exp(-(n-2)\delta^2/2)\Big)\\
&\leq 1-\frac{\delta^2}{4}\quad (\text{if } n\geq n(\delta)\sim 100/\delta^2).
\end{aligned}
$$

Therefore,

$$\mathbf{E}\Big(\min_{x\in(A_{2\delta})^c}|G(x)|\Big) \geq a_k - a_n\Big(1-\frac{\delta^2}{4}\Big)$$
$$> \sqrt{k-1} - \sqrt{n}\Big(1-\frac{\delta^2}{4}\Big)$$

which is positive if $k-1 > n(1-\delta^2/4)^2$; i.e., if $l =: n-k \leq [\delta^2 n/4] - 1$. Hence, there is a subspace E; namely, $G^{-1}(0)$, of dimension l (since a.e. $G^{-1}(0)$ has dimension l) which misses the set $(A_{2\delta})^c$; i.e., $E \cap S^{n-1} \subset A_{2\delta}$. This completes the proof of the lemma and the theorem.

Remark. In Theorem 2.1 we could have chosen the function f to be

$$f(x) =: \Big\|\sum_{i=1}^m \xi_i e_i\Big\|_p^p;$$

then obviously the same proof would work with some slight changes. Clearly, $\omega_f(\delta) \leq \delta^p$ (since $|\cdot| \geq \|\cdot\|_p$), but in (2.7) we would get

$$\mu(f) \geq 2^{p-1}\int_{S^{n-1}} \max_{1\leq i\leq m} \|\xi_i e_i\|_p^p \, d\mu,$$

and we can estimate $\|e_i\|_p^p$ from below by

$$\|e_i\|_p^p \geq \|\omega_{i,i}e_i\|_p^p \geq \|y_i\|_p^p - \Big\|\sum_{j=1}^{i-1}\omega_{i,j}e_j\Big\|_p^p$$
$$\geq 1 - \Big(\sum_{j=1}^{i-1}\omega_{i,j}^2\Big)^{p/2} > 1 - \Big(\frac{m}{n}\Big)^{p/2}, \quad i = 1,2,\ldots,m.$$

The final result is then a weaker estimate for the value l of Theorem 2.1 in terms of $p > 0$ only. However, if X_p has cotype 2; for example, when X_p is any n-dimensional subspace of l_p $(0 < p < 2)$, then this choice

$$f(x) =: \Big\|\sum_{i=1}^m \xi_i e_i\Big\|_p^p$$

with $m = [n/2]$ would yield a better result for l in terms of n, and we would have the conclusion of Theorem 2.1 for $l = [c_p n\varepsilon^{2/p}]$, where $c_p > 0$ depends only on p and the cotype 2 constant of X_p.

COROLLARY 2.3. *In the notation of Theorem* 2.1 *and inequality* (1.4), *if* $\|\cdot\|$ *is a quasi-norm on* R^n *with constant* $C_X =: 2^{1/p}$, *then there is an* l*-dimensional subspace* E *and a linear operator* T *on* R^n *so that*

$$1 - \varepsilon \leq \|Tx\|^p \leq 4(1 + \varepsilon) \quad \textit{for all } x \in E \cap S^{n-1}.$$

The constant 4 above cannot be replaced by 1 for a general quasi-norm $\|\cdot\|$, but we can show that every n-dimensional quasi-normed space X contains a big l-dimensional subspace E with the property that close to any point in E there is a point in X where the norm is "almost" Euclidean. To make this precise, denote by $\mathscr{E}_X$ the ellipsoid of maximal volume contained in the unit ball of $X_p = (R^n, \|\cdot\|_p)$, and let ρ_X be the geodesic metric on $\mathscr{E}_X$.

THEOREM 2.4. *Let* $0 < \delta, \varepsilon < 1$, $0 < \lambda < 1/2$, $0 < p \leq 1$ *and* n *be an integer so that*

$$n\lambda^2\delta^4 \geq 100 \ln 5 \quad \textit{and} \quad \varepsilon > c(\ln(n+1))^{-p/2}.$$

Let $X = (R^n, \|\cdot\|)$ *be a quasi-normed space with* $C_X = 2^{1/p}$. *Then there is a subspace* E *of dimension* $l = [\delta^2\lambda n/4]$ *and an operator* T *on* R^n *so that for every* $y \in E \cap \partial\mathscr{E}_X$ *there is a point* $x \in \partial\mathscr{E}_X$ *such that*

(1) $\rho_X(x, y) \leq \delta$,

(2) $(1 - \varepsilon)(1 - 2\lambda) \leq \|Tx\|^p \leq (1 + \varepsilon)(1 + 2\lambda)$.

The proof of this is lengthy and requires some additional results based on measure estimates for Gaussian processes.

REMARK 2.5. If in the definition of the quasi-norm we replace (1.2) with the weaker condition

$$(1.2^*) \qquad \|tx\| = t\|x\|, \quad 0 \leq t \in R,\ x \in X,$$

this implies that B_X is no longer assumed to be symmetric about the origin. Then all the results above still remain true. Thus we can start by taking ∂B_X to be any bounded starshaped surface in R_n, not necessarily continuous, so that B_X contains a Euclidean ball about 0 with positive radius, and Theorems 2.1–2.4 hold for the corresponding quasi-normed spaces $X = (R^n, \|\cdot\|)$ and $X_p = (R^n, \|\cdot\|_p)$.

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