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RATIONAL PERIOD FUNCTIONS AND INDEFINITE BINARY QUADRATIC FORMS, II

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1. Introduction

The classification of rational period functions of positive weight was begun in [CP]. This was accomplished by looking at possible minimal or irreducible systems of poles. Knopp in [Kn2] showed that the finite poles of rational period functions occur only at zero or at real quadratic irrationals and then found all rational period functions with poles only at zero. Of interest now is the case where the minimal pole set contains quadratic irrationals. If the minimal pole set exhibits algebraic symmetry, that is, if the pole set contains α', the conjugate of α, whenever it contains α, then the rational period function of weight $2k$, k odd, with poles only at elements of the pole set is of the form [CP]

$$\sum_{\substack{a>0>c \\ [a,b,c]\in\mathscr{A}}} \frac{1}{(az^2 - bz + c)^k}$$

where the sum is over primitive indefinite quadratic forms in a narrow equivalence class $\mathscr{A}$.

The situation when the irreducible pole set fails to be symmetric is studied in this paper. Given a pole set which is minimal and not symmetric, it is always possible to construct a rational period function of weight $2k$ with quadratic irrational poles only at elements of that pole set whenever $k = 1$, 2, 3, 4, 5 or 7. For all other weights certain "obstructions" may occur in the construction which act as barriers to the existence of rational period functions. In fact such obstructions are contained in a space isomorphic to $S_{2k} \oplus S_{2k}$ where S_{2k} is the space of cusp forms of weight $2k$. By way of example an infinite family of irreducible pole sets is given for which it is impossible to construct a rational period function of weight twelve with quadratic irrational poles in exactly one of the pole sets.

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2. Rational period functions, pole sets, and indefinite binary quadratic forms

DEFINITION. Let F be meromorphic in the complex upper half-plane $\mathscr{H}$, and, for $k \in \mathbf{Z}$, let F satisfy

$$F(z+1) = F(z), \qquad z^{-2k}F(-1/z) = F(z) + q(z). \tag{2.1}$$

If $q(z)$ is a rational function of z and if F is also meromorphic at $i\infty$, then F is called a *modular integral of weight* $2k$ *with a rational period function* $q(z)$. If F is holomorphic in $\mathscr{H}$ and at $i\infty$, then F is called entire. In addition, if $q \equiv 0$, then F is an entire modular form; and if F is zero at $i\infty$, then F is a cusp form.

The slash operator, defined for $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$ in $\Gamma(1)$, the modular group, by

$$F|_{2k}M = (cz+d)^{-2k}F(Mz)$$

is often used so that (2.1) becomes

$$F|_{2k}S = F, \qquad F|_{2k}T = F + q$$

where $Sz = z+1$ and $Tz = -1/z$ are the translation and inversion which generate $\Gamma(1)$. For notational convenience the weight $2k$ of the operator is frequently suppressed.

If $q(z)$ is the rational period function of a modular integral, $q(z)$ satisfies two relations determined by the defining relations for $\Gamma(1)$, namely, $T^2 = U^3 = I$, where $U = ST$ and I and $-I$ are identified as linear fractional transformations. The relations for $q(z)$ are

$$q|(I+T) = 0, \qquad q|(I+U+U^2) = 0. \tag{2.2}$$

In [Kn1] Knopp showed, by using generalized Eichler-Poincaré series, that the existence of an entire modular integral with rational period function $q(z)$ is equivalent to the existence of a rational function satisfying (2.2). As a result, when studying existence questions one need only search for solutions to (2.2).

It is natural to search for and classify rational period functions of positive weight by looking at the finite poles which necessarily occur at zero or a real quadratic irrational. Those rational period functions of positive weight with poles only at zero are given by [Kn3].

$$q(z) = \begin{cases} c(1-z^{-2k}) & \text{if } k > 1, \\ c_1(1-z^{-2}) + c_2 z^{-1} & \text{if } k = 1. \end{cases} \tag{2.3}$$

In [H] Hawkins describes carefully the sets of quadratic irrationals which are poles of a rational period function. (See also [CP].) These pole sets are the disjoint union of irreducible systems of poles where *irreducible systems of poles* are minimal sets of quadratic irrational poles forced by the relations (2.2) to occur together. The elements of an ireducible system of poles are all equivalent, that is, given α, β in the pole set, there is $M \in \Gamma(1)$ with $M\alpha = \beta$. The irreducible systems may be written in the form

$$\left\{\alpha_i, -\frac{1}{\alpha_i} : \alpha_i > 0\right\}_{i=1}^{p}$$

with

$$0 < \alpha_1 < \alpha_2 < \cdots < \alpha_\omega < 1 < \alpha_{\omega+1} < \cdots < \alpha_p$$

where there is always at least one pole between zero and one and another pole greater than one. In addition, there is a natural correspondence between an irreducible system of poles and a narrow equivalence class of primitive indefinite binary quadratic forms.

Let $Q(X,Y) = aX^2 + bXY + cY^2$ be a quadratic form with integral coefficients and positive non-square discriminant $D = b^2 - 4ac$. Assume also that $Q(X,Y)$ is primitive, that is, $(a,b,c) = 1$. For convenience, denote $Q(X,Y)$ by Q or by $[a,b,c]$.

For

$$M \in \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma(1)$$

set

$$X' = \alpha X + \beta Y, Y' = \gamma X + \delta Y$$

and

$$Q \circ M = Q(X', Y').$$

Then Q_1 and Q_2 are *equivalent in the narrow sense*, written as $Q_1 \cong Q_2$, if there is $M \in \Gamma(1)$ with $Q_1 \circ M = Q_2$. Also, $Q = [a,b,c]$ is called *reduced* if $\alpha > 0$, $c > 0$, and $b > a + c$. Each narrow equivalence class contains at least one reduced form. Also, associated to each narrow equivalence class $\mathscr{A}$ are three other related equivalence classes of quadratic forms:

$$\mathscr{A}' = \{[a,-b,c] : [a,b,c] \in \mathscr{A}\}$$
$$\theta\mathscr{A} = \{[-a,b,-c] : [a,b,c] \in \mathscr{A}\}$$
$$\theta\mathscr{A}' = \{[-a,-b,-c] : [a,b,c] \in \mathscr{A}\}$$

The correspondence alluded to earlier between irreducible systems of poles and narrow equivalence classes of primitive quadratic forms is obtained as follows. If $\alpha_1, \dots, \alpha_\omega$ are the positive poles less than one, set $\beta_i = TS^{-1}\alpha_i$, $i = 1, \dots, \omega$. Then the β_i are the larger roots of $a_i z^2 - b_i z + c_i = 0$ where $[a_i, b_i, c_i]$ is a reduced form in a class $\mathscr{A}$ and there are exactly ω reduced forms in $\mathscr{A}$. This connection is made using the negative continued fraction expressions for the β_i. (See [H] or [CP] for details.)

3. Rational period functions and irreducible systems of poles with $\mathscr{A} \neq \theta\mathscr{A}'$

Let P be an irreducible system of poles and let $\mathscr{A}$ be the narrow equivalence class of primitive quadratic forms associated with P. The question of interest is for which positive weights $2k$, if any, do there exist rational period functions with quadratic irrational poles only at P. If $\mathscr{A} = \theta\mathscr{A}'$ and k is odd, in [CP] it was shown that

$$q_{\mathscr{A},k}(z) = \sum_{\substack{a>0>c \\ [a,b,c]\in\mathscr{A}}} \frac{1}{(az^2 - bz + c)^k} - \sum_{\substack{a<0<c \\ [a,b,c]\in\mathscr{A}}} \frac{1}{(az^2 - bz + c)^k}$$

is a nontrivial rational period function of weight $2k$ with poles only at elements of P. In particular, P is symmetric in that if $\alpha \in P$, then α', the conjugate of α, is also an element of P. Important in the proof was the alternative expression for $q_{\mathscr{A},k}(z)$ as

(3.1)

$$q_{\mathscr{A},k}(z) = \left(\sum_{\substack{[a,b,c]\in\mathscr{A} \\ [a,b,c] \text{ reduced}}} \frac{1}{(az^2 - bz + c)^k} - (-1)^k \sum_{\substack{[a,b,c]\in\theta\mathscr{A}' \\ [a,b,c] \text{ reduced}}} \frac{1}{(az^2 - bz + c)^k} \right) \| (-U + U^2).$$

When $\mathscr{A} \neq \theta\mathscr{A}'$, $q_{\mathscr{A},k}(z)$ is still always a nontrivial rational period function. As the following theorem shows, if $\mathscr{A} \neq \theta\mathscr{A}'$, P is asymmetric and the poles of $q_{\mathscr{A},k}(z)$ lie in two distinct irreducible systems of poles. Useful in the proof is the following result of Shintani [Sh].

LEMMA 3.1. *Let $\beta = (b + \sqrt{D})/2a$ where $[a, b, c]$ is a reduced quadratic form of discriminant D. Such a β is called a reduced number. Let $[(n_1, \dots, n_w)]$ be the negative continued fraction expansion of β. Denote by ε the fundamental* *unit of* $Q(\sqrt{D})$. *Then*:

(1) *If* $N(\varepsilon) = -1$, $n_1 + n_2 + \cdots + n_w = 3w$.

(2) *If* $N(\varepsilon) = +1$ *and* $\gamma = [\overline{(m_1, \ldots, m_v)}]$ *is a reduced number equivalent to* $1/\beta$,

$$n_1 + n_2 + \cdots + n_w + m_1 + m_2 + \cdots + m_v = 3(w + v).$$

THEOREM 3.2. *Let P be an irreducible system of poles corresponding to the narrow equivalence class* $\mathscr{A}$ *of primitive quadratic forms. Assume that* $\mathscr{A} \neq \theta\mathscr{A}'$. *Then*:

(1) *If* $\alpha \in P$, $\alpha' \notin P$,

(2) $P' = \{\alpha' \colon \alpha \in P\}$ *is an irreducible system of poles corresponding to* $\theta\mathscr{A}'$,

(3) *The set of poles of* $q_{\mathscr{A},k}(z)$ *is* $P \cup P'$.

Proof. The poles of $q_{\mathscr{A},k}(z)$ occur at the zeros of quadratic forms in $\mathscr{A}$. If α and α' are the zeros of $az^2 - bz + c$ where $[a, b, c] \in \mathscr{A}$, and if α and α' were in the same irreducible system of poles, then by Lemma 5.2 in [CP], $[a, b, c] \cong [-a, -b, -c]$ and $\mathscr{A} = \theta\mathscr{A}'$. Since $\mathscr{A} \neq \theta\mathscr{A}'$, the poles of $q_{\mathscr{A},k}(z)$ split into at least two irreducible systems; and each irreducible system contains no conjugate pairs.

For notational convenience, let $P = P(\mathscr{A})$ and $P(\theta\mathscr{A}')$ denote the irreducible systems of poles corresponding to $\mathscr{A}$ and $\theta\mathscr{A}'$. Set

$$P(\mathscr{A}) = \left\{\alpha_j, -\frac{1}{\alpha_j}\right\}_{j=1}^{p}$$

where $0 < \alpha_1 < \cdots < \alpha_\omega < 1 < \alpha_{\omega+1} < \cdots < \alpha_p$ and

$$P(\theta\mathscr{A}') = \left\{\gamma_j, -\frac{1}{\gamma_j}\right\}_{j=1}^{p'}$$

where $0 < \gamma_1 < \cdots < \gamma_{\omega'} < 1 < \gamma_{\omega'+1} < \cdots < \gamma_{p'}$. Then there are ω reduced forms in $\mathscr{A}$ and ω' reduced forms in $\theta\mathscr{A}'$. As a result, it follows from (3.1) that $q_{\mathscr{A},k}(z)$ has $4(\omega + \omega')$ poles. On the other hand, it is clear from (3.1) and the correspondence between irreducible systems of poles and narrow equivalence classes that $q_{\mathscr{A},k}(z)$ has poles at $P(\mathscr{A}) \cup P(\theta\mathscr{A}')$. Since $P(\mathscr{A})$ has $2p$ elements and $P(\theta\mathscr{A}')$ has $2p'$ elements, the set of poles of $q_{\mathscr{A},k}(z)$ will be precisely $P(\mathscr{A}) \cup P(\theta\mathscr{A}')$ if

$$2p + 2p' = 4(\omega + \omega'). \tag{3.2}$$

It will then follow that $P(\theta\mathscr{A}') = P'$.

To prove (3.2) we make use of the negative continued fraction expansion of the reduced number $\beta_1 = TS^{-1}\alpha_1$ and Shintani's lemma applied to $1/\beta_1'$. First, consider the case where $N(\varepsilon) = 1$. As seen in [CP], the continued fraction expansion for β_1 is of the form $\beta_1 = [\overline{(n_1, n_2, \ldots, n_\omega)}]$ where $n_i \geq 2$, $i = 1, \ldots, \omega$ and $n_1 + \cdots + n_\omega = p + \omega$. It then follows that

$$\frac{1}{\beta_1'} = [\overline{(n_\omega, n_{\omega-1}, \ldots, n_1)}]$$

and that $1/\beta_1'$ is also a reduced number. However, since β_1' corresponds to a quadratic form in $\theta\mathscr{A}'$, if γ is a reduced number such that $\gamma \sim \beta_1'$, then

$$\gamma = [\overline{(m_1, \ldots, m_{\omega'})}]$$

where $m_i \geq 2$, $i = 1, \ldots, \omega'$ and $m_1 + \cdots + m_{\omega'} = p' + \omega'$. Shintani's lemma, as applied to $1/\beta_1'$, then gives

$$m_1 + \cdots + m_{\omega'} + n_1 + \cdots + n_\omega = 3(\omega + \omega')$$

or $p + p' = 2(\omega + \omega')$.

Now assume that $N(\varepsilon) = -1$. If β_1 is the larger root of $az^2 - bz + c = 0$ where $[a, b, c]$ is a reduced form in $\mathscr{A}$, then $1/\beta_1'$ is the larger root of $cz^2 - bz + a = 0$ where $[c, b, a]$ is a reduced form in $\mathscr{A}'$. Since $N(\varepsilon) = -1$, $\theta\mathscr{A}' = \mathscr{A}'$. As a result, $1/\beta_1'$ has a continued fraction expansion of the form $[\overline{(m_1, \ldots, m_{\omega'})}]$ with $m_1 + \cdots + m_{\omega'} = p' + \omega'$. However we already know that

$$1/\beta_1' = [\overline{(n_\omega, \ldots, n_1)}]$$

where $n_1 + \cdots + n_\omega = p + \omega$. Therefore, $\omega = \omega'$ and $p = p'$; and, by Shintani's lemma applied to $1/\beta_1'$, $p + \omega = 3\omega$ or $p = 2\omega$ which is (3.2) in this case. □

Now that the structure of the pole set of $q_{\mathscr{A},k}(z)$ is determined, the following result is easily obtained.

COROLLARY 3.3. *Suppose that $q(z)$ is a nontrivial rational period function of positive weight $2k$ whose set of quadratic irrational poles is the irreducible system of poles $P(\mathscr{A})$ where $\mathscr{A} \neq \theta\mathscr{A}'$. Suppose also that*

$$\lim_{z\to\alpha_1} (z - a_1)^k q(z) = \lim_{z\to\alpha_1} (z - \alpha_1)^k q_{\mathscr{A},k}(z) \tag{3.3}$$

where α_1 is the smallest positive pole in $P(\mathscr{A})$. Then $q_{\mathscr{A},k}(z) - q(z)$ is a rational period function whose set of quadratic irrational poles is $P(\theta\mathscr{A}')$.

Proof. Since the quadratic irrational poles of a rational period function have order k (see [A], [H]), $q_{\mathscr{A},k}(z) - q(z)$ is a rational period function which no longer has a pole at α_1. However, since $P(\mathscr{A})$ is minimal, $q_{\mathscr{A},k}(z) - q(z)$ no longer has poles at the remaining elements of $P(\mathscr{A})$. As a result, $q_{\mathscr{A},k}(z) - q(z)$ has quadratic irrational poles only at elements of $P(\theta\mathscr{A}')$. □

In essence, the preceding corollary says that if a rational period function exists with quadratic irrational poles only at elements of $P(\mathscr{A})$, $\mathscr{A} \neq \theta\mathscr{A}'$, then the rational period function, when normalized as in (3.3), is the sum of principal parts of $q_{\mathscr{A},k}(z)$ at elements of $P(\mathscr{A})$ with the possible addition of a rational function having a pole at zero or infinity. To take advantage of the observation, we write

$$q_{\mathscr{A},k}(z) = \sum_{\substack{a>0>c \\ [a,b,c]\in\mathscr{A}}} \frac{1}{(az^2 - bz + c)^k} - \sum_{\substack{a<0<c \\ [a,b,c]\in\mathscr{A}}} \frac{1}{(az^2 - bz + c)^k}$$

$$= \sum_{\substack{a>0>c \\ [a,b,c]\in\mathscr{A}}} \frac{1}{(az^2 - bz + c)^k}\Bigg|(I - T).$$

Set

$$r_{\mathscr{A},k}(z) = \sum_{\substack{a>0>c \\ [a,b,c]\in\mathscr{A}}} \frac{1}{(az^2 - bz + c)^k}.$$

We now use the partial fraction decomposition of each of the terms $(az^2 - bz + c)^{-k}$ to split $r_{\mathscr{A},k}(z)$ into two sums, one with poles in $P(\mathscr{A})$ and the other with poles in $P(\theta\mathscr{A}')$. Given $Q_1 = [a_1, b_1, c_1]$ and $Q_2 = [a_2, b_2, c_2]$ in $\mathscr{A}$ with $a_1 > 0 > c_1, a_2 > 0 > c_2, Q_1 \cong Q_2$ which implies by Lemma 5.2 of [CP] that

$$\frac{b_1 + \sqrt{D}}{2a_1} \sim \frac{b_2 + \sqrt{D}}{2a_2}.$$

As a result of this fact and Theorem 3.2 all the poles with "+" signs lie in one irreducible system of poles, call it $P(\mathscr{A})$, and their conjugates lie in the other irreducible system $P(\theta\mathscr{A}')$. Next note that

$$\frac{1}{(az^2 - bz + c)^k} = \frac{1}{a^k}\sum_{s=1}^{k}\binom{2k-1-s}{k-s}\frac{(\alpha'-\alpha)^{s-2k}(-1)^k}{(z-\alpha)^s} + \frac{1}{a^k}\sum_{s=1}^{k}\binom{2k-1-s}{k-s}\frac{(\alpha-\alpha')^{s-2k}(-1)^k}{(z-\alpha')^s}$$

where

$$\alpha = \frac{b+\sqrt{D}}{2a} \text{ and } \alpha' = \frac{b-\sqrt{D}}{2a}.$$

Set

$$q_1(z) = \sum_{\substack{a>0>c\\ [a,b,c]\in\mathscr{A}}} \frac{1}{a^k}\sum_{s=1}^{k}\binom{2k-1-s}{k-s}\frac{(\alpha'-\alpha)^{s-2k}(-1)^k}{(z-\alpha)^s}$$

and

$$q_2(z) = \sum_{\substack{a>0>c\\ [a,b,c]\in\mathscr{A}}} \frac{1}{a^k}\sum_{s=1}^{k}\binom{2k-1-s}{k-s}\frac{(\alpha-\alpha')^{s-2k}(-1)^k}{(z-\alpha')^s}.$$

Then

$$\begin{aligned} q_{\mathscr{A},k}(z) &= (q_1(z) + q_2(z))|(I-T) \\ &= q_1(z)|(I-T) + q_2(z)|(I-T) \end{aligned}$$

where $q_1(z)|(I-T)$ has poles at all points of $P(\mathscr{A})$ and possibly at zero and $q_2(z)|(I-T)$ has poles at all points of $P(\theta\mathscr{A}')$ and possibly at zero.

The following theorem is the basis of a constructive approach to the question of the existence of rational period functions of positive weight $2k$ whose quadratic irrational pole set is $P(\mathscr{A})$.

THEOREM 3.4. *There exists a rational period function of weight $2k$ with quadratic irrational poles only at points of $P(\mathscr{A})$, $\mathscr{A} \neq \theta\mathscr{A}'$, if and only if there exists a rational function $F_{\mathscr{A},k}$, which may have poles at* 0 *and* ∞, *such that*

(3.4) $$F_{\mathscr{A},k}|(I+T) = 0$$

and

$$F_{\mathscr{A},k}|(I + U + U^2) = H_{\mathscr{A},k} \tag{3.5}$$

where $H_{\mathscr{A},k} = (q_1|T - q_1)|(I + U + U^2)$.

Proof. Suppose first that such an $F_{\mathscr{A},k}$ exists. Set $q = q_1|(I - T) + F_{\mathscr{A},k}$. Then q is a rational function with poles at elements of $P(\mathscr{A})$ and possibly at 0 and ∞ and q satisfies

$$q|(I + T) = F_{\mathscr{A},k}|(I + T) = 0$$

and

$$q|(I + U + U^2) = -H_{\mathscr{A},k} + F_{\mathscr{A},k}|(I + U + U^2) = 0.$$

In other words, q is a rational period function.

Now suppose that there exists a rational period function q whose quadratic irrational pole set is $P(\mathscr{A})$ and that q is normalized as in (3.3). Then, by Corollary 3.3 and the decomposition of $q_{\mathscr{A},k}(z)$,

$$q = q_1|(I - T) + F_{\mathscr{A},k}$$

where $F_{\mathscr{A},k}$ is a rational function with possible poles at 0 and ∞. Since $q|(I + T) = 0$, $F_{\mathscr{A},k}|(I + T) = 0$; since $q|(I + U + U^2) = 0$, $0 = -H_{\mathscr{A},k} + F_{\mathscr{A},k}|(I + U + U^2)$. $F_{\mathscr{A},k}$ satisfies both (3.4) and (3.5). □

4. Reduction

According to Theorem 3.4 the existence of a rational period function of positive weight whose quadratic irrational pole set is $P(\mathscr{A})$ depends on solving two functional equations (3.4) and (3.5). Actually, (3.4) and (3.5) yield an overdetermined system of linear equations. The point of this section is to show that solving this system is equivalent to determining whether a specific rational function, depending on the weight and the class $\mathscr{A}$, is identically zero. What we have then is an algorithmic procedure for determining when a rational period function exists whose quadratic irrational pole set is the irreducible system corresponding to $\mathscr{A}$, $\mathscr{A} \neq \theta\mathscr{A}'$.

First, however, we need more information about $H_{\mathscr{A},k}$.

LEMMA 4.1. $H_{\mathscr{A},k} = (q_1|T - q_1)|(I + U + U^2)$ *satisfies*:
(i) $H_{\mathscr{A},k}|U = H_{\mathscr{A},k}$;
(ii) $H_{\mathscr{A},k}$ *is a rational function which may have poles only at* 0 *and* 1;
(iii) *The order of the poles at* 0 *and* 1 *is at most* $2k - 2$;
(iv) $H_{A,1} \equiv 0$.

Proof. From the definition of $H_{\mathscr{A},k}$ it is clear that $H_{\mathscr{A},k}|U = H_{\mathscr{A},k}$ and that $H_{\mathscr{A},k}$ may have poles at points of $P(\mathscr{A})$ in addition to poles at zero and one which are introduced under the action of the slash operator. On the other hand, since $q_1|T - q_1 = q_{\mathscr{A},k} + q_2|(I - T)$, $H_{\mathscr{A},k} = (q_1|T - q_1)|(I + U + U^2) = (q_2 - q_2|T)|(I + U + U^2)$. From this alternative expression for $H_{\mathscr{A},k}$, it is clear that $H_{\mathscr{A},k}$ may have poles at points of $P(\theta\mathscr{A}')$, zero and one. Since $P(\theta\mathscr{A}')$ and $P(\mathscr{A})$ are disjoint, the only possible poles of $H_{\mathscr{A},k}$ occur at zero and one.

To get the order of the poles at zero and one we need a more explicit representation for $H_{\mathscr{A},k}$. The only terms in the definition of $H_{\mathscr{A},k}$ which contribute to poles at zero and one are $q_1|T$, $q_1|TU$, $-q_1|U$, and $-q_1|U^2$. A rather tedious calculation which involves yet another partial fraction decomposition of each of these four expressions gives that

$$H_{\mathscr{A},k}(z) = \sum_{\substack{a>0>c \\ [a,b,c]\in\mathscr{A}}} \frac{1}{a^k} \sum_{s=1}^{k} \binom{2k-1-s}{k-s} (\alpha' - \alpha)^{s-2k} (-1)^k (-1)^s$$
$$\times \sum_{t=1}^{2k-s} \binom{2k-1-t}{2k-s-t} \left\{ \frac{1}{z^t} \left(\left(-\frac{1}{\alpha} \right)^{s+t-2k} - \left(-\frac{1}{\alpha-1} \right)^{s+t-2k} \right) \right.$$
$$\left. + \frac{1}{(z-1)^t} \left(\left(-\frac{1}{\alpha+1} \right)^{s+t-2k} - \left(-\frac{1}{\alpha} \right)^{s+t-2k} \right) \right\}.$$

It is now easy to check that the order of the pole at both zero and one is at most $2k - 2$. It is also clear that when $k = 1$ $H_{\mathscr{A},1}(z) \equiv 0$. □

Let us now look more closely at the form of solutions $F_{\mathscr{A},k}$ of (3.4) and (3.5). When $k = 1$, $H_{\mathscr{A},1} \equiv 0$; and the solutions to (3.4) and (3.5) are the rational period functions

$$F_{\mathscr{A},1}(z) = c_1 z^{-1} + c_2(1 - z^{-2}).$$

When $k > 1$, any two solutions will differ by a rational period function $q(z) = c(1 - z^{-2k})$. As a result, we normalize $F_{\mathscr{A},k}$ so that the coefficient of z^{-2k} is zero in the Laurent series expansion of $F_{\mathscr{A},k}$ about zero. Then the solution, if it exists, will be unique.

LEMMA 4.2. *Let $k \geq 2$. Suppose $F_{\mathscr{A},k}$ is a rational function with possible poles at zero and infinity, normalized so that the coefficient of z^{-2k} is zero in the Laurent series expansion about zero. Then, if $F_{\mathscr{A},k}$ satisfies*

$$(3.4) \qquad F_{\mathscr{A},k}|(I + T) = 0$$

and

$$F_{\mathcal{A},k}|(I + U + U^2) = H_{\mathcal{A},k}, \tag{3.5}$$

$F_{\mathcal{A},k}$ *is of the form*

$$F_{\mathcal{A},k}(z) = \sum_{s=1}^{k-1} A_s\left(\frac{1}{z^s} - \frac{(-1)^s}{z^{2k-s}}\right) + \frac{A_k}{z^k}$$

where $A_k = 0$ *if* k *is even.*

Proof. Write

$$F_{\mathcal{A},k}(z) = \sum_{j=m}^{n} \frac{A_j}{z^j}.$$

Since

$$F_{\mathcal{A},k}|(I + T) = \sum_{j=m}^{n} A_j\left(\frac{1}{z^j} + \frac{(-1)^j}{z^{2k-j}}\right)$$
$$= \sum_{j=m}^{n} A_j/z^j + \sum_{j=2k-n}^{2k-m} \frac{A_{2k-j}(-1)^j}{z^j},$$

if $F_{\mathcal{A},k}|(I + T) = 0$, $n + m = 2k$, $n \geq k$, and $A_{2k-j} = -(1)^j A_j$. In particular, $A_k = -(-1)^k A_k$ which implies that $A_k = 0$ if k is even. The fact that $F_{\mathcal{A},k}$ satisfies (3.5) can be used to show that $m \geq 1$ since the terms corresponding to $j = m$ and $j = 2k - m$ given rise, under the action of the slash operator, to a pole of order $2k - m - 1$ at zero. However, $H_{\mathcal{A},k}$ has a pole at zero of order at most $2k - 2$; and $m \geq 1$. □

It now remains to write down the system of equations to be satisfied by the coefficients of $F_{\mathcal{A},k}$ if $F_{\mathcal{A},k}$ is to satisfy both (3.4) and (3.5). It is convenient to isolate part of the calculation in the next lemma.

LEMMA 4.3. *If*

$$F_{\mathcal{A},k}(z) = \sum_{s=1}^{k-1} A_s\left(\frac{1}{z^s} - \frac{(-1)^s}{z^{2k-s}}\right) + \frac{A_k}{z^k}, \quad k \geq 2,$$

with $A_k = 0$ if k is even, then

$$(4.1)\quad F_{\mathscr{A},k}|(I+U+U^2)$$
$$= \sum_{r=1}^{k-2} \frac{1}{z^r}\left(\sum_{s=1}^{k} A_s(-1)^s \binom{2k-1-r}{2k-s-r} - \sum_{s=r+1}^{k-1} A_s \binom{2k-1-r}{s-r}\right)$$
$$+ \sum_{r=k-1}^{2k-2} \frac{1}{z^r}\left(\sum_{s=1}^{2k-r-1} A_s(-1)^s \binom{2k-1-r}{2k-s-r}\right)$$
$$+ \sum_{r=1}^{k-2} \frac{1}{(z-1)^r}\left(-\sum_{s=1}^{k} A_s(-1)^r \binom{2k-1-r}{2k-s-r}\right.$$
$$\left. + \sum_{s=r+1}^{k-1} A_s(-1)^{r+s} \binom{2k-1-r}{s-r}\right)$$
$$+ \sum_{r=k-1}^{2k-2} \frac{1}{(z-1)^r}\left(-\sum_{s=1}^{2k-r-1} A_s(-1)^r \binom{2k-1-r}{2k-s-r}\right).$$

Proof. First note that

$$(4.2)\quad F_{\mathscr{A},k}|(I+U+U^2)$$
$$= \sum_{s=1}^{k-1} A_s\left(\frac{1}{z^s} - \frac{(-1)^s}{z^{2k-s}}\right) + \frac{A_k}{z^k}$$
$$+ \sum_{s=1}^{k-1} A_s\left(\frac{1}{z^{2k-s}(z-1)^s} - \frac{(-1)^s}{z^s(z-1)^{2k-s}}\right) + \frac{A_k}{z^k(z-1)^k}$$
$$+ \sum_{s=1}^{k-1} A_s\left(\frac{(-1)^s}{(z-1)^{2k-s}} - \frac{1}{(z-1)^s}\right) + \frac{A_k(-1)^k}{(z-1)^k}.$$

But,

$$\frac{1}{z^{2k-s}(z-1)^s} = \sum_{r=1}^{2k-s} \binom{2k-1-r}{2k-s-r}\frac{(-1)^s}{z^r} + \sum_{r=1}^{s} \binom{2k-1-r}{s-r}\frac{(-1)^{r+s}}{(z-1)^r},$$
$$\frac{(-1)^s}{z^s(z-1)^{2k-s}} = \sum_{r=1}^{2k-s} \binom{2k-1-r}{2k-s-r}\frac{(-1)^r}{(z-1)^r} + \sum_{r=1}^{s} \binom{2k-1-r}{s-r}\frac{1}{z^r},$$

and

$$\frac{1}{z^k(z-1)^k} = \sum_{r=1}^{k} \binom{2k-1-r}{k-r}\left(\frac{(-1)^k}{z^r} + \frac{(-1)^{k+r}}{(z-1)^r}\right).$$

Substituting these expressions into (4.2) and collecting like terms yields (4.1). □

When $k \geq 3$, for k odd consider the system of equations

$$(4.3)\qquad \sum_{s=1}^{2k-r-1} A_s(-1)^s \binom{2k-1-r}{2k-s-r} = B_r, \quad r = k-1, \dots, 2k-2$$

where the B_r are the coefficients of z^{-r} in

$$H_{\mathscr{A},k}(z) = \sum_{r=1}^{2k-2} \frac{Br}{z^r} + \frac{C_r}{(z-1)^r}.$$

Since this is a triangular system in the variables A_s, $s = 1, \dots, k$ with non-zero determinant, it is possible to solve it. Denote the solutions by $A_1^*, A_2^*, \dots, A_k^*$. Denote by $F_{\mathscr{A},k}^*$ the function

$$F_{\mathscr{A},k}^*(z) = \sum_{s=1}^{k-1} A_s^* \left(\frac{1}{z^s} - \frac{(-1)^s}{z^{2k-s}} \right) + \frac{A_k^*}{z^k}$$

and set

$$H_{\mathscr{A},k}^*(z) = F_{\mathscr{A},k}^* | (I + U + U^2).$$

From Lemma 4.3 it is clear that $H_{\mathscr{A},k}^*$ is of the form

$$H_{\mathscr{A},k}^*(z) = \sum_{r=1}^{2k-2} \frac{B_r^*}{z^r} + \frac{C_r^*}{(z-1)^r}$$

with $B_r^* = B_r$ when $r = k-1, \dots, 2k-2$.

Similarly, when $k \geq 2$, k even, let $A_1^*, \dots, A_{k-1}^*$ be the solutions of

$$(4.4)\qquad \sum_{s=1}^{2k-r-1} A_s(-1)^s \binom{2k-1-r}{2k-s-r} = B_r, \quad r = k, \dots, 2k-2.$$

Set

$$F_{\mathscr{A},k}^*(z) = \sum_{s=1}^{k-1} A_s^* \left(\frac{1}{z^s} - \frac{(-1)^s}{z^{2k-s}} \right)$$

and

$$H^*_{\mathscr{A},k}(z) = F^*_{\mathscr{A},k}|(I + U + U^2)$$
$$= \sum_{r=1}^{2k-2} \frac{B^*_r}{z^r} + \frac{C^*_r}{(z-1)^r}$$

so that $B^*_r = B_r$ for $r = k, \dots, 2k-2$.

THEOREM 4.4. *For $k \geq 2$ there exists a rational period function $q(z)$ of weight $2k$ whose set of quadratic irrational poles is the irreducible system of poles $P(\mathscr{A})$, $\mathscr{A} \neq \theta\mathscr{A}'$, if, and only if, $H_{\mathscr{A},k} - H^*_{\mathscr{A},k} \equiv 0$.*

Proof. If $H_{\mathscr{A},k} - H^*_{\mathscr{A},k} \equiv 0$, then $F^*_{\mathscr{A},k}$ satisfies both (3.4) and (3.5) and, by Theorem 3.4, such a rational period function exists.

On the other hand, if the rational period function $q(z)$ exists, then by Theorem 3.4 and Lemma 4.2 there exists $F_{\mathscr{A},k}$ of the form

$$F_{\mathscr{A},k}(z) = \sum_{s=1}^{k-1} A_s\left(\frac{1}{z^s} - \frac{(-1)^s}{z^{2k-s}}\right) + \frac{A_k}{z^k}, \quad A_k = 0 \text{ if } k \text{ is even},$$

satisfying $F_{\mathscr{A},k}|(I + U + U^2) = H_{\mathscr{A},k}$. By Lemma 4.3 for k odd the coefficients $A_1, \dots, A_k$ are determined by (4.3) and for k even the coefficients $A_1, \dots, A_{k-1}$ are determined by (4.4). Hence, $F_{\mathscr{A},k} = F^*_{\mathscr{A},k}$ and $H_{\mathscr{A},k} = H^*_{\mathscr{A},k}$. □

Remark. Theorem 4.4 gives an algorithmic procedure for determining whether or not there exists a rational period function of weight $2k$, $k \geq 2$, whose quadratic irrational pole set is the irreducible system of poles $P(\mathscr{A})$. Given the pole set, one can easily find $q_{\mathscr{A},k}(z)$ and then $q_1(z)$. Given $q_1(z)$ one calculates $H_{\mathscr{A},k}$ from which one can find $F^*_{\mathscr{A},k}$ and then $H^*_{\mathscr{A},k}$.

5. Obstructions

For $k \geq 2$, set $K_{\mathscr{A},k}(z) = H_{\mathscr{A},k}(z) - H^*_{\mathscr{A},k}(z)$. Whenever $K_{\mathscr{A},k}(z)$ is not identically zero, no rational period function exists whose quadratic irrational role set is $P(\mathscr{A})$. Since these functions $K_{\mathscr{A},k}(z)$ block the existence of rational period functions, it is entirely appropriate to call them obstructions. When will obstructions occur? The answer, interestingly enough, is that obstructions may occur whenever the space of cusp forms of weight $2k$ is nontrivial.

LEMMA 5.1. *$K_{\mathscr{A},k}(z)$ has the following properties.*

(5.1) $K_{\mathscr{A},k}|U = K_{\mathscr{A},k}$

(5.2) *$K_{\mathscr{A},k}(z)$ is a rational function with possible poles only at 0 and 1. The poles at 0 and 1 are both of the same order.*

(5.3) *If $K_{\mathscr{A},k}(z)$ has a pole at 0, its order is at most $k - 2$ if k is odd and $k - 1$ is k is even.*

Proof. (5.1) is obvious. (5.3) follows immediately from the definition of $H^*_{\mathscr{A},k}(z)$. That the poles at 0 and 1 have the same order comes from $K_{\mathscr{A},k}|U = K_{\mathscr{A},k}$. □

We now define a space of function which contains all of the functions $K_{\mathscr{A},k}(z)$, $\mathscr{A} \neq \theta\mathscr{A}'$. For $k \geq 2$, let $R_k = \{K(z)$: $K(z)$ satisfies (5.1), (5.2), and (5.3)$\}$. The next theorem shows that the space R_k is trivial when $k = 2$, 3, 4, 5 and 7. It then follows from Theorem 3.4 that for these values of k there will always exist a rational period function with $P(\mathscr{A})$ as its quadratic irrational pole set. The rational period function is then given by $q(z) = q_1(z)|(I - T) + F^*_{\mathscr{A},k}(z)$.

THEOREM 5.2. *$R_k = \{0\}$ if $k = 2, 3, 4, 5$ or 7.*

Proof. If $K(z) \in R_k$, then

$$K(z) = \sum_{j=1}^{k-2} \frac{B_j}{z^j} + \frac{C_j}{(z-1)^j} \quad \text{if } k \text{ is odd}$$

and

$$K(z) = \sum_{j=1}^{k-1} \frac{B_j}{z^j} + \frac{C_j}{(z-1)^j} \quad \text{if } k \text{ is even.}$$

Since $K|U = K$, by comparing the principal parts of K and $K|U$ at zero and one, we get equations relating the coefficients B_j and C_j. For instance, when $k = 4$,

$$K(z) = \sum_{j=1}^{2} \frac{B_j}{z^j} + \frac{C_j}{(z-1)^j}$$

and

$$K|U = \sum_{j=1}^{2} \frac{B_j}{(z-1)^j z^{8-j}} + \frac{C_j(-1)^j}{z^{8-j}}$$

$$= \sum_{j=1}^{2} B_j \left(\sum_{r=1}^{8-j} \binom{7-r}{8-j-r} \frac{(-1)^j}{z^r} + \sum_{r=1}^{j} \binom{7-r}{j-r} \frac{(-1)^{r+j}}{(z-1)^r} \right)$$

$$+ \sum_{j=1}^{2} \frac{C_j(-1)^j}{z^{8-j}}$$

$$= \sum_{r=1}^{5} \frac{-B_1 + (7-r)B_2}{z^r} + \frac{1}{z^6}(-B_1 + B_2 + C_2) + \frac{1}{z^7}(-B_1 - C_1)$$

$$+ \frac{B_1 - 6B_2}{z-1} + \frac{B_2}{(z-1)^2}.$$

Since $K|U = K$,

$$-B_1 - C_1 = 0$$
$$-B_1 + B_2 + C_2 = 0$$
$$-B_1 + 2B_2 = 0$$
$$-B_1 + 3B_2 = 0$$
$$B_2 = C_2.$$

The only solution to this system of equations is $B_1 = B_2 = C_1 = C_2 = 0$. A similar exercise in solving simultaneous equations shows that $K(z) \equiv 0$ if $k = 2, 3, 5$ or 7. □

The next theorem gives more information on the structure of the space R_k. The connection with the space S_{2k} of cusp forms of weight $2k$ has also been noticed by Ash [A].

THEOREM 5.3. *$R_k \cong R_k^+ \oplus R_k^-$ where*

$$R_k^+ = \{K \in R_k : K(1-z) = K(z)\}$$
$$R_k^- = \{K \in R_k : K(1-z) = -K(z)\}.$$

Also, $R_k^+ \cong R_k^- \cong S_{2k}$. *In particular*,

$$\dim R_k = \begin{cases} 2\left[\dfrac{k}{6}\right] & \text{if } k \not\equiv 1 \pmod 6 \\ 2\left[\dfrac{k}{6}\right] - 2 & \text{if } k \equiv 1 \pmod 6). \end{cases} \tag{5.4}$$

Proof. We first show that (5.4) holds. Set $R_k^* = \{K\colon K \in R_k$ and the pole at zero has order $\le k-3$ if k is even and order $\le k-4$ if k is odd$\}$,

$$K_k^+(z) = \begin{cases} \dfrac{1}{z^4(z-1)^{k-2}} + \dfrac{1}{z^{k-2}(z-1)^4} + \dfrac{1}{z^{k-2}(z-1)^{k-2}} & \text{if } k \text{ is even,} \\ \dfrac{1}{z^6(z-1)^{k-3}} + \dfrac{1}{z^{k-3}(z-1)^6} + \dfrac{1}{z^{k-3}(z-1)^{k-3}} & \text{if } k \text{ is odd} \end{cases}$$

and

$$K_k^-(z) = \begin{cases} (2z^3 - 3z^2 - z + 2)\left(\dfrac{1}{z^5(z-1)^{k-1}} + \dfrac{1}{z^{k-1}(z-1)^5} + \dfrac{1}{z^{k-1}(z-1)^{k-1}}\right) & \text{if } k \text{ is even} \\ (2z^3 - 3z^2 - 3z + 2)\left(\dfrac{1}{z^7(z-1)^{k-2}} + \dfrac{1}{z^{k-2}(z-1)^7} + \dfrac{1}{z^{k-2}(z-1)^{k-2}}\right) & \text{if } k \text{ is odd.} \end{cases}$$

Note that if k is even, $k \ge 6$, $K_k^+ \in R_k^+$ with a pole at zero of order $k-2$ whereas $K_k^- \in R_k^-$ with a pole at zero of order $k-1$. If k is odd, $k \ge 9$, $K_k^+ \in R_k^+$ with a pole at zero of order $k-3$ whereas $K_k^- \in R_k^-$ with a pole at zero of order $k-2$. As a result, for $k \ge 8$ it is easy to see that

$$R_k = R_k^* \oplus \mathbf{C} \cdot K_k^+ \oplus \mathbf{C} \cdot K_k^-. \tag{5.5}$$

The next observation is that for $k \ge 8 R_{k-6} \cong R_k^*$ where the linear transformation

$$\varphi\colon R_{k-6} \to R_k^*$$

defined by $\varphi(K) = K_6^+ K$ establishes the isomorphism. Since $K_6^+ \in R_6$, φ maps one-to-one into R_k^*; and it suffices to check that φ is onto. In other words, given $K^* \in R_k^*$, $K^* \not\equiv 0$, it suffices to show that $K = K^*/K_6^+$ belongs to R_{k-6}. Since $K|_{2k-12}U = K$, we need to check that K has poles only at zero and one whose order is of the appropriate size. Since $K^* \in R_k^*$,

$K(z)$ can be written as

$$K(z) = \frac{P(z)}{z^{m-4}(z-1)^{m-4}}$$

where $P(z)$ is a polynomial of degree at most $2m$ and $P(0) \neq 0$, $P(1) \neq 0$. In addition, since $K|_{2k-12}U = K$, $z^{3m-2k}(-1)^m P(1 - 1/z) = P(z)$. If $P(z) = a_0 + a_1 z + \cdots + a_s z^5$,

$$z^{2m-2k}(-1)^m P\left(\frac{z-1}{z}\right) = (-1)^m P(1) z^{3m-2k} + \cdots + a_s(-1)^{s+m} z^{3m-2k-s};$$

and it follows that $s = 3m - 2k$. Then, since $k \geq 8, m \geq 6$ and it is now easy to check that the only poles of $K(z)$ occur at zero and one and that the order of the poles is sufficiently small.

Finally, it follows from (5.5) that for $k \geq 8 \dim R_k = \dim R_{k-6} + 2$. For $k = 2$, 3, 4, 5 and 7 $R_k = \{0\}$ whereas an easy calculation in the spirit of Theorem 5.2 shows that R_6 has dimension two. In particular, (5.4) holds for $k = 2, \ldots, 7$. The since the formula in (5.4) and $\dim R_k$ both increase by two as k increases by six, (5.4) holds for all $k \geq 2$.

We now show that $R_k^+ \cong R_k^- \cong S_{2k}$ from which it follows that $R_k = R_k^+ \oplus R_k^-$. From (5.5) for $k \geq 8$ we have

$$R_k^+ = (R_k^+)^* \oplus \mathbf{C}K_k^+ \quad \text{and} \quad R_k^- = (R_k^-)^* \oplus \mathbf{C}K_k^-$$

where $(R_k^+)^* = R_k^+ \cap R_k^*$ and $(R_k^-)^* = R_k^- \cap R_k^*$. Multiplication by K_6^+ shows that for $k \geq 8$,

$$(R_k^+)^* \cong K_6^+ R_{k-6}^+ \quad \text{and} \quad (R_k^-)^* \cong K_6^+ R_{k-6}^-.$$

As a result,

$$R_k^+ \cong K_6^+ R_{k-6}^+ \oplus \mathbf{C}K_k^+,$$
$$R_k^- \cong K_6^+ R_{k-6}^- \oplus \mathbf{C}K_k^-.$$

Recall that for $k \geq 8$,

$$S_{2k} \cong \Delta S_{2k-12} \oplus \mathbf{C} \cdot \Delta G_{2k-12}$$

where Δ is the famous discriminant function of weight twelve and G_{2k-12} is the Eisenstein series of weight $2k - 12$. Since $R_k^+ \cong R_k^- \cong S_{2k}$ if $2 \leq k \leq 7$, an inductive argument, with K_6^+ mapping to Δ, now shows that $R_k^+ \cong R_k^- \cong S_{2k}$ for $k \geq 2$. □

COROLLARY 5.3. *Given* $N = \dim R_k + 1$ *classes* $\mathscr{A}_1, \ldots, \mathscr{A}_N$ *with* $\mathscr{A}_i \neq \theta\mathscr{A}_i'$, *there exists a nontrivial rational period function of weight* $2k$ *whose pole set is contained in* $P(\mathscr{A}_1) \cup \cdots \cup P(\mathscr{A}_N) \cup \{0\}$.

Proof. Since $K_{\mathscr{A}_i, k} \in R_k$, $i = 1, \ldots, N$ and $N > \dim R_k$, there exist constants $c_1, \ldots, c_N$, not all zero, such that $\sum_{i=1}^N c_i K_{\mathscr{A}_{ij}k} = 0$. Now set

$$q = \sum_{i=1}^{N} c_i q_1^i | (I - T) + \sum_{i=1}^{N} c_i F^*_{\mathscr{A}_i, k}$$

where q_1^i is the rational function derived from $q_{\mathscr{A}_i, k}$ with poles at elements of $P(\mathscr{A}_i)$. Then $q(z)$ is a nontrivial rational period function of weight $2k$ since $q|(I + T) = 0$ and

$$\begin{aligned} q|(I + U + U^2) &= -\sum_{i=1}^{N} c_i H_{\mathscr{A}_i, k} + \sum_{i=1}^{N} c_i H^*_{\mathscr{A}_i, k} \\ &= -\sum_{i=1}^{N} c_i K_{\mathscr{A}_i, k} = 0. \end{aligned}$$

□

Remark. It is interesting to note that the classes in Corollary 5.3 may be chosen so that the irreducible systems of poles lie in distinct real quadratic fields.

6. Examples of obstructions

Obstructions to the existence of rational period functions with quadratic irrational poles in a single irreducible system of poles do occur. Here we give an infinite family of classes $\mathscr{A}$ with $\mathscr{A} = \mathscr{A}'$ but $\mathscr{A} \neq \theta\mathscr{A}'$ for which there is no rational period function of weight twelve with quadratic irrational poles lying only in the irreducible system $P(\mathscr{A})$. We begin with some general results which are of independent interest.

THEOREM 6.1. *Suppose* $\mathscr{A} = \mathscr{A}'$ *but* $\mathscr{A} \neq \theta\mathscr{A}'$. *Then for* $k \geq 2$,

$$H_{\mathscr{A}, k}(1 - z) = H_{\mathscr{A}, k}(z) \tag{6.1}$$

and, if

$$F_{\mathscr{A}, k}(z) = \sum_{s=1}^{k-1} A_s \left(\frac{1}{z^s} - \frac{(-1)^s}{z^{2k-s}} \right) + \frac{A_k}{z^k}$$

satisfies $F_{\mathscr{A},k}|(I+U+U^2)=H_{\mathscr{A},k}$, *then*

$$A_s = 0 \text{ whenever } s \text{ is even.} \tag{6.2}$$

Proof. We begin by proving (6.1). In Lemma 4.1 it was shown that

$$\begin{aligned}(6.3)\quad & H_{\mathscr{A},k}(z)\\ &= \sum_{\substack{a>0>c\\ [a,b,c]\in\mathscr{A}}} \frac{1}{a^k}\sum_{s=1}^{k}\binom{2k-1-s}{k-s}(\alpha'-\alpha)^{s-2k}(-1)^k(-1)^s\\ &\quad\times\sum_{t=1}^{2k-s}\binom{2k-1-t}{2k-s-t}\left\{\frac{1}{z^t}\left(\left(-\frac{1}{\alpha}\right)^{s+t-2k}-\left(\frac{-1}{\alpha-1}\right)^{s+t-2k}\right)\right.\\ &\quad\left.+\frac{1}{(z-1)^t}\left(\left(-\frac{1}{\alpha+1}\right)^{s+t-2k}-\left(-\frac{1}{\alpha}\right)^{s+t-2k}\right)\right\}.\end{aligned}$$

This was derived using the formula $H_{\mathscr{A},k}=(q_1|T-q_1)|(I+U+U^2)$. An alternative expression can be derived using the fact that $H_{\mathscr{A},k} = (q_2-q_2|T)|(I+U+U^2)$; one merely interchanges the role of α and α' and introduces a minus sign to get

$$\begin{aligned}(6.4)\quad & H_{\mathscr{A},k}(z)\\ &= -\sum_{\substack{a>0>c\\ [a,b,c]\in\mathscr{A}}} \frac{1}{a^k}\sum_{s=1}^{k}\binom{2k-1-s}{k-s}(\alpha'-\alpha)^{s-2k}(-1)^k\\ &\quad\times\sum_{t=1}^{2k-s}\binom{2t-1-t}{2k-s-t}\left\{\frac{1}{z^t}\left(\left(-\frac{1}{\alpha'}\right)^{s+t-2k}-\left(\frac{-1}{\alpha'-1}\right)^{s+t-2k}\right)\right.\\ &\quad\left.+\frac{1}{(z-1)^t}\left(\left(-\frac{1}{\alpha'+1}\right)^{s+t-2k}-\left(-\frac{1}{\alpha'}\right)^{s+t-2k}\right)\right\}.\end{aligned}$$

On the other hand, since $\mathscr{A}=\mathscr{A}'$, using (6.3) and summing initially over $[a,-b,c]$ but still calling

$$\alpha=\frac{b+\sqrt{D}}{2a}$$

gives yet another expression for $H_{\mathscr{A},k}$:

(6.5) $H_{\mathscr{A},k}(z)$

$$= \sum_{\substack{a>0>c \\ [a,b,c]\in\mathscr{A}}} \frac{1}{a^k} \sum_{s=1}^{k} \binom{2k-1-s}{k-s} (\alpha'-\alpha)^{s-2k}(-1)^k$$

$$\times \sum_{t=1}^{2k-s} \binom{2k-1-t}{2k-s-t} \left\{ \frac{(-1)^t}{z^t} \left(\left(-\frac{1}{\alpha'}\right)^{s+t-2k} - \left(\frac{-1}{\alpha'-1}\right)^{s+t-2k} \right) \right.$$

$$\left. + \frac{(-1)^t}{(z-1)^t} \left(\left(-\frac{1}{\alpha'-1}\right)^{s+t-2k} - \left(-\frac{1}{\alpha'}\right)^{s+t-2k} \right) \right\}.$$

It is now clear from (6.4) and (6.5) that $H_{\mathscr{A},k}(1-z) = H_{\mathscr{A},k}(z)$.

Next, since $H_{\mathscr{A},k}(1-z) = H_{\mathscr{A},k}(z)$, $H_{\mathscr{A},k}(z)$ can be written as

$$H_{\mathscr{A},k}(z) = \sum_{r=1}^{2k-2} Br\left(\frac{1}{z^r} + \frac{(-1)^r}{(z-1)^r} \right).$$

Then if $F_{\mathscr{A},k}|(I+U+U^2) = H_{\mathscr{A},k}$, it follows from (4.1) by looking first at the principal part at zero and then at the principal part at one, that for $k-1 \le r \le 2k-2$,

$$\sum_{s=1}^{2k-r-1} A_s(-1)^s \binom{2k-1-r}{2k-s-r} = B_r$$

and

$$\sum_{s=1}^{2k-r-1} A_s \binom{2k-1-r}{2k-s-r} = B_r.$$

It is now immediate that $A_s = 0$ if s is even. □

COROLLARY 6.2. *If* $\mathscr{A} = \mathscr{A}'$, $\mathscr{A} \neq \theta\mathscr{A}'$, *then* $K_{\mathscr{A},k}(z) = H_{\mathscr{A},k}(z) - H^*_{\mathscr{A},k}(z) \in R_k^+$.

Proof. Since we already know that $H_{\mathscr{A},k}(z)$ satisfies $H_{\mathscr{A},k}(1-z) = H_{\mathscr{A},k}(z)$, it suffices to check that

$$H^*_{\mathscr{A},k}(1-z) = H^*_{\mathscr{A},k}(z).$$

Recall that

$$H^*_{\mathscr{A},k}(z) = \sum_{r=1}^{2k-2} \frac{B^*_r}{z^r} + \frac{C^*_r}{(z-1)^r} = F^*_{\mathscr{A},k}|(I+U+U^2)$$

where

$$F^*_{\mathscr{A},k}(z) = \sum_{s=1}^{k-1} A^*_s\left(\frac{1}{z^s} - \frac{(-1)^s}{z^{2k-2}}\right) + \frac{A^*_k}{z^k}$$

with, for k odd, $A^*_s, s = 1,\dots,k$, determined by

$$\sum_{s=1}^{2k-r-1} A^*_s(-1)^s \binom{2k-1-r}{2k-s-r} = B_r \tag{6.6}$$

for $r = k-1,\dots,2k-2$ and, for k even, $\mathscr{A}^*_k = 0$ and $\mathscr{A}^*_s, s = 1,\dots, k-1$ are given by (6.6) for $r = k,\dots,2k-2$. In particular, $B_r = B^*_r$ for $r = p,\dots,2k-2$ with $p = k-1$ if k is odd and $p = k$ if k is even. Therefore,

$$K_{\mathscr{A},k}(z) = \sum_{r=1}^{p-1} \frac{B_r - B^*_r}{z^r} + \sum_{r=1}^{2k-2} \frac{B_r(-1)^r - C^*_r}{(z-1)^r}.$$

However, by Lemma 5.1 the order of the pole of $K_{\mathscr{A},k}(z)$ at one is the same as the order of the pole at zero. As a result, $C^*_r = B_r(-1)^r, p \le r \le 2k-2$. Then by Lemma 4.3, for $p \le r \le 2k-2$,

$$B^*_r = B_r = \sum_{s=1}^{2k-r-1} A^*_s(-1)^s \binom{2k-1-r}{2k-s-r},$$

$$C^*_r = (-1)^r B_r = -\sum_{s=1}^{2k-r-1} A^*_s(-1)^s \binom{2k-1-r}{2k-s-r}$$

and $A^*_s = 0$ if s is even. It is then clear from (4.1) that

$$H^*_{\mathscr{A},k}(1-z) = H^*_{\mathscr{A},k}(z)$$

since $H^*_{\mathscr{A},k} = F^*_{\mathscr{A},k}|(I+U+U^2)$. □

It follows from the preceding corollary and Theorem 5.3 that when $k = 6$ there is one potential obstruction to the existence of a rational period function of weight twelve with quadratic irrational poles at $P(\mathscr{A})$, where $\mathscr{A} = \mathscr{A}'$, $\mathscr{A} \neq \theta\mathscr{A}'$. As a result, the necessary and sufficient condition for the existence of such a rational period function can be rephrased as a single identity for the coefficients of $H_{\mathscr{A},k}$ in terms of a sum over the class $\mathscr{A}$.

LEMMA 6.3. *Let* $\mathscr{A} = \mathscr{A}'$, $\mathscr{A} \neq \theta\mathscr{A}'$, $H_{\mathscr{A},6}(z) = \sum_{r=1}^{10} B_r(1/z^r + (-1)^r/(z-1)^r)$. *Then*

$$B_{10} = \frac{-252}{D^{11/12}} \sum_{\substack{a>0>c\\ [a,b,c]\in\mathscr{A}}} a^5,$$

$$B_8 = \frac{21}{D^{11/2}} \sum_{\substack{a>0>c\\ [a,b,c]\in\mathscr{A}}} (Da^3 - 9a^3b^2 - 12a^5),$$

$$B_6 = \frac{1}{D^{11/2}} \sum_{\substack{a>0>c\\ [a,b,c]\in\mathscr{A}}} \left(\frac{-315}{4}ab^4 + \frac{105}{2}Dab^2 - \frac{15}{4}D^2a - 630a^3b^2 + 70Da^3 - 151a^5\right),$$

$$B_4 = \frac{1}{D^{11/2}} \sum_{\substack{a>0>c\\ [a,b,c]\in\mathscr{A}}} \left(-\frac{441}{16}\frac{b^6}{a} + \frac{735}{16}D\frac{b^4}{a} - \frac{315}{16}\frac{D^2b^2}{a} + \frac{21}{16}\frac{D^3}{a} - \frac{2205}{4}ab^4 + \frac{735}{2}Dab^2 - \frac{105}{4}D^2a - 1323a^3b^2 + 147Da^3 - 252a^5\right)$$

Proof. It follows from (6.3) that

$$B_r = \frac{1}{D^6} \sum_{\substack{a>0>c\\ [a,b,c]\in\mathscr{A}}} a^6 \sum_{s=1}^{6} \binom{11-s}{6-s}\binom{11-r}{12-s-r}\left(\frac{\sqrt{D}}{a}\right)^2 \times\left((-\alpha)^{12-s-r} - (1-\alpha)^{12-s-r}\right).$$

These expressions are then simplified by replacing α with $(b+\sqrt{D})/2a$ and noting that since $[a,b,c]$ and $[a,-b,c]$ are both in $\mathscr{A}$, sums containing odd powers of b are zero. □

THEOREM 6.4. *Let* $\mathscr{A} = \mathscr{A}'$, $\mathscr{A} \neq \theta\mathscr{A}'$, *and*

$$H_{\mathscr{A},6}(z) = \sum_{r=1}^{10} B_r\left(\frac{1}{z^r} + \frac{(-1)^r}{(z-1)^r}\right).$$

There exists a rational period function of weight twelve whose set of quadratic irrational poles is $P(\mathscr{A})$ *iff*

$$68B_{10} - 105B_8 + 42B_6 - 5B_4 = 0 \tag{6.6}$$

iff

$$\sum_{\substack{a>0>c \\ [a,b,c]\in\mathscr{A}}} (c-a)(b^4 + 6ab^2c + 2a^2c^2) = 0. \tag{6.7}$$

Proof. If such a rational period function exists, by Theorems 3.4 and 6.1 and Lemmas 4.2 and 4.3 there exists

$$F_{\mathscr{A},6}(z) = \sum_{s=1}^{5} A_s\left(\frac{1}{z^s} - \frac{(-1)^s}{z^{12-s}}\right)$$

with $A_2 = A_4 = 0$,

$$\sum_{s=1}^{11-r} A_s(-1)^s\binom{11-r}{12-s-r} = B_r,$$

$6 \le r \le 10$, and

$$\sum_{s=1}^{5} A_s(-1)^s\binom{7}{8-s} - 7A_5 = B_4. \tag{6.8}$$

As a result, $A_1 = -B_{10}$, $A_3 = \frac{1}{3}(B_{10} - B_8)$, and $A_5 = \frac{1}{15}(-7B_{10} + 10B_8 - 3B_6)$. Substituting into (6.8) and simplifying yields (6.6).

Now assume that (6.6) holds. We will show that $K_{\mathscr{A},6}(z) \equiv 0$ from which it follows by Theorem 4.4 that the desired rational period function exists. From Theorem 5.3 and Corollary 6.2 we know that

$$K_{\mathscr{A},6}(z) = \frac{c}{z^4(z-1)^4}$$

for some $c \in \mathbf{C}$. On the other hand we also know that

$$\lim_{z\to 0} z^4 K_{\mathscr{A},6}(z) = B_4 - B_4^*$$

so that $c = B_4 - B_4^*$. From the definition of $H^*_{\mathscr{A},6}(z)$ it follows that

$$B_4^* = \frac{68}{5}B_{10} - 21B_8 + \frac{42}{5}B_6.$$

Then

$$5c = 5(B_4 - B_4^*) = 5B_4 - 68B_{10} + 105B_8 - 42B_6 = 0$$

by (6.6). Finally that (6.6) and (6.7) are equivalent follows from Lemma 6.3. □

Using Theorem 6.4 it is now easy to check, given a class $\mathscr{A}$, whether or not an obstruction to the existence of a rational period function of weight twelve occurs. For the following infinite family of classes obstructions do occur. Set $D_t = t^2 + 4t, t = 2, 3, \ldots$ and let $\mathscr{A}_t$ be the narrow equivalence class whose cycle of reduced forms is given by $[a_p, b_p, c_p], 0 \le p \le t-1$ where

$$a_p = (p+1)t - p^2,$$

$$b_p = (2p+3)t - 2(p^2+p),$$

$$c_p = (p+2)t - (p+1)^2.$$

Then $A_t = A_t'$ but $A_t \neq \theta\mathscr{A}_t'$ since there is exactly one reduced form in $\theta\mathscr{A}_t'$, namely $[1, t+2, 1]$. This family was studied extensively by Kramer in [Kr] who showed that the cusp forms $f_{k, D_t, \mathscr{A}_t}$ span S_{2k}. The same classes show up implicitly in [C].

COROLLARY 6.5. *Let D_t and $\mathscr{A}_t$ be defined as above. There is no rational period function of weight twelve whose set of quadratic irrational poles is $P(\mathscr{A}_t)$.*

Proof. For notational convenience set $F(a, b, c) = (c-a)(b^4 + 6ab^2c + 2a^2c^2)$. By Lemma 3.1 of [CP],

$$\sum_{\substack{a>0>c \\ [a,b,c]\in\mathscr{A}}} F(a,b,c) = \sum_{\substack{[a,b,c]\in\mathscr{A} \\ \text{Reduced}}} F(c, 2c-b, a-b+c) + \sum_{\substack{[a,b,c]\in\theta\mathscr{A}' \\ \text{Reduced}}} F(-a+b-c, b-2a, -a).$$

For $\mathscr{A}_t$ we then have

$$\sum_{\substack{a>0>c \\ [a,b,c]\in\mathscr{A}_t}} F(a,b,c) = \sum_{s=0}^{t} F((s+1)t - s^2, t-2s, -1).$$

When simplified, it turns out that

$$\sum_{\substack{a>0>c \\ [a,b,c]\in\mathscr{A}_t}} F(a,b,c) = 0 \text{ iff } t(t+1)(t+2)(t+3)(t+4) = 0.$$

Since $t \geq 2$, (6.7) is never true. □

It is clear that Theorem 6.4 can be used to generate other examples of classes $\mathscr{A}$ where obstructions occur. It is also clear that analogous results can be worked out for higher weights. In addition, it is interesting to note that if $\mathscr{A} = \theta\mathscr{A}$, $\mathscr{A} \neq \mathscr{A}'$, the analogues of Theorem 6.1 and Corollary 6.2 are that $H_{\mathscr{A},k}(1-z) = -H_{\mathscr{A},k}(z)$ and $K_{\mathscr{A},k} \in R_k^-$.

7. Conclusion

In [CP] we showed the existence of a rational period function of weight $2k$, k odd whose quadratic irrational pole set is the irreducible system $P(\mathscr{A})$ where $\mathscr{A} = \theta\mathscr{A}'$. In this paper we discuss the existence of a rational period function whose quadratic irrational pole set is $P(\mathscr{A})$ where $\mathscr{A} \neq \theta\mathscr{A}'$. In this case rational period functions of weight $2k$ always exist whenever the space of cusp forms of weight $2k$ is trivial. However, whenever S_{2k} is nontrivial, obstructions may occur and rational period functions with quadratic irrational poles in a single irreducible system need not exist as shown by examples for weight twelve. The more general question of whether the space of obstructions is always equal to R_k is of interest. It also remains to examine more carefully the case where $\mathscr{A} = \theta\mathscr{A}'$ and k is even. Given that Hawkins [H] has an example of the nonexistence of a rational period function of weight twelve with poles at $\{\pm(1 \pm \sqrt{5})/2\}$, it appears that obstructions also occur.

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