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A THEOREM OF NEHARI TYPE

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Introduction

The Bergman space A^2 is the space of all analytic functions f defined on the open unit disk D such that they are square integrable with respect to the area measure $dA = (1/\pi)dy\,dx$. The Bergman space A^2 is a closed subspace of L^2 and the polynomials are dense in A^2. Let $\phi \in L^\infty(D)$, the Hankel operator H_ϕ is defined on A^2 by $H_\phi(f) = P(J(\phi f))$, where J is the unitary operator defined on L^2 by $J(f(z)) = f(\bar{z})$ and P is the orthogonal projection of L^2 onto A^2. It is easily established that $H_\phi T_z = T_{\bar{z}} H_\phi$, where T_z is the operator defined on A^2 by $T_z(f) = zf$ and $T_{\bar{z}} f = P(\bar{z} f)$. Thus, the Hankel operators H_ϕ are special instances of solutions of the operator equation

$$ST_z = T_{\bar{z}} S \tag{1}$$

where S is a bounded operator on A^2. From (1), it is easily established that $\langle Spq^+, 1\rangle = \langle Sp, q\rangle$, where p and q are polynomials in z, and $p^+(z) = \overline{p(\bar{z})}$. Thus, it follows that

$$\langle Sb_\xi^{1/2}, \left(b_\xi^{1/2}\right)^+ \rangle = \langle Sb_\xi, 1\rangle$$

$$\text{where } b_\xi(z) = \left(1 - |\xi|^2\right)\left(1 - \bar{\xi} z\right)^{-3}, \xi, z \in D.$$

In this paper a Nehari type theorem is proved. In particular it is shown that if S is a bounded operator on A^2 which satisfies (1), then $S = H_\phi$ for some $\phi \in L^\infty(D)$. However, it should be mentioned that in [3] the symbol ϕ was determined explicitly whenever S is of finite rank. The theorem above is of Nehari type due to the fact that Nehari [4] proved that if S is a bounded operator on the Hardy space H^2 such that $ST_{e^{i\theta}} = T_{e^{-i\theta}} S$, then $S = H_\phi$ for some $\phi \in L^\infty(\partial D)$, moreover, ϕ can be chosen such that $\|H_\phi\| = \|\phi\|_\infty$.

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1. The main result

To establish the result and for the sake of completeness the following is needed.

DEFINITION 1. A collection of points $\{\xi_i\}_{i=1}^{\infty}$ in D is called an η-lattice if the hyperbolic balls $\{z\colon d(z,\xi_i) < \eta\}$ cover D, and the ξ_i are separated in the same sense that $d(\xi_i, \xi_j) \geq c_0\eta$, $i \neq j$. Here c_0 is a constant associated with the η-lattice and $d(z,w) = |z-w|/|1-\overline{w}z|$.

THEOREM 1. *Let $\{\xi_i\}_{i=1}^{\infty}$ be an η-lattice with η sufficiently small then for $g \in A^1$ (that is, g is analytic in D and $\int_D |g|\,dA < \infty$), there exist $\{\lambda_i\}$ in l^1 such that*

(i) $g = \sum_i \lambda_i b_{\xi_i}$

(ii) $\sum_i |\lambda_i| \leq \alpha \|g\|_1$, *where α is a constant which depends only on η.*

Remark 1. Theorem 1 is a special case of Theorem 1.3 proved in [2]. To prove it, take the Hilbert space H in Theorem 1.3 to be $\mathbf{C}$, and follow the same techniques of the proof of Theorem 1.3. In [1], a theorem similar to Theorem 1 is proved for the case $g \in A^2$, and thus it can be concluded that $\operatorname{span}\{b_{\xi_i}\}$ is dense in A^1 and A^2. Also, it is known that $\|b_\xi\|_1 \leq c$, $\xi \in D$ and c is a constant. For a proof of the last statement see [2].

THEOREM 2. *Let S be a bounded operator defined on the Bergman space A^2 such that $ST_z = T_{\bar{z}}S$. Then there exist $\phi \in L^\infty(D)$ such that $S = H_\phi$.*

Proof. Let $M = \operatorname{span}\{b_{\xi_i}\}$ where $\{\xi_i\}$ is an η-lattice. Define the linear functional G on M by $G(f) = \langle Sf, 1\rangle$. Note that $M \subset A^2$ and hence is contained in A^1. From Theorem 1, given $f \in M$ there exist $\{\lambda_i\}$ in l^1 such that $f = \sum_i \lambda_i b_{\xi_i}$ and $\sum |\lambda_i| \leq \alpha \|f\|_1$.

Given $0 < r < 1$, note that $\|b_{\xi_i}(rz)\|_2 \leq k(r)$. Thus, with $f_r(z) = f(rz)$ we see that

$$\begin{aligned}
\langle Sf_r, 1\rangle &= \Big\langle S\Big(\sum_i \lambda_i b_{\xi_i}(rz)\Big), 1\Big\rangle \\
&= \sum_i \lambda_i \big\langle S\big(b_{\xi_i}(rz)\big), 1\big\rangle \\
&= \sum_i \lambda_i \Big\langle S\big(b_{\xi_i}^{1/2}(rz)\big), \big(b_{\xi_i}^{1/2}(rz)\big)^{+}\Big\rangle.
\end{aligned}$$

Therefore,

$$|\langle Sf_r, 1\rangle| \leq \sum_i |\lambda_i|\, \|S\| \cdot \sup_{\xi_i} \|b_{\xi_i}(rz)\|_1.$$

Consequently, it follows from previous discussion and Remark 1 that

$$|\langle Sf_r, 1\rangle| \leq \alpha c \|f\|_1.$$

But $f_r \to f$ in A^2. Thus, by the continuity of G it follows that $|G(f)| \leq \beta \|f\|_1$ for some constant β. Since span$\{b_{\xi_i}\}$ is dense in A^1 it follows that G extends by continuity to an element of $(A^1)^*$, and consequently, by the Hahn Banach Theorem to an element of $(L^1)^* = L^\infty(D)$. Therefore, there exists $\phi \in L^\infty(D)$, such that

$$\begin{aligned}\langle Sf, 1\rangle &= \langle \phi f, 1\rangle \\ &= \langle J(\phi f), 1\rangle \\ &= \langle P(J\phi f), 1\rangle \\ &= \langle H_\phi f, 1\rangle.\end{aligned}$$

Moreover, by [1], span$\{b_{\xi_i}\}$ is dense in A^2. Thus, it follows that $\langle H_\phi h, 1\rangle = \langle Sh, 1\rangle$, $h \in A^2$. Using the fact that $\langle Spq^+, 1\rangle = \langle Sp, q\rangle$ where p, q are polynomials in z, it follows that $\langle Sp, q\rangle = \langle H_\phi p, q\rangle$, and hence, $S = H_\phi$, and this ends the proof of the theorem.

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