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PETTIS INTEGRALS AND SINGULAR INTEGRAL OPERATORS

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Introduction

The present note is concerned with conditions guaranteeing the integrability of operator valued functions acting on spaces $L^p(\mathbf{R}^n)$ for $1 < p < \infty$. To set the stage, some definitions and notation need to be fixed. Let E be a locally convex space. Let $(\Omega, \mathscr{S}, \mu)$ be a measure space. A scalarly μ-measurable function $\Psi\colon \Omega \to E$ is said to be *Pettis μ-integrable* if

$$\int_\Omega |\langle \Psi(\omega), \xi \rangle| \, d\mu(\omega) < \infty,$$

for every $\xi \in E'$, and for every $A \in \mathscr{S}$, there exists a vector $\Psi\mu(A) \in E$ such that $\langle \Psi\mu(A), \xi \rangle = \int_A \langle \Psi(\omega), \xi \rangle \, d\mu(\omega)$ for all $\xi \in E'$. In the context of Banach spaces, the notion of a vector valued function being integrable in this sense is, perhaps, less widely used than the familiar notion of Bochner integrability. If X is a Banach space with norm $\|\cdot\|$, then a function $\Psi\colon \Omega \to X$ is said to be *strongly μ-measurable* if it is the limit μ-a.e. of a sequence of X-valued $\mathscr{S}$-simple functions. If Ψ is strongly μ-measurable, then the function $\|\Psi\|\colon \Omega \to [0, \infty)$ defined by $\|\Psi\|(\omega) = \|\Psi(\omega)\|$ for all $\omega \in \Omega$ is μ-measurable, and Ψ is said to be *Bochner μ-integrable* if $\int_\Omega \|\Psi\| \, d\mu < \infty$. In this case, there exist X-valued $\mathscr{S}$-simple functions s_m, $m = 1, 2, \ldots$, such that $\lim_{m \to \infty} \int_\Omega \|\Psi - s_m\| \, d\mu = 0$, and so for each $A \in \mathscr{S}$, if $\Psi\mu(A) \in E$ is defined by

$$\Psi\mu(A) = \lim_{m \to \infty} \int_A s_m \, d\mu,$$

then

$$\langle \Psi\mu(A), \xi \rangle = \int_\Omega \langle \Psi(\omega), \xi \rangle \, d\mu(\omega)$$

for all $\xi \in X'$, as required for the definition of Pettis integrability.

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The difference between Pettis integrability and Bochner integrability is essentially the difference between the absolute summability of a sequence and the unconditional summability of a sequence in a Banach space. Explicitly, if Ω is the set $\mathbf{N}$ of counting numbers, $\mathscr{S}$ is the collection of all subsets of $\mathbf{N}$ and μ is counting measure, then a sequence Ψ: $\mathbf{N} \to X$ with values in the Banach space X is Bochner μ-integrable if and only if $\{\Psi(m)\}_{m=1}^{\infty}$ is absolutely summable in X, that is, $\sum_{m=1}^{\infty}\|\Psi(m)\| < \infty$. On the other hand, the function Ψ is Pettis μ-integrable if and only if $\{\Psi(m)\}_{m=1}^{\infty}$ is unconditionally summable in X, which is to say that there exists $\sum_{m\in\mathbf{N}}\Psi(m) \in X$ such that for every neighbourhood U of 0 in X, there exists a finite subset J of $\mathbf{N}$ such that $\sum_{m\in\mathbf{N}}\Psi(m) - \sum_{m\in K}\Psi(m) \in U$ for every finite set $K \supseteq J$. A theorem of Dvoretsky-Rogers ensures that in every infinite-dimensional Banach space, there exists an unconditionally summable sequence which fails to be absolutely summable [Day].

The Pettis integral has been used to explore the geometric properties of Banach spaces [Tal] and there is a sense that it is an integration process "deeper" than the Bochner integral; for example, G.A. Edgar [Edg, Theorem 4.2] has shown that the assertion "for every Banach space X, each bounded and scalarly measurable function Ψ: $[0,1] \to X$ is Pettis integrable with respect to Lebesgue measure" is independent of the usual axioms of set theory (ZFC).

The purpose of this note is to indicate that even for the spaces $L^p(\mathbf{R}^n)$, $1 < p < \infty$, which are well understood from the point of view of Banach space geometry, conditions ensuring the Pettis integrability of operator valued functions acting on $L^p(\mathbf{R}^n)$, $1 < p < \infty$, are closely related to fundamental problems in harmonic analysis—the boundedness of singular integral operators. The observation may not be new, although as far as we know it was first exploited in [Jef]. However, it does seem worthwhile to make an explicit note of it in a context more general than that considered in [Jef].

In section 1, the integrability of convolution-valued functions on $L^p(\mathbf{R}^n)$, $1 \le p < \infty$, is deduced from an obvious estimate in terms of the Bochner norm; see Proposition 1.1. This contrasts with the main result, Theorem 2.7 in Section 2, where the convolution-valued function acting on $L^p(\mathbf{R}^n)$, $1 < p < \infty$, is only Pettis integrable, and its integral is associated with a singular integral operator. The relationship between the Pettis integral and singular integral operators is made explicit in Section 3 with the example of the Hilbert transform.

The techniques used in proving our results are an unusual combination of arguments concerning summability in a Banach space, applied here to the spaces $L^p(\mathbf{R}^n)$, $1 < p < \infty$, and methods from real-variable harmonic analysis. The main idea is to use Marcinkeiwicz' interpolation theorem to provide a candidate for the indefinite integral. We then need to check that the operators so defined are actually the values of the integral. It is feasible that general arguments guaranteeing the Pettis integrability of Banach space valued functions, see [Tal], could profitably be applied to the proof of the boundedness of singular integral operators by exploiting the connection outlined in this note.

1. Integrability of convolution-valued functions

Let E be a locally convex space. Let $(\Omega, \mathscr{S}, \mu)$ be a measure space. A function Ψ: $\Omega \to E$ is said to be *scalarly μ-measurable* if the function $\omega \mapsto \langle \Psi(\omega), \xi \rangle$, $\omega \in \Omega$, is μ-measurable for each $\xi \in E'$. The function Ψ is said to be *scalarly μ-integrable* if it is scalarly μ-measurable and $\int_\Omega |\langle \Psi(\omega), \xi \rangle| \, d\mu(\omega) < \infty$, for every $\xi \in E'$.

In the case that X is a Banach space and $E = \mathscr{L}(X)$, the space of continuous linear operators on X endowed with the strong operator topology, the notion of Pettis integrability translates as follows. A scalarly μ-measurable function Ψ: $\Omega \to \mathscr{L}(X)$ is Pettis μ-integrable whenever

$$\int_\Omega |\langle \Psi(\omega)x, \xi \rangle| \, d\mu(\omega) < \infty$$

for every $x \in X$ and $\xi \in X'$, and for every $A \in \mathscr{S}$, there exists an operator $\Psi\mu(A) \in \mathscr{L}(X)$ such that $\langle \Psi\mu(A)x, \xi \rangle = \int_\Omega \langle \Psi(\omega)x, \xi \rangle \, d\mu(\omega)$ for all $x \in X$ and $\xi \in X'$. The existence of $\Psi\mu(A)x \in X$ satisfying the preceding equality for each $x \in X$ and $A \in \mathscr{S}$ ensures that the linear map $x \mapsto \Psi\mu(A)x$, $x \in X$, is continuous, by the closed graph theorem. If for each $x \in X$, the function defined by $\Psi x(\omega) = \Psi(\omega)x$, $\omega \in \Omega$, is Bochner μ-integrable, then it follows that Ψ is Pettis μ-integrable in $\mathscr{L}(X)$. A simple condition ensuring Bochner integrability is given in this section.

Let n be a positive integer. The inner product $\sum_{j=1}^n x_j y_j$ of two vectors $x = (x_1, \dots, x_n)$, and $y = (y_1, \dots, y_n)$, in $\mathbf{R}^n$ is denoted by $\langle x, y \rangle$. The associated norm of $\mathbf{R}^n$ is written as $|\cdot|$. Where necessary, the Lebesgue measure on $\mathbf{R}^n$ is denoted by λ; it is taken to be defined on the collection $\mathscr{B}(\mathbf{R}^n)$ of all Borel subsets of $\mathbf{R}^n$. For each $1 \le p \le \infty$, the space $L^p(\mathbf{R}^n, \mathscr{B}(\mathbf{R}^n), \lambda)$ is denoted simply by $L^p(\mathbf{R}^n)$; the norm of $L^p(\mathbf{R}^n)$ is denoted by $\|\cdot\|_p$, as usual. The occasional confounding of a function with its equivalence class modulo null functions causes no confusion in the present context.

For a function $f \in L^2(\mathbf{R}^n) \cap L^1(\mathbf{R}^n)$, the Fourier transform $\hat{f} \in L^2(\mathbf{R}^n)$ of f is defined by

$$\hat{f}(x) = \int_{\mathbf{R}^n} e^{-i\langle x, \xi \rangle} f(\xi) \, d\xi, \quad x \in \mathbf{R}^n$$

and the Fourier transform is extended to all of $L^2(\mathbf{R}^n)$ by continuity. We set

$\mathscr{F}f = \hat{f}$ for all $f \in L^2(\mathbf{R}^n)$. The same notation is adopted for functions $f \in L^1(\mathbf{R}^n)$, so, where appropriate, $\mathscr{F}$ also denotes a continuous linear map from $L^1(\mathbf{R}^n)$ to the space of continuous functions vanishing at infinity.

For the $L^2(\mathbf{R}^n)$ case, on denoting the adjoint of the bounded linear operator

$$\mathscr{F}: L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$$

by $\mathscr{F}^*$, the Fourier-Plancherel theorem asserts that

$$\mathscr{F}^{-1} = \frac{1}{(2\pi)^n}\mathscr{F}^*.$$

For each $j = 1, \dots, n$, let D_j be the self adjoint extension of the operator $(1/i)\partial/\partial x_j$ defined on the space of all smooth functions of compact support and set $D = (D_1, \dots, D_n)$. For all $f, g \in L^2(\mathbf{R}^n)$, set $(f, g) = \int_{\mathbf{R}^n} f(x)\overline{g(x)}\,dx$.

Given a bounded Borel measurable function $\phi: \mathbf{R}^n \to \mathbf{C}$, the operator

$$\phi(D): L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$$

is defined by

$$[\phi(D)f]\hat{} = \phi \cdot \hat{f} \quad \text{for all } f \in L^2(\mathbf{R}^n).$$

The operator $\phi(D)$ corresponds to the operator obtained from the functional calculus for the n-tuple D of commuting self adjoint operators. Suppose that ψ belongs to the space $L^1(\mathbf{R}^n)$. Then $\hat{\psi}$ belongs to the space $C_0(\mathbf{R}^n)$ of continuous functions on $\mathbf{R}^n$ vanishing at infinity, and we have

$$\left[\hat{\psi}(D)f\right]\hat{} = \hat{\psi}.\hat{f}, \quad f \in L^2(\mathbf{R}^n).$$

Because $[\psi * f]\hat{} = \hat{\psi}.\hat{f}$, we have $\hat{\psi}(D)f = \psi * f$ for all $f \in L^2(\mathbf{R}^n)$. Recall that for all $1 \le p \le \infty$, $\psi \in L^1(\mathbf{R}^n)$ and $f \in L^p(\mathbf{R}^n)$, the function $\psi * f$ belongs to $L^p(\mathbf{R}^n)$ and

$$\|\psi * f\|_p \le \|\psi\|_1 \|f\|_p \quad \text{[Hew-Ro, p 298]}.$$

Define $\hat{\psi}(D)f = \psi * f$ for all $f \in L^p(\mathbf{R}^n)$, that is, it will prove convenient to denote the bounded linear operator of convolution with ψ, defined on each space $L^p(\mathbf{R}^n)$, $1 \le p \le \infty$, by $\hat{\psi}(D)$; the identities above, valid on $L^2(\mathbf{R}^n)$, explain the origin of the notation.

Let $(\Omega, \mathscr{S}, \mu)$ be a σ-finite measure space. Let $\Phi: \Omega \to L^1(\mathbf{R}^n)$ be a function. Let $\mathscr{F}\Phi: \Omega \to L^\infty(\mathbf{R}^n)$ be the function defined by $\mathscr{F}\Phi(\omega) =$

$\mathscr{F}[\Phi(\omega)]$ for every $\omega \in \Omega$. Set $\hat{\Phi}_\omega(D)f = \Phi(\omega) * f$, for every $f \in L^p(\mathbf{R}^n)$, $1 \leq p \leq \infty$, and every $\omega \in \Omega$.

The following basic result gives a sufficient conditions for the integral of a convolution-valued function to define a bounded linear operator on $L^p(\mathbf{R}^n)$.

1.1 PROPOSITION. *Let $\Phi\colon \Omega \to L^1(\mathbf{R}^n)$ be a Bochner μ-integrable function. Let $1 \leq p < \infty$. Then for each $f \in L^p(\mathbf{R}^n)$, the function $\omega \mapsto \hat{\Phi}_\omega(D)f$, $\omega \in \Omega$, is Bochner integrable in $L^p(\mathbf{R}^n)$.*

Proof. The product σ-algebra of $\mathscr{S}$ and $\mathscr{B}(\mathbf{R}^n)$ is denoted by $\mathscr{S} \otimes \mathscr{B}(\mathbf{R}^n)$. The space $L^1(\Omega, L^1(\mathbf{R}^n))$ of $L^1(\mathbf{R}^n)$-valued Bochner μ-integrable functions is isomorphic to

$$L^1(\Omega \times \mathbf{R}^n, \mathscr{S} \otimes \mathscr{B}(\mathbf{R}^n), \mu \otimes \lambda) \quad \text{[D − U, Example VII.1.10]},$$

so corresponding to $\Phi\colon \Omega \to L^1(\mathbf{R}^n)$, there exists an integrable function $\phi\colon \Omega \times \mathbf{R}^n \to \mathbf{C}$ such that for each $f \in L^p(\mathbf{R}^n)$ and for almost every $\omega \in \Omega$, the function $\phi(\omega, \cdot) * f$ is a representative of the equivalence class $\hat{\Phi}_\omega(D)f$.

The function $(\omega, x) \mapsto [\phi(\omega, \cdot) * f](x)$, $\omega \in \Omega$, $x \in \mathbf{R}^n$, is jointly measurable and

$$\int_\Omega \|\phi(\omega, \cdot) * f\|_p \, d\mu(\omega) \leq \|f\|_p \int_\Omega \|\phi(\omega, \cdot)\|_1 \, d\mu(\omega)$$
$$= \|f\|_p \int_\Omega \|\Phi(\omega)\|_1 \, d\mu(\omega)$$

by [Hew-Ro, p 298]. It follows that the function $\omega \mapsto \hat{\Phi}_\omega(D)f$, $\omega \in \Omega$, is strongly measurable in $L^p(\mathbf{R}^n)$ and

$$\int_\Omega \|\hat{\Phi}_\omega(D)f\|_p \, d\mu(\omega) = \int_\Omega \|\phi(\omega, \cdot) * f\|_p \, d\mu(\omega) < \infty.$$

1.2 *Remark.* In Section 3, we shall see that in the cases where the integral of the function $\Phi\colon \Omega \to L^1(\mathbf{R}^n)$ is associated with a singular integral operator, there exists an element $f \in L^p(\mathbf{R}^n)$ for which the function $\omega \mapsto \hat{\Phi}_\omega(D)f$, $\omega \in \Omega$, is *not* Bochner integrable in $L^p(\mathbf{R}^n)$. □

2. Pettis integrability of convolution-valued functions

Let $(\Omega, \mathscr{S}, \mu)$ be a σ-finite measure space. We first establish conditions for which a function $\Phi\colon \Omega \to L^1(\mathbf{R}^n)$ defines a Pettis integrable operator valued function $\omega \mapsto \hat{\Phi}_\omega(D)$, $\omega \in \Omega$, acting on $L^2(\mathbf{R}^n)$.

The space $L^\infty(\mathbf{R}^n)$ endowed with the weak*-topology $\sigma(L^\infty(\mathbf{R}^n), L^1(\mathbf{R}^n))$ is written simply as $L^\infty(\mathbf{R}^n)_\sigma$.

2.1 LEMMA. (i) *If* $\Phi\colon \Omega \to L^1(\mathbf{R}^n)$ *is scalarly measurable function, then the function* $\mathscr{F}\Phi\colon \Omega \to L^\infty(\mathbf{R}^n)$ *is scalarly measurable in* $L^\infty(\mathbf{R}^n)_\sigma$.

(ii) *If* $\mathscr{F}\Phi$ *is scalarly integrable in* $L^\infty(\mathbf{R}^n)_\sigma$, *then* $\mathscr{F}\Phi$ *is Pettis integrable in* $L^\infty(\mathbf{R}^n)_\sigma$ *and*

$$\sup\left\{\int_\Omega |\langle \mathscr{F}\Phi(\omega), f\rangle|\, d\mu(\omega) : f \in L^1(\mathbf{R}^n),\ \|f\|_1 \le 1\right\} < \infty.$$

Proof. Assume that $\Phi\colon \Omega \to L^1(\mathbf{R}^n)$ is scalarly measurable. The adjoint of the linear map

$$\mathscr{F}\colon L^1(\mathbf{R}^n) \to L^\infty(\mathbf{R}^n)_\sigma$$

is again the Fourier transform $\mathscr{F}$, so

$$\langle \mathscr{F}\Phi(\omega), f\rangle = \langle \Phi(\omega), \mathscr{F}f\rangle$$

for all $f \in L^1(\mathbf{R}^n)$ and $\omega \in \Omega$. If Φ is scalarly measurable, then it follows that $\mathscr{F}\Phi$ is scalarly measurable.

The proof of the second assertion follows from the closed graph theorem [D-U, p. 53], but it may also be seen directly, as follows.

There exists a countable subset H of the closed unit ball of $L^1(\mathbf{R}^n)$ so that

$$\|g\|_\infty = \sup\{|\langle g, f\rangle| : h \in H\}$$

for all $g \in L^\infty(\mathbf{R}^n)$. If $\mathscr{F}\Phi$ is scalarly measurable in $L^\infty(\mathbf{R}^n)_\sigma$, then it follows that the function $\|\mathscr{F}\Phi\|_\infty$ is measurable. Set $\Omega_m = \{\omega \in \Omega : \|\mathscr{F}\Phi(\omega)\|_\infty \le m\}$, $m = 1, 2, \ldots$. Then for each $A \in \mathscr{S}$ and $m = 1, 2, \ldots$, the linear functional

$$f \mapsto \int_{\Omega_m \cap A} \langle \mathscr{F}\Phi(\omega), f\rangle\, d\mu(\omega), \quad f \in L^1(\mathbf{R}^n),$$

is continuous on $L^1(\mathbf{R}^n)$, so there corresponds a vector

$$\int_{\Omega_m \cap A} \mathscr{F}\Phi(\omega)\, d\mu(\omega) \in L^\infty(\mathbf{R}^n)$$

such that

$$\left\langle \int_{\Omega_m \cap A} \mathscr{F}\Phi(\omega)\, d\mu(\omega), f \right\rangle = \int_{\Omega_m \cap A} \langle \mathscr{F}\Phi(\omega), f\rangle\, d\mu(\omega)$$

for all $f \in L^1(\mathbf{R}^n)$. Moreover, the scalar integrability of $\mathscr{F}\Phi$ ensures that the sequence of vectors

$$\int_{\Omega_m \cap A} \mathscr{F}\Phi(\omega)\, d\mu(\omega), \quad m = 1, 2, \ldots,$$

is Cauchy in $L^\infty(\mathbf{R}^n)_\sigma$, which is a sequentially complete space, so there exists a unique limit

$$\int_A \mathscr{F}\Phi(\omega)\, d\mu(\omega)$$

as $m \to \infty$ with the property that

$$\left\langle \int_A \mathscr{F}\Phi(\omega)\, d\mu(\omega), f \right\rangle = \int_A \langle \mathscr{F}\Phi(\omega), f\rangle\, d\mu(\omega)$$

for all $f \in L^1(\mathbf{R}^n)$.

It follows that $\mathscr{F}\Phi$ is Pettis μ-integrable. Consequently, the set

$$W = \left\{ \int_A \mathscr{F}\Phi(\omega)\, d\mu(\omega) \colon A \in \mathscr{S} \right\}$$

is bounded in $L^\infty(\mathbf{R}^n)_\sigma$, and so norm bounded in $L^\infty(\mathbf{R}^n)$. Therefore,

$$\sup\left\{ \int_\Omega |\langle \mathscr{F}\Phi(\omega), f\rangle|\, d\mu(\omega) \colon f \in L^1(\mathbf{R}^n),\ \|f\|_1 \le 1 \right\} \le 4 \sup_{g \in W} \|g\|_\infty < \infty. \quad \square$$

2.2 *Note.* If Φ is scalarly integrable in $L^1(\mathbf{R}^n)$, then $\mathscr{F}\Phi$ is scalarly integrable in $L^\infty(\mathbf{R}^n)_\sigma$. However, the converse may not hold, in general. This is typical of the situation where the integral of the operator valued function $\omega \mapsto \hat{\Phi}_\omega(D)$, $\omega \in \Omega$, acting on $L^2(\mathbf{R}^n)$ is associated with a singular integral operator; see Example 3.1. $\square$

2.3 THEOREM. *If $\mathscr{F}\Phi \colon \Omega \to L^\infty(\mathbf{R}^n)$ is scalarly integrable in $L^\infty(\mathbf{R}^n)_\sigma$, then the operator valued function $\omega \mapsto \hat{\Phi}_\omega(D)$, $\omega \in \Omega$, is Pettis integrable in $\mathscr{L}(L^2(\mathbf{R}^n))$, and for all $A \in \mathscr{S}$,*

$$\left\| \int_A \hat{\Phi}_\omega(D)\, d\mu(\omega) \right\| \le \sup\left\{ \int_\Omega |\langle \mathscr{F}\Phi(\omega), f\rangle|\, d\mu(\omega) \colon f \in L^1(\mathbf{R}^n),\ \|f\|_1 \le 1 \right\}.$$

Furthermore, for each $f \in L^2(\mathbf{R}^d)$ and $A \in \mathscr{S}$, the equality

$$\mathscr{F}\int_A \hat{\Phi}_\omega(D) f\, d\mu(\omega) = \textstyle\int_A \mathscr{F}\Phi(\omega)\, d\mu(\omega) . \hat{f}$$

holds.

Proof. First, $\mathscr{F}\Phi$ is Pettis integrable in $L^\infty(\mathbf{R}^n)_\sigma$ by Lemma 2.1. Because $L^2(\mathbf{R}^n)$ is a separable reflexive Banach space, it suffices to prove that for each $f \in L^2(\mathbf{R}^n)$, the function $\omega \mapsto \hat{\Phi}_\omega(D)f$, $\omega \in \Omega$, is scalarly integrable in $L^2(\mathbf{R}^n)$; see, for example, [D-U, Theorem II.3.7].

Let f and g be elements of the closed unit ball of $L^2(\mathbf{R}^n)$. Then

$$\begin{aligned}\int_\Omega \left|\left(\hat{\Phi}_\omega(D)f, g\right)\right| d\mu(\omega) &= (2\pi)^{-n} \int_\Omega \left|\left(\hat{\Phi}_\omega(D)f, \mathscr{F}^*\mathscr{F}g\right)\right| d\mu(\omega)\\ &= (2\pi)^{-n} \int_\Omega \left|\left\langle \mathscr{F}\Phi(\omega), \hat{f}.\bar{\hat{g}}\right\rangle\right| d\mu(\omega).\end{aligned}$$

In the last term, $\hat{f}.\bar{\hat{g}} \in L^1(\mathbf{R}^n)$, where the brackets represent the duality between $L^\infty(\mathbf{R}^n)$ and $L^1(\mathbf{R}^n)$. Furthermore, $\|\hat{f}\cdot\bar{\hat{g}}\|_1 \le (2\pi)^n$, so the required operator bound follows as f and g range over the unit ball of $L^2(\mathbf{R}^n)$.

Let $f \in L^2(\mathbf{R}^n)$. The equalities

$$\begin{aligned}\int_\Omega \left(\hat{\Phi}_\omega(D)f, g\right) d\mu(\omega) &= (2\pi)^{-n} \int_\Omega \left(\hat{\Phi}_\omega(D)f, \mathscr{F}^*\mathscr{F}g\right) d\mu(\omega)\\ &= (2\pi)^{-n} \int_\Omega \left(\mathscr{F}\left(\hat{\Phi}_\omega(D)f\right), \mathscr{F}g\right) d\mu(\omega)\\ &= (2\pi)^{-n} \int_\Omega \left((\mathscr{F}\Phi(\omega)) \cdot \hat{f}, \hat{g}\right) d\mu(\omega)\end{aligned}$$

for $g \in L^2(\mathbf{R}^n)$ ensure that $\mathscr{F}\int_A \hat{\Phi}_\omega(D) f \, d\mu(\omega) = \int_A \mathscr{F}\Phi(\omega)\, d\mu(\omega) \cdot \hat{f}$ holds because,

$$\begin{aligned}\left(\mathscr{F}\int_A \hat{\Phi}_\omega(D) f \, d\mu(\omega), h\right) &= \left(\int_A \mathscr{F}\left(\hat{\Phi}_\omega(D)f\right) d\mu(\omega), h\right)\\ &= \int_A \left(\mathscr{F}\left(\hat{\Phi}_\omega(D)f\right), h\right) d\mu(\omega)\end{aligned}$$

for all $h \in L^2(\mathbf{R}^n)$. □

2.4 Corollary. *There exists a constant* $M > 0$ *such that the inequality*

$$\lambda\left(\left\{x \in \mathbf{R}^n : \left|\int_A \hat{\Phi}_\omega(D) f \, d\mu(\omega)\right|(x) > \alpha\right\}\right) \le \frac{M^2}{\alpha^2}\|f\|_2$$

holds for all $\alpha > 0$, $f \in L^2(\mathbf{R}^n)$ *and* $A \in \mathscr{S}$; *that is, the family* $\{\int_A \hat{\Phi}_\omega(D)\, d\mu(\omega) : A \in \mathscr{S}\}$ *of operators is uniformly of weak-type* (2, 2).

The following result of Benedek-Calderón-Panzone [Tor, Theorem XI.1.1] is basic to proving Pettis integrability in the present context. For each

$x_0 \in \mathbf{R}^n$ and $R > 0$, let

$$B(x_0, R) = \{x \in \mathbf{R}^n : |x - x_0| < R\}.$$

The collection, modulo null functions, of all bounded measurable functions with compact support in $\mathbf{R}^n$ is denoted by $L_c^\infty(\mathbf{R}^n)$.

2.5 THEOREM. *Let T be a sublinear operator from $L_c^\infty(\mathbf{R}^n)$ into the space of measurable functions. Suppose that T satisfies the following two conditions:*

(i) *There exists a constant r satisfying $1 < r < \infty$, for which T is of weak-type (r, r); more precisely, there exists a constant $c_1 > 0$ such that for all $f \in L_c^\infty(\mathbf{R}^n)$ and $\alpha > 0$,*

$$\alpha^r \lambda(\{|Tf| > \alpha\}) \le c_1^r \|f\|_r^r.$$

(ii) *There exist constants $c_2 > 1$ and $c_3 > 0$ such that given $x_0 \in \mathbf{R}^n$ and $R > 0$, the inequality*

$$\int_{\mathbf{R}^n \setminus B(x_0, c_2 R)} |Tf(x)|\, dx \le c_3 \|f\|_1$$

holds for every $f \in L_c^\infty(\mathbf{R}^n)$ with the property that $\operatorname{supp} f \subseteq B(x_), R)$ *and*

$$\int_{B(x_0, R)} f(x)\, dx = 0.$$

Then T is an operator of weak-type $(1, 1)$; *that is, there exists a constant $c_4 > 0$ such that*

$$\alpha \lambda\{|Tf| > \alpha\} \le c_4 \|f\|_1$$

for all $f \in L_c^\infty(\mathbf{R}^n)$ and $\alpha > 0$. Moreover, c_4 depends only on c_1, c_2, c_3 and n.

The translate $x \mapsto f(x + a)$, $x \in \mathbf{R}^n$, of a function $f : \mathbf{R}^n \to \mathbf{C}$ by an element $a \in \mathbf{R}^n$ is denoted by f_a. Given a property $P(x)$ concerning elements x of $\mathbf{R}^n$, it proves convenient to denote the set $\{x \in \mathbf{R}^n : P(x)\}$ by $\{P(x)\}$, with the understanding that x is a generic element of $\mathbf{R}^n$.

2.6 LEMMA. *Let $\Phi : \Omega \to L^1(\mathbf{R}^n)$ be a scalarly measurable function. The function*

$$(\omega, y) \mapsto \left\langle \Phi(\omega)_{-y} - \Phi(\omega), \chi_{\{|x| \ge 2|y|\}} \right\rangle, \ \omega \in \Omega,\ y \in \mathbf{R}^n,$$

is jointly $\mathcal{S} \otimes \mathcal{B}(\mathbf{R}^n)$-measurable.

Proof. The result is easily seen to be true if Φ is an $L^1(\mathbf{R}^n)$-valued simple function. Because $L^1(\mathbf{R}^n)$ is separable, a scalarly measurable function with values in $L^1(\mathbf{R}^n)$ is strongly measurable, so the general result follows on taking limits in $L^1(\mathbf{R}^n)$ of $L^1(\mathbf{R}^n)$-valued simple functions. □

2.7 THEOREM. *Let $\Phi\colon \Omega \to L^1(\mathbf{R}^n)$ be a scalarly measurable function. Suppose that the following two conditions hold:*

(i) *$\mathscr{F}\Phi\colon \Omega \to L^\infty(\mathbf{R}^n)$ is scalarly integrable in $L^\infty(\mathbf{R}^n)_\sigma$.*

(ii) *There exists a constant $c > 0$ such that*

$$\int_\Omega \left| \left\langle \Phi(\omega)_{-y} - \Phi(\omega), \chi_{\{|x| \geq 2|y|\}} \right\rangle \right| d\mu(\omega) \leq c \text{ for all } y \neq 0.$$

Then, the operator valued function $\omega \mapsto \hat{\Phi}_\omega(D)$, $\omega \in \Omega$, is Pettis integrable in $\mathscr{L}(L^p(\mathbf{R}^n))$ whenever $1 < p < \infty$.

Proof. According to Theorem 2.3, the function $\omega \mapsto \hat{\Phi}_\omega(D)f$, $\omega \in \Omega$, is Pettis μ-integrable in $L^2(\mathbf{R}^n)$ for each $f \in L_c^\infty(\mathbf{R}^n)$, so set $T_A f = \int_A \hat{\Phi}_\omega(D) f \, d\mu(\omega)$ for every $f \in L_c^\infty(\mathbf{R}^n)$ and $A \in \mathscr{S}$. For each $A \in \mathscr{S}$, the function $T_A f$ is square-integrable on $\mathbf{R}^n$, and by Corollary 2.4, T_A is a weak-type $(2,2)$ operator from $L_c^\infty(\mathbf{R}^n)$ to the space of measurable functions. We see that T_A is of weak type $(1,1)$ by appealing to Theorem 2.5, as follows.

Let $x_0 \in \mathbf{R}^n$ and $R > 0$. Let f be a bounded measurable function such that $\operatorname{supp} f \subseteq B(x_0, R)$ and $\int_{B(x_0, R)} f(x)\,dx = 0$. Then for each $\omega \in \Omega$,

$$\begin{aligned}
\hat{\Phi}_\omega(D)f(x) &= \int_{\mathbf{R}^n} \Phi(\omega)(x-y) f(y)\,dy \\
&= \int_{B(x_0,R)} \Phi(\omega)(x-y) f(y)\,dy \\
&= \int_{B(x_0,R)} (\Phi(\omega)(x-y) - \Phi(\omega)(x-x_0)) f(y)\,dy.
\end{aligned}$$

Let u_R be the characteristic function of the set $\{x \in \mathbf{R}^n : |x - x_0| > 2R\}$. If I_1 denotes the integral $\int_{\mathbf{R}^n \setminus B(x_0, 2R)} |T_A f(x)|\,dx$, then

$$I_1 = \|u_R . T_A f\|_1 = \left\| u_R \cdot \int_A \hat{\Phi}_\omega(D) f \, d\mu(\omega) \right\|_1 .$$

The set of all $g \in L^\infty(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$ satisfying $\|g\|_\infty \leq 1$ is a norming set for

$L^1(\mathbf{R}^n)$, so it follows that

$$I_1 = \sup\left\{\left|\left(u_R \cdot \int_A \hat{\Phi}_\omega(D) f \, d\mu(\omega), g\right)\right| : g \in L^\infty(\mathbf{R}^n) \cap L^2(\mathbf{R}^n), \|g\|_\infty \le 1\right\}$$
$$\le \sup\left\{\int_\Omega \left|\left(\hat{\Phi}_\omega(D) f, u_R g\right)\right| d\mu(\omega) : g \in L^\infty(\mathbf{R}^n) \cap L^2(\mathbf{R}^n), \|g\|_\infty \le 1\right\}.$$

Suppose that g is an element of $L^\infty(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$ with $\|g\|_\infty \le 1$. Let I_2 denote the integral

$$\int_\Omega \left|\left(\hat{\Phi}_\omega(D) f, u_R g\right)\right| d\mu(\omega).$$

Then

$$I_2 \le \int_\Omega \int_{\mathbf{R}^n} \left| u_R(x) g(x) \left[\int_{\mathbf{R}^n} \Phi(\omega)(x-y) f(y)\, dy\right]\right| dx\, d\mu(\omega)$$
$$= \int_\Omega \int_{\mathbf{R}^n} \left| u_R(x) g(x) \left[\int_{B(x_0, R)} (\Phi(\omega)(x-y) - \Phi(\omega)(x - x_0)) f(y)\, dy\right]\right| dx\, d\mu(\omega).$$

The function $(x, y) \mapsto u_R(x) g(x)(\Phi(\omega)(x-y) - \Phi(\omega)(x - x_0)) f(y)$, $x, y \in \mathbf{R}^n$, is Borel measurable for each $\omega \in \Omega$, because $\Phi(\omega) \in L^1(\mathbf{R}^n)$. Hence, by Tonelli's theorem,

$$I_2 \le \int_\Omega \int_{\mathbf{R}^n} |u_R(x) g(x)| \left[\int_{B(x_0, R)} |\Phi(\omega)(x-y) - \Phi(\omega)(x - x_0)| |f(y)|\, dy\right] dx\, d\mu(\omega)$$
$$= \int_\Omega \int_{B(x_0, R)} \left[\int_{\mathbf{R}^n} |u_R(x) g(x)| |\Phi(\omega)(x-y) - \Phi(\omega)(x - x_0)|\, dx\right] |f(y)|\, dy\, d\mu(\omega)$$
$$\le \int_\Omega \int_{B(x_{)}, R)} \left[\int_{\mathbf{R}^n} |u_R(x)| |\Phi(\omega)(x-y) - \Phi(\omega)(x - x_0)|\, dx\right] |f(y)|\, dy\, d\mu(\omega)$$
$$\le \int_\Omega \int_{B(x_0, R)} \left[\int_{|x| > 2|y - x_0|} |\Phi(\omega)(x - y + x_0) - \Phi(\omega)(x)|\, dx\right] |f(y)|\, dy\, d\mu(\omega).$$

Now the function

$$(\omega, y) \mapsto \left\langle \Phi(\omega)_{-y} - \Phi(\omega), \chi_{\{|x| \geq 2|y|\}} \right\rangle, \ \omega \in \Omega, \ y \in \mathbf{R}^n,$$

is jointly $\mathscr{S} \otimes \mathscr{B}(\mathbf{R}^n)$-measurable according to Lemma 2.6, so by Tonelli's theorem,

$$\int_\Omega \int_{B(x_0, R)} \left[\int_{|x| > 2|y - x_0|} |\Phi(\omega)(x - y + x_0) - \Phi(\omega)(x)| \, dx \right] |f(y)| \, dy \, d\mu(\omega)$$

$$= \int_{B(x_0, R)} |f(y)| \int_\Omega \left[\int_{|x| > 2|y - x_0|} |\Phi(\omega)(x - y + x_0) - \Phi(\omega)(x)| \, dx \right] d\mu(\omega) \, dy$$

$$\leq c \int_{B(x_0, R)} |f(y)| \, dy.$$

Therefore, condition (ii) of Theorem 2.5 is satisfied with c_3 independent of $A \in \mathscr{S}$, and T_A is an operator of weak-type $(1, 1)$, with the constant c_4 of Theorem 2.5 independent of $A \in \mathscr{S}$. The proof of Theorem 2.7 is completed by appealing to the following lemmas.

The following result is proved, in greater generality, in [Th1, 0.2] by appealing to the Banach-Dieudonné theorem.

2.8 Lemma. *Let $1 \leq p < \infty$ and let $1 < q \leq \infty$ be defined by $1/p + 1/q = 1$. Suppose that H is a subset of $L^q(\mathbf{R})$ with dense linear span. Let $\{f_j\}_{j=1}^\infty$ be a sequence of functions in $L^p(\mathbf{R}^n)$ satisfying the property that for every subsequence $s = \{f_{j_k}\}_{k=1}^\infty$ of $\{f_j\}_{j=1}^\infty$, there exists a function $g_s \in L^p(\mathbf{R}^n)$ such that $\langle g_s, h \rangle = \sum_{k=1}^\infty \langle f_{j_k}, h \rangle$ for all $h \in H$.*

Then $\{f_j\}_{j=1}^\infty$ is unconditionally summable in $L^p(\mathbf{R}^n)$ and $g_s = \sum_{k=1}^\infty f_{j_k}$ for every subsequence $s = \{f_{j_k}\}_{k=1}^\infty$ of $\{f_j\}_{j=1}^\infty$.

2.9 Lemma. *Let $1 < p \leq 2$. Then for each $f \in L_c^\infty(\mathbf{R}^n)$, the function $\omega \mapsto \hat{\Phi}_\omega(D) f$, $\omega \in \Omega$, is Pettis μ-integrable in $L^p(\mathbf{R}^n)$. Furthermore, there is a constant $\alpha_p > 0$ such that*

$$\left\| \int_A \hat{\Phi}_\omega(D) f \, d\mu(\omega) \right\|_p \leq \alpha_p \|f\|_p$$

for every $f \in L_c^\infty(\mathbf{R}^n)$ and $A \in \mathscr{S}$.

Proof. In view of the conclusion of Theorem 2.5, the result follows from the Marcinkiewicz interpolation theorem [Tor, Theorem IV.4.1], given that T_A is of weak-type $(1,1)$ and weak-type $(2,2)$, uniformly for each $A \in \mathscr{S}$. However, a small modification (implicit in [Tor, Theorem XI.3.1]) needs to be made to the proof. We then need to check that the operators so obtained are actually the values of an indefinite integral.

Let f be an element of $L_c^\infty(\mathbf{R}^n)$. Then $T_A f = \int_A \hat{\Phi}_\omega(D) f \, d\mu(\omega)$ is a square-integrable function on $\mathbf{R}^n$, for every $A \in \mathscr{S}$. If f is real valued, then for any $\gamma > 0$, the truncation $\chi_F f$ of f on the set $F = \{x \in \mathbf{R}^n\colon |f| \le \gamma\}$ belongs to $L_c^\infty(\mathbf{R}^n)$ too. For complex valued f, write it as the sum of its real and imaginary parts. The proof of [Tor, Theorem IV.4.1] now establishes that for any $1 < p < 2$, there is a constant $\alpha_p > 0$ such that

$$\|T_A f\|_p \le \alpha_p \|f\|_p$$

for every $A \in \mathscr{S}$. Furthermore, the constant α_p is independent of the choice of $f \in L_c^\infty(\mathbf{R}^n)$. In particular, the assumption in [Tor, Theorem IV.4.1] that T_A is already defined on $L^1(\mathbf{R}^n) + L^2(\mathbf{R}^n)$ is superfluous; the weak-type $(1,1)$ and weak-type $(2,2)$ constants do not depend on $A \in \mathscr{S}$, so α_p does not depend on $A \in \mathscr{S}$ either.

It still needs to be established that the function $\omega \mapsto \hat{\Phi}_\omega(D) f$, $\omega \in \Omega$,is Pettis μ-integrable in $L^p(\mathbf{R}^n)$, and that for every $A \in \mathscr{S}$ and every $g \in L^q(\mathbf{R}^n)$ with $1/p + 1/q = 1$, the equality $\langle T_A f, g\rangle = \int_A \langle \hat{\Phi}_\omega(D) f, g\rangle \, d\mu(\omega)$ is valid. This follows from [Th2, Corollary 5.1], but we give a direct proof in the present context.

According to Theorem 2.3,

$$\langle T_A f, g\rangle = \int_A \left\langle \hat{\Phi}_\omega(D) f, g\right\rangle d\mu(\omega)$$

for all $g \in L^q(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$ and $A \in \mathscr{S}$. The function $\omega \mapsto \|\hat{\Phi}_\omega(D) f\|_p$, $\omega \in \Omega$, is $\mathscr{S}$-measurable, because $L^p(\mathbf{R}^n)$ has a countable norming set consisting of elements of $L^q(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$. Now the number $\|\hat{\Phi}_\omega(D) f\|_p$ is finite for all $\omega \in \Omega$ and the measure space $(\Omega, \mathscr{S}, \mu)$ is σ-finite, so there exists an increasing family of sets $\Omega_k \in \mathscr{S}$, $k = 1, 2, \ldots$, such that $\Omega = \bigcup_{k=1}^\infty \Omega_k$, and for each $k = 1, 2, \ldots$, the set Ω_k has finite μ-measure and $\|\hat{\Phi}_\omega(D) f\|_p \le k$ for every $\omega \in \Omega_k$. Let $\Sigma_1 = \Omega_1$ and $\Sigma_k = \Omega_k \setminus \Omega_{k-1}$, $k = 2, 3, \ldots$.

Then for each $k = 1, 2, \ldots$, the function $\omega \mapsto \hat{\Phi}_\omega(D) f$, $\omega \in \Sigma_k$, is bounded and strongly measurable in $L^p(\mathbf{R}^n)$, so it is Bochner μ-integrable on Σ_k, and so Pettis μ-integrable on Σ_k. Given $k = 1, 2, \ldots$, the equality

$$\left\langle T_{A\cap\Sigma_k} f, g\right\rangle = \int_{A\cap\Sigma_k} \left\langle \hat{\Phi}_\omega(D) f, g\right\rangle d\mu(\omega)$$

holds for all $g \in L^q(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$ and $A \in \mathscr{S}$, so

$$T_{A \cap \Sigma_k} f = \int_{A \cap \Sigma_k} \hat{\Phi}_\omega(D) f \, d\mu(\omega)$$

because $L^q(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$ is dense in $L^q(\mathbf{R}^n)$. We show that the vectors $\int_{A \cap \Sigma_k} \hat{\Phi}_\omega(D) f \, d\mu(\omega)$, $k = 1, 2, \ldots$, are unconditionally summable in $L^p(\mathbf{R}^n)$.

For each $g \in L^q(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$,

$$\begin{aligned}\sum_{k=1}^{\infty} \left| \left\langle \int_{A \cap \Sigma_k} \hat{\Phi}_\omega(D) f \, d\mu(\omega), g \right\rangle \right| &\le \sum_{k=1}^{\infty} \int_{A \cap \Sigma_k} \left| \langle \hat{\Phi}_\omega(D) f, g \rangle \right| d\mu(\omega) \\ &= \int_A \left| \langle \hat{\Phi}_\omega(D) f, g \rangle \right| d\mu(\omega) < \infty\end{aligned}$$

because the function $\omega \mapsto \hat{\Phi}_\omega(D) f$, $\omega \in \Omega$, is Pettis μ-integrable in $L^2(\mathbf{R})$ by Theorem 2.3. Moreover, if $\{\Sigma_{j_k}\}_{k=1}^{\infty}$ is a subsequence of $\{\Sigma_j\}_{j=1}^{\infty}$, then

$$\langle T_B f, g \rangle = \sum_{k=1}^{\infty} \left\langle \int_{A \cap \Sigma_{j_k}} \hat{\Phi}_\omega(D) f \, d\mu(\omega), g \right\rangle$$

for the set $B = \cup_{k=1}^{\infty} (A \cap \Sigma_{j_k})$, so by Lemma 2.8, the vectors $\int_{A \cap \Sigma_k} \hat{\Phi}_\omega(D) f$, $k = 1, 2, \ldots$, form an unconditionally summable sequence in $L^p(\mathbf{R}^n)$. Hence,

$$\langle T_A f, g \rangle = \sum_{k=1}^{\infty} \int_{A \cap \Sigma_k} \langle \hat{\Phi}_\omega(D) f, g \rangle \, d\mu(\omega)$$

for all $g \in L^q(\mathbf{R}^n)$ and $A \in \mathscr{S}$.

By choosing A to be the set of all $\omega \in \Omega$ for which $\Re(\langle \hat{\Phi}_\omega(D) f, g \rangle) \ge 0$, and then those for which $\Re(\langle \hat{\Phi}_\omega(D) f, g \rangle) < 0$, it follows that

$$\int_\Omega \left| \Re\left(\langle \hat{\Phi}_\omega(D) f, g \rangle \right) \right| d\mu(\omega) < \infty.$$

Similarly,

$$\int_\Omega \left| \Im\left(\langle \hat{\Phi}_\omega(D) f, g \rangle \right) \right| d\mu(\omega) < \infty,$$

so that $\int_\Omega |\langle \hat{\Phi}_\omega(D)f, g\rangle| \, d\mu(\omega) < \infty$ and the equality

$$\langle T_A f, g\rangle = \int_A \left\langle \hat{\Phi}_\omega(D)f, g\right\rangle d\mu(\omega)$$

holds for every $g \in L^q(\mathbf{R}^n)$ and $A \in \mathscr{S}$ by the Beppo-Levi convergence theorem. The function $\omega \mapsto \hat{\Phi}_\omega(D)f$, $\omega \in \Omega$, is therefore Pettis μ-integrable in $L^p(\mathbf{R}^n)$, as required. □

The following elementary lemma asserts that the collection of all $\phi \in L_c^\infty(\mathbf{R}^n)$ satisfying $\|\phi\|_q \le 1$ is a norming set for $L^p(\mathbf{R}^n)$.

2.10 LEMMA. *Let $1 < p < \infty$ and $q = p/(p-1)$. Let $\psi \in L^1_{\text{loc}}(\mathbf{R}^n)$ be a function such that*

$$\alpha = \sup\left\{\left|\int_{R^n} \phi\psi \, d\lambda\right| : \phi \in L_c^\infty(\mathbf{R}^n), \|\phi\|_q \le 1\right\} < \infty.$$

Then $\psi \in L^p(\mathbf{R}^n)$ and $\|\psi\|_p = \alpha$.

2.11 LEMMA. *Let $2 < p < \infty$ and $q = p/(p-1)$. Then for each $f \in L_c^\infty(\mathbf{R}^n)$, the function $\omega \mapsto \hat{\Phi}_\omega(D)f$, $\omega \in \Omega$, is Pettis μ-integrable in $L^p(\mathbf{R}^n)$. Furthermore, there is a constant $\alpha_p > 0$ such that*

$$\left\|\int_A \hat{\Phi}_\omega(D) f \, d\mu(\omega)\right\|_p \le \alpha_p \|f\|_p$$

for every $f \in L_c^\infty(\mathbf{R}^n)$ and $A \in \mathscr{S}$.

Proof. The reflection operator Λ: $L^1(\mathbf{R}^n) \to L^1(\mathbf{R}^n)$ is defined by $\Lambda g(x) = g(-x)$ for all $g \in L^1(\mathbf{R}^n)$ and for almost all $x \in \mathbf{R}^n$. Set $\tilde{\Phi}_\omega(D) = [\Lambda\Phi(\omega)]^\wedge(D)$ for each $\omega \in \Omega$; then

$$\left\langle \hat{\Phi}_\omega(D)f, \phi\right\rangle = \left\langle f, \tilde{\Phi}_\omega(D)\phi\right\rangle$$

for every $f \in L_c^\infty(\mathbf{R}^n)$ and $\phi \in L_c^\infty(\mathbf{R}^n)$.

Let ϕ be an element of $L_c^\infty(\mathbf{R}^n)$ satisfying $\|\phi\|_q \le 1$. Because $1 < q < 2$, it is possible to apply Lemma 2.9 to the function $\omega \mapsto \Lambda\Phi(\omega)$, $\omega \in \Omega$, to deduce that the $L^q(\mathbf{R}^n)$-valued function $\omega \mapsto \tilde{\Phi}_\omega(D)\phi$, $\omega \in \Omega$, is Pettis μ-integrable in $L^q(\mathbf{R}^n)$, and there exist a constant $\beta_q > 0$ such that $\|\int_A \tilde{\Phi}_\omega(D)\phi \, d\mu(\omega)\|_q \le \beta_q \|\phi\|_q$ for each $A \in \mathscr{S}$. In particular,

$$\left|\int_A \left\langle \hat{\Phi}_\omega(D)f, \phi\right\rangle d\mu(\omega)\right| = \left|\int_A \left\langle f, \tilde{\Phi}_\omega(D)\phi\right\rangle d\mu(\omega)\right| \le \beta_q \|f\|_p \|\phi\|_q$$

for all $A \in \mathscr{S}$. The function $\omega \mapsto \hat{\Phi}_\omega(D)f$, $\omega \in \Omega$, is Pettis μ-integrable in $L^2(\mathbf{R}^n)$ by Theorem 2.3. Since $\phi \in L_c^\infty(\mathbf{R}^n) \subseteq L^2(\mathbf{R}^n)$, we have

$$\int_A \langle \hat{\Phi}_\omega(D)f, \phi \rangle \, d\mu(\omega) = \left\langle \int_A \hat{\Phi}_\omega(D) f \, d\mu(\omega), \phi \right\rangle$$

for each $A \in \mathscr{S}$. The function $\int_A \hat{\Phi}_\omega(D) f \, d\mu(\omega)$, being in $L^2(\mathbf{R}^n)$, belongs to $L^1_{\text{loc}}(\mathbf{R}^n)$, so an appeal to Lemma 2.10 proves that $\int_A \hat{\Phi}_\omega(D) f \, d\mu(\omega) \in L^p(\mathbf{R}^n)$ as well, and

$$\left\| \int_A \hat{\Phi}_\omega(D) f \, d\mu(\omega) \right\|_p \leq \beta_q \|f\|_p.$$

As yet, we only have the equality

$$\int_A \langle \hat{\Phi}_\omega(D)f, \phi \rangle \, d\mu(\omega) = \left\langle \int_A \hat{\Phi}_\omega(D) f \, d\mu(\omega), \phi \right\rangle$$

for all $A \in \mathscr{S}$ and all $\phi \in L_c^\infty(\mathbf{R}^n)$. However, [Th2, Corollary 5.1] or the direct argument of Lemma 2.9 establishes that $\int_\Omega |\langle \hat{\Phi}_\omega(D)f, \phi \rangle| \, d\mu(\omega) < \infty$ and

$$\int_A \langle \hat{\Phi}_\omega(D)f, \phi \rangle \, d\mu(\omega) = \left\langle \int_A \hat{\Phi}_\omega(D) f \, d\mu(\omega, \phi) \right\rangle$$

for all $A \in \mathscr{S}$ and all $\phi \in L^q(\mathbf{R}^n)$. □

The next lemma provides a convenient condition guaranteeing that a function is Pettis μ-integrable.

2.12 LEMMA. *Let X be a Banach space. Suppose that Ψ_j, $j = 1, 2, \ldots$, are strongly measurable Pettis μ-integrable functions converging μ-a.e. in X to a function Ψ: $\Omega \to X$. If the X-valued vector measures $\Psi_j \mu$, $j = 1, 2, \ldots$, are uniformly countably additive, then Ψ is Pettis μ-integrable and* $\lim_{j \to \infty} \sup_{A \in \mathscr{S}} \|\Psi\mu(A) - \Psi_j \mu(A)\| = 0$.

Proof. By virtue of Egorov's measurability theorem, there is an increasing collection of non-μ-null sets $\Omega_k \in \mathscr{S}$ of finite μ-measure, $k = 1, 2, \ldots$, which cover Ω, such that on each set Ω_k, $k = 1, 2, \ldots$, the functions Ψ_j, $j = 1, 2, \ldots$, converge uniformly to Ψ. Given $\varepsilon > 0$, choose $k = 1, 2, \ldots$, so large that $\|\Psi_j \mu((\Omega \setminus \Omega_k) \cap A)\| < \varepsilon/4$ for all $A \in \mathscr{S}$ and $j = 1, 2, \ldots$. There exists a positive integer J such that $\|\Psi(\omega) - \Psi_j(\omega)\| < \varepsilon/(4\mu(\Omega_k))$

for all $j > J$ and $\omega \in \Omega_k$. For all $A \in \mathscr{S}$ and $m, j > J$,

$$\begin{aligned}
&\|\Psi_m\mu(A) - \Psi_j\mu(A)\| \\
&\quad\le \|\Psi_m\mu(\Omega_k \cap A) - \Psi_j\mu(\Omega_k \cap A)\| \\
&\qquad + \|\Psi_m\mu((\Omega \setminus \Omega_k) \cap A) - \Psi_j\mu((\Omega \setminus \Omega_k) \cap A)\| \\
&\quad\le \|\Psi_m\mu(\Omega_k \cap A) - \Psi\mu(\Omega_k \cap A)\| \\
&\qquad + \|\Psi\mu(\Omega_k \cap A) - \Psi_j\mu(\Omega_k \cap A)\| + \varepsilon/2 \\
&\quad\le \int_{\Omega_k \cap A} \|\Psi_m - \Psi\|\, d\mu + \int_{\Omega_k \cap A} \|\Psi - \Psi_j\|\, d\mu + \varepsilon/2 < \varepsilon;
\end{aligned}$$

that is, $\lim_{m, j \to \infty} \sup_{A \in \mathscr{S}} \|\Psi_m\mu(A) - \Psi_j\mu(A)\| = 0$. In particular, for each $A \in \mathscr{S}$, the vectors $\Psi_j\mu(A)$, $j = 1, 2, \ldots$, are convergent in X.

To show the Pettis μ-integrability of Ψ, let $\xi \in X'$. Then the scalar function $\langle \Psi, \xi \rangle$ is μ-integrable because the functions $\langle \Psi_j, \xi \rangle$, $j = 1, 2, \ldots$, form a Cauchy sequence in mean by the above argument and $\lim_{j\to\infty} \langle \Psi_j(\omega), \xi \rangle = \langle \Psi(\omega), \xi \rangle$ for μ-almost every $\omega \in \Omega$, by assumption. Consequently, for every $A \in \mathscr{S}$,

$$\begin{aligned}
\int_A \langle \Psi, \xi \rangle\, d\mu &= \lim_{j \to \infty} \int_A \langle \Psi_j, \xi \rangle\, d\mu \\
&= \lim_{j \to \infty} \left\langle \Psi_j\mu(A), \xi \right\rangle \\
&= \left\langle \lim_{j \to \infty} \Psi_j\mu(A), \xi \right\rangle,
\end{aligned}$$

which proves that Ψ is Pettis μ-integrable. □

Proof of Theorem 2.7. Let $1 < p < \infty$. By Lemmas 2.9 and 2.11, for each $f \in L_c^\infty(\mathbf{R}^n)$ the function $\omega \mapsto \hat{\Phi}_\omega(D)f$, $\omega \in \Omega$, is Pettis μ-integrable in $L^p(\mathbf{R}^n)$. Furthermore, there is a constant $\alpha_p > 0$ such that $\|\int_A \hat{\Phi}_\omega(D) f\, d\mu(\omega)\|_p \le \alpha_p \|f\|_p$ for every $f \in L_c^\infty(\mathbf{R}^n)$ and $A \in \mathscr{S}$.

Let $f \in L^p(\mathbf{R}^n)$ and choose functions $f_k \in L_c^\infty(\mathbf{R}^n)$, $k = 1, 2, \ldots$, convergent to f in $L^p(\mathbf{R}^n)$. Given $\omega \in \Omega$, we have

$$\begin{aligned}
\|\hat{\Phi}_\omega(D)f - \hat{\Phi}_\omega(D)f_k\|_p &= \|\Phi(\omega) * f - \Phi(\omega) * f_k\|_p \\
&\le \|\Phi(\omega)\|_1 \|f - f_k\|_p \to 0 \quad \text{as } k \to \infty
\end{aligned}$$

and

$$\left\| \int_A \hat{\Phi}_\omega(D) f_k \, d\mu(\omega) - \int_A \hat{\Phi}_\omega(D) f_m \, d\mu(\omega) \right\|_p \leq \alpha_p \|f_k - f_m\|_p$$

for all $A \in \mathscr{S}$ and $k, m = 1, 2, \dots$. Then by Lemma 2.12, the function $\omega \mapsto \hat{\Phi}_\omega(D) f$, $\omega \in \Omega$, is Pettis μ-integrable in $L^p(\mathbf{R}^n)$ and

$$\int_A \hat{\Phi}_\omega(D) f \, d\mu(\omega) = \lim_{k \to \infty} \int_A \hat{\Phi}_\omega(D) f_k \, d\mu(\omega)$$

for each $A \in \mathscr{S}$. In particular, $\|\int_A \hat{\Phi}_\omega(D) f \, d\mu(\omega)\|_p \leq \alpha_p \|f\|_p$ for every $f \in L^p(\mathbf{R}^n)$ and $A \in \mathscr{S}$. □

2.13 *Remark*. The mean value theorem ensures that condition (ii) of Theorem 2.7 holds whenever the function $(\omega, x) \mapsto \Phi(\omega)(x)$, $\omega \in \Omega$, $x \in \mathbf{R}^n$, is jointly measurable, the function Φ_ω is smooth on $\mathbf{R}^n \setminus \{0\}$ for almost all $\omega \in \Omega$, and there exists a constant $B > 0$ such that $\int_\Omega |\nabla \Phi_\omega(x)| \, d\mu(\omega) \leq B/|x|^{n+1}$ for all $x \neq 0$. The analogue of Theorem 2.7 for non-convolution operators may be formulated in terms of this condition.

3. The Hilbert transform and Pettis integrals

The *Hilbert transform* H: $L^2(\mathbf{R}) \to L^2(\mathbf{R})$, is defined by $(Hf)\hat{} = \operatorname{sgn} \hat{f}$ for every $f \in L^2(\mathbf{R})$, so we have $H = \operatorname{sgn}(D)$ in the operational calculus for self adjoint operators. Here sgn: $\mathbf{R} \to \{-1, 0, 1\}$ is the signum function, $\operatorname{sgn}(x) = x/|x|$, $x \neq 0$, $\operatorname{sgn}(0) = 0$. The operator H is actually a singular integral operator in the following sense. For every $\varepsilon > 0$ and $f \in L^2(\mathbf{R})$, set

$$H_\varepsilon f(x) = \frac{i}{\pi} \int_{|x-y| > \varepsilon} \frac{1}{x - y} f(y) \, dy \quad \text{for all } x \in \mathbf{R}.$$

The integral exists by virtue of the Cauchy-Schwarz inequality and the so defined operator H_ε on $L^2(\mathbf{R})$ is continuous for every $\varepsilon > 0$. Then the operator H on $L^2(\mathbf{R})$ is also given by $Hf = \lim_{\varepsilon \to 0} H_\varepsilon f$ for each $f \in L^2(\mathbf{R})$. It is well-known that the limit converges in $L^2(\mathbf{R})$ (see, for example, [B-N, Theorem 8.1.7]), or alternatively it can be deduced from Proposition 3.2 and Lemma 3.3 below.

It is easy to check that H can be represented as a genuine Pettis integral

$$H = \int_0^\infty \hat{F}_t(D) \, dt$$

with respect to the function $F_t\colon \mathbf{R} \to \mathbf{R}$ defined by $F_t(x) = i\,\mathrm{sgn}(x)e^{-|x|t}/\pi$ for all $t > 0$ and $x \in \mathbf{R}$. The identity

$$\frac{2}{\pi}\int_0^\infty \frac{\xi}{t^2+\xi^2}\,dt = \mathrm{sgn}(\xi)$$

for every $\xi \neq 0$ together with Theorem 2.3 serve to establish this.

However, this observation alone does not explain why the introduction of a suitable auxiliary parameter enables the replacement of a principal value integral by an operator valued Pettis integral. Furthermore, the question arises of what is the relationship between the convergence of the principle value integral to the convergence of the Pettis integral. Rather than providing more results applying to the general framework, we illustrate the situation more completely in the following example.

3.1 *Example.* Given $t > 0$ and $\varepsilon \geq 0$, let $F_t^\varepsilon(x) = i\,\mathrm{sgn}(x)e^{-|x|t}/\pi$ for all $x \in \mathbf{R}$ such that $|x| > \varepsilon$, and $F_t^\varepsilon(x) = 0$ for all $x \in \mathbf{R}$ such that $|x| \leq \varepsilon$. We calculate the Fourier transform $\hat{F}_t^\varepsilon$ of F_t^ε for all $t > 0$ and $\varepsilon \geq 0$:

$$\begin{aligned}\hat{F}_t^\varepsilon(\xi) &= \int_{\mathbf{R}} F_t^\varepsilon(x)e^{-ix\xi}\,dx\\ &= \frac{i}{\pi}\left[\int_\varepsilon^\infty e^{-xt}e^{-ix\xi}\,dx - \int_{-\infty}^{\varepsilon} e^{xt}e^{-ix\xi}\,dx\right]\\ &= \frac{2e^{-\varepsilon t}}{\pi}\left[\frac{t\sin(\varepsilon\xi)+\xi\cos(\varepsilon\xi)}{t^2+\xi^2}\right], \quad \xi \in \mathbf{R}.\end{aligned}$$

Now for all $\varepsilon, t > 0$ and $\xi \in \mathbf{R}$,

$$\begin{aligned}\left|\frac{2}{\pi}\left[\frac{e^{-\varepsilon t}\,t\sin(\varepsilon\xi)}{t^2+\xi^2}\right]\right| &= \left|\frac{2}{\pi}(\varepsilon t e^{-\varepsilon t})\frac{\sin(\varepsilon\xi)}{\varepsilon\xi}\left[\frac{\xi}{t^2+\xi^2}\right]\right|\\ &\leq C_1\left[\frac{|\xi|}{t^2+\xi^2}\right], \qquad (1)\end{aligned}$$

where

$$C_1 = \frac{2}{\pi}\sup_{u>0} ue^{-u} \cdot \sup_{v>0}\frac{\sin v}{v};$$

also,

$$\left|\frac{2e^{-\varepsilon t}}{\pi}\left[\frac{\xi\cos(\varepsilon\xi)}{t^2+\xi^2}\right]\right| \leq \frac{2}{\pi}\frac{|\xi|}{t^2+\xi^2}. \qquad (2)$$

Because

$$\frac{2}{\pi}\int_0^\infty \frac{|\xi|}{t^2+\xi^2}\,dt = 1$$

for every $\xi \neq 0$, it follows from Theorem 2.3 that for every $\varepsilon \geq 0$, the function $t \mapsto \hat{F}_t^\varepsilon(D)$, $t > 0$, is Pettis integrable in $\mathscr{L}(L^2(\mathbf{R}))$. According to Remark 2.13 and Theorem 2.7, the function $t \mapsto \hat{F}_t^\varepsilon(D)$, $t > 0$, is Pettis integrable in $\mathscr{L}(L^p(\mathbf{R}))$ whenever $\varepsilon \geq 0$ and $1 < p < \infty$. □

Let $F_t = F_t^0$ for every $t > 0$.

3.2 Proposition. *The limit* $\lim_{\varepsilon\to 0}\int_A \hat{F}_t^\varepsilon(D)\,dt = \int_A \hat{F}_t(D)\,dt$ *converges in the strong operator topology of* $\mathscr{L}(L^2(\mathbf{R}))$, *uniformly for* $A \in \mathscr{B}(\mathbf{R})$.

Proof. Let

$$\nu(A)(\xi) = \int_A |\xi|/(t^2+\xi^2)\,dt$$

for all $A \in \mathscr{B}(\mathbf{R})$ and $\xi \in \mathbf{R}$. Then $|\nu(A)(\xi)| \leq \pi/2$ for all $A \in \mathscr{B}(\mathbf{R})$ and $\xi \in \mathbf{R}$, so it follows by dominated convergence that $A \mapsto \nu(A)\cdot f$, $A \in \mathscr{B}(\mathbf{R})$, is an $L^2(\mathbf{R})$-valued measure for each $f \in L^2(\mathbf{R})$. Now by (1), $|\hat{F}_t^\varepsilon(\xi)| \leq C_1|\xi|/(t^2+\xi^2)$ for all $\varepsilon, t > 0$ and $\xi \in \mathbf{R}$. By the Plancherel formula

$$\left|\left\langle \int_A \hat{F}_t^\varepsilon(D) f\,dt, g\right\rangle\right| \leq C_1(2\pi)^{-1}\|\nu(A)\hat{f}\|_2\|g\|_2, \quad A \in \mathscr{B}(\mathbf{R}),$$

for all $f, g \in L^2(\mathbf{R})$ and for all $\varepsilon > 0$. Therefore, for each $f \in L^2(\mathbf{R})$, the family of $L^2(\mathbf{R})$-valued vector measures $A \mapsto \int_A \hat{F}_t^\varepsilon(D) f\,dt$, $A \in \mathscr{B}(\mathbf{R})$, $\varepsilon > 0$, is uniformly countably additive. By virtue of the estimate (1), dominated convergence and the Plancherel formula, $\hat{F}_t^\varepsilon(D) \to \hat{F}_t(D)$ in the strong operator topology on $\mathscr{L}(L^2(\mathbf{R}))$ as $\varepsilon \to 0$, for each $t > 0$. It follows from Lemma 2.12 that the function $t \mapsto \hat{F}_t(D)$, $t > 0$, is Pettis integrable in $\mathscr{L}(L^2(\mathbf{R}))$, and given $f \in L^2(\mathbf{R})$, we have $\lim_{\varepsilon\to 0}\int_A \hat{F}_t^\varepsilon(D) f\,dt = \int_A \hat{F}_t(D) f\,dt$ in the norm topology of $L^2(\mathbf{R})$, uniformly for $A \in \mathscr{S}$. □

The following lemma is most easily proved by computing Fourier transforms and appealing to Theorem 2.3. The following proof illustrates directly how the choice of the functions F_t^ε, $\varepsilon \geq 0$, $t > 0$ was dictated by the requirement that

$$\int_0^\infty F_t^\varepsilon(x)\,dt = \frac{i}{\pi x}\chi_{(\varepsilon,\infty)}(|x|),$$

for all $x \neq 0$, in conjunction with the application of Fubini's theorem.

3.3 LEMMA. *For every $\varepsilon > 0$, the identity $H_\varepsilon = \int_0^\infty \hat{F}_t^\varepsilon(D)\,dt$ holds.*

Proof. Let $f \in L^2(\mathbf{R})$ and $g \in L^2(\mathbf{R}) \cap L^1(\mathbf{R})$. Then

$$\begin{aligned}\int_0^\infty \left(\hat{F}_t^\varepsilon(D)f, g\right) dt &= \int_0^\infty (F_t^\varepsilon * f, g)\, dt \\ &= \int_0^\infty \int_{\mathbf{R}} \left[\int_{\mathbf{R}} F_t^\varepsilon(x-y) f(y)\, dy\right] \overline{g(x)}\, dx\, dt.\end{aligned}$$

Now for each $t > 0$, the function $(x, y) \to \overline{g(x)} F_t^\varepsilon(x-y) f(y)$, $x, y \in \mathbf{R}$, is integrable by the Tonelli's theorem, and

$$\int_{\mathbf{R}^2} \left|\overline{g(x)} F_t^\varepsilon(x-y) f(y)\right| dx\, dy \leq \|g\|_2 \|F_t^\varepsilon\|_1 \|f\|_2.$$

Let Λ: $L^2(\mathbf{R}) \to L^2(\mathbf{R})$ be the reflection operator $\Lambda f(x) = f(-x)$, $x \in \mathbf{R}$, $f \in L^2(\mathbf{R})$ on $L^2(\mathbf{R})$. Then by a change of variables, we have

$$\int_{\mathbf{R}} \left[\int_{\mathbf{R}} F_t^\varepsilon(x-y) f(y)\, dy\right] \overline{g(x)}\, dx = \int_{\mathbf{R}} F_t^\varepsilon(u) (\Lambda f) * \bar{g}(u)\, du.$$

Now $(\Lambda f) * \bar{g} \in L^2(\mathbf{R})$, and for each $u \in \mathbf{R}$ with $\mu \neq 0$,

$$\int_0^\infty F_t^\varepsilon(u)\, dt = \frac{i}{\pi u} \chi_{(\varepsilon, \infty)}(|u|)$$

so again, by the Fubini-Tonelli theorem, $(t, u) \mapsto F_t^\varepsilon(u)(\Lambda f) * \bar{g}(u)$, $t > 0$, $u \in \mathbf{R}$, is integrable on $[0, \infty) \times \mathbf{R}$, and we have

$$\begin{aligned}&\int_0^\infty \left[\int_{\mathbf{R}} F_t^\varepsilon(u)(\Lambda f) * \bar{g}(u)\, du\right] dt \\ &\quad= \int_{\mathbf{R}} \left[\int_0^\infty F_t^\varepsilon(u)(\Lambda f) * \bar{g}(u)\, dt\right] du \\ &\quad= \frac{i}{\pi} \int_{\mathbf{R}} \frac{1}{u} \chi_{(\varepsilon, \infty)}(|u|)(\Lambda f) * \bar{g}(u)\, du \\ &\quad= \frac{i}{\pi} \int_{\mathbf{R}} \left[\int_{\mathbf{R}} \chi_{(\varepsilon, \infty)}(|x-y|) \frac{1}{x-y} f(y)\, dy\right] \bar{g}(x)\, dx \\ &\quad= (H_\varepsilon f, g).\end{aligned}$$

The equality

$$\int_0^\infty \left(\hat{F}_t^\varepsilon(D) f, g\right) dt = (H_\varepsilon f, g)$$

therefore follows for every $f \in L^2(\mathbf{R})$ and every $g \in L^2(\mathbf{R}) \cap L^1(\mathbf{R})$. Because $L^2(\mathbf{R}) \cap L^1(\mathbf{R})$ is dense in $L^2(\mathbf{R})$, we have $H_\varepsilon = \int_0^\infty \hat{F}_t^\varepsilon(D)\,dt$. □

3.4 THEOREM. *The equality $H = \int_0^\infty \hat{F}_t(D)\,dt$ holds in the space $\mathscr{L}(L^2(\mathbf{R}))$.*

Proof. The statement is a consequence of Proposition 3.2 and Lemma 3.3 because

$$H = \lim_{\varepsilon \to 0} H_\varepsilon \quad \text{and} \quad \lim_{\varepsilon \to 0} \int_0^\infty \hat{F}_t^\varepsilon(D)\,dt = \int_0^\infty \hat{F}_t(D)\,dt$$

in the strong operator topology. □

As mentioned previously, for each $1 < p < \infty$, the function $t \mapsto \hat{F}_t(D)$, $t > 0$, is Pettis integrable in $\mathscr{L}(L^p(\mathbf{R}))$. The Hilbert transform H and the integral $\int_0^\infty \hat{F}_t(D)\,dt$ are also equal as operators on the space $L^p(\mathbf{R})$.

The next example shows that $\int_0^\infty \hat{F}_t(D)\,dt$ is a genuine *Pettis* integral, not a Bochner integral, that is, there exists a function $f \in L^2(\mathbf{R})$ such that $\int_0^\infty \|\hat{F}_t(D)f\|_2\,dt = \infty$.

3.5 *Example.* Suppose that $\hat{f}(x) = x^{-1/2}|\ln(x) - 1|^{-1}\chi_{(0,1]}(x)$ for all $x \in \mathbf{R}$. Then

$$\begin{aligned}
\int_0^\infty \|\hat{F}_t \cdot \hat{f}\|_2\,dt &= \frac{2}{\pi}\int_0^\infty \left[\int_0^1 \frac{\xi|\ln \xi - 1|^{-2}}{(\xi^2 + t^2)^2}\,d\xi\right]^{1/2} dt \\
&= \frac{2}{\pi}\int_0^\infty \frac{1}{t}\left[\int_0^{1/t} \frac{\xi}{(\xi^2 + 1)^2|\ln(\xi t) - 1|^2}\,d\xi\right]^{1/2} dt \\
&\geq \frac{2}{\pi}\int_0^{1/2} \frac{1}{t}\left[\int_1^2 \frac{\xi}{(\xi^2 + 1)^2|\ln(\xi t) - 1|^2}\,d\xi\right]^{1/2} dt \\
&\geq \frac{2\sqrt{2}}{5\pi}\int_0^{1/2} \frac{1}{t|\ln(t) - 1|}\,dt = \infty
\end{aligned}$$

The function f belongs to $L^2(\mathbf{R})$ and it follows from Plancherel's theorem that

$$\int_0^\infty \|\hat{F}_t(D)f\|_2\,dt = \infty. \qquad \square$$

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