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A NEW PROOF OF THE RESTRICTION THEOREM FOR WEAK TYPE (1,1) MULTIPLIERS ON $\mathbf{R}^n$

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1 Introduction

In [4, Problem 5], A Pelczynski asked the following question is it true that if a multiplier function m continuous at lattice points determines a weak type $(1, 1)$ multiplier operator $Mf = (m\hat{f})^{\vee}$ then the restriction m' of m to these points determines a weak type $(1, 1)$ multiplier operator $M'f = (m'\hat{f})^{\vee}$ on $\mathbf{T}^n$? In other words, can we extend the classical de Leeuw theorem [3, Proposition 3 3] to the weak $(1, 1)$ case? The positive answer was given in [1, Theorem 1 1] In this paper the authors have shown (Theorem 1 2) that convolutions with L_1 functions leave the space of weak type multipliers invariant This allows us to consider only functions m such that kernel $m^{\vee}$ has compact support and then to use the classical argument of Calderón [2] However the consequence of using this strong result is that one gets the estimate $\|M'\| \leq C\|M\|$ for some constant $C > 1$ which appears in the formulation of the Theorem 1 2 and it seems that in the case of that theorem this constant must be greater then 1 The purpose of this work is to give a direct proof of Pelczynski conjecture with $C = 1$ The main idea is based on taking the averages over big subsets of $\mathbf{R}^n$ and to some extend is simlar to the method of Calderón

2 Notation

$\mathbf{R}^n$ stands for the n-dimensional vector space and $\mathbf{Z}^n$ for the sublattice of $\mathbf{R}^n$ consisting of the points with integer-valued coordinates The dual group of $\mathbf{Z}^n$—the n–dimensional torus $\mathbf{T}^n$—will be identified with the cube $[-\pi, \pi]^n \subset \mathbf{R}^n$, whose boundary points are identified in the standard way The symbols λ_n, μ_n stand for invariant measures on $\mathbf{R}^n$ and $\mathbf{T}^n$ respectively, determined by conditions $\lambda_n(\{|x|_\infty \leq 1/2\}) = 1$, $\mu_n(\mathbf{T}^n) = (2\pi)^n$, where $|x|_\infty = \max_{1\leq j\leq n} |x_j|$ for $x = (x_j) \in \mathbf{R}^n$ The symbols $L_p(\mathbf{R}^n) = L_p(\mathbf{R}^n, \lambda_n)$, $L_p(\mathbf{T}^n) = L_p(\mathbf{T}^n, \mu_n)$ have usual meanings and denote complex valued functions The norms in this spaces will be denoted by $\| \ \|_{p\,\mathbf{R}}$ and $\| \ \|_{p\,\mathbf{T}}$ respectively By $L_1^*(\mathbf{R}^n)$, $L_1^*(\mathbf{T}^n)$ we denote the weak L_1 spaces

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of functions for which the quasinorms

$$\|\phi\|_{\mathbf{R}}^{*} = \sup_{c>0} c\lambda_n(\{|\phi| > c\}) \qquad \text{for } \phi\colon \mathbf{R}^n \to \mathbf{C},$$

$$\|f\|_{\mathbf{T}}^{*} = \sup_{c>0} c\lambda_n(\{|f| > c\}) \qquad \text{for } f\colon \mathbf{T}^n \to \mathbf{C}$$

are bounded. The weak norm $\|\cdot\|^*$ of an operator $M\colon L_1(\mathbf{X}) \to \mathbf{L}_1^*(\mathbf{X})$ for $\mathbf{X} = \mathbf{R}^n, \mathbf{Z}^n$ is defined in a standard way as $\|M\|^* = \sup_{x\in L_1(\mathbf{X})\setminus\{0\}} \|M(x)\|_{\mathbf{X}}^*/\|x\|_{1,\mathbf{X}}$. By $Trig(\mathbf{T}^n)$ we denote the space of trigonometric polynomials on $\mathbf{T}^n$, i.e., the space of all finite linear combinations of the exponents $e^{i\langle\cdot,a\rangle}$ for $a \in \mathbf{Z}^n$. By $\langle x, y\rangle = \sum_{j=1}^n x_j y_j$ we denote the usual scalar product of vectors x and y in $\mathbf{R}^n$. For $\phi \in L_1(\mathbf{R}^n)$ and for $f \in Trig(\mathbf{T}^n)$ the Fourier Transforms $\hat{\phi}$ and $\tilde{f}$ are defined by

$$\hat{\phi}(\xi) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} \phi(x)e^{-i\langle x,\xi\rangle}dx \qquad \text{for } \xi \in \mathbf{R}^n,$$

$$\tilde{f}(a) = \frac{1}{(2\pi)^n}\int_{\mathbf{T}^n} f(x)e^{-i\langle x,a\rangle}dx \qquad \text{for } a \in \mathbf{Z}^n.$$

Finally letter $\mathcal{D}$ stands for the Schwartz class—the space of infinitely many times differentiable functions on $\mathbf{R}^n$ with compact support.

3. Weak type multipliers on $\mathbf{R}^n$ and $\mathbf{Z}^n$

The main result of the present paper is essentially contained in the following:

PROPOSITION 1. *Let $m \in L_\infty(\mathbf{R}^n)$ be continuous at the points of $\mathbf{Z}^n$. Define the operators $M_{\mathbf{R}}\colon L_1(\mathbf{R}^n) \cap L_2(\mathbf{R}^n) \to L_2(\mathbf{R}^n)$ and $M_{\mathbf{T}}\colon Trig(\mathbf{T}^n) \to Trig(\mathbf{T}^n)$ by*

$$[M_{\mathbf{R}}(\phi)]\widehat{\ } = m\hat{\phi}$$

and

$$[M_{\mathbf{T}}(f)]\widetilde{\ }(a) = m(a)\tilde{f}(a) \qquad \textit{for } a \in \mathbf{Z}^n.$$

Then the relation

$$\text{(1)} \qquad \sup_{c>0} c\lambda_n(\{|M_{\mathbf{R}}\phi| > c\}) \le \|\phi\|_{1,\mathbf{R}} \qquad \textit{for } \phi \in L_1(\mathbf{R}^n) \cap L_2(\mathbf{R}^n)$$

implies

$$\text{(2)} \qquad \sup_{c>0} c\mu_n(\{|M_{\mathbf{T}}f| > c\}) \le \|f\|_{1,\mathbf{T}} \qquad \textit{for } f \in Trig(\mathbf{T}^n).$$

Before proving Proposition 1 we will introduce the following notation: for $f \in Trig(\mathbf{T}^n)$, $f(\cdot) = \sum_{a\in A} \hat{f}(a)e^{i\langle a,\cdot\rangle}$, $A \subset \mathbf{Z}^n$, denote by F the periodic extension of f on $\mathbf{R}^n$; i.e.,

$$F(x) = \sum_{a\in A} \hat{f}(a)e^{i\langle a,x\rangle} \qquad \text{for } x \in \mathbf{R}^n,$$

and for $g = M_{\mathbf{T}}(f) = \sum_{a\in A} m(a)\hat{f}(a)e^{i\langle a,\cdot\rangle}$ denote by G the analogous periodic extension of g. Now for $k = 1, 2, \ldots$ pick an real-valued function $H_k \in \mathcal{D}$ such that

$$0 \le H_k(x) \le 1 \qquad \text{for } x \in \mathbf{R}^n, \tag{3}$$

$$H_k(x) = 1 \qquad \text{for } |x|_\infty \le k\pi, \tag{4}$$

$$H_k(x) = 0 \qquad \text{for } |x|_\infty \ge (k+1)\pi. \tag{5}$$

Then for $\varepsilon > 0$ define $H_k^\varepsilon(x) = H_k(\varepsilon x)$ for $x \in \mathbf{R}^n$ and finally let

$$R_k^\varepsilon = M_{\mathbf{R}}(H_k^\varepsilon F) - H_k^\varepsilon G$$

(this definition makes sense as $H_k^\varepsilon F \in L_1(\mathbf{R}^n) \cap L_2(\mathbf{R}^n)$). We have:

PROPOSITION 2. *$R_k^\varepsilon \in L_\infty(\mathbf{R}^n)$ and*

$$\lim_{\varepsilon\to 0} \|R_k^\varepsilon\|_{\infty,\mathbf{R}} = 0 \qquad \textit{for} \quad k = 1, 2, \ldots. \tag{6}$$

Proof. The inverse formula for Fourier transform yields

$$\|R_k^\varepsilon\|_{\infty,\mathbf{R}} \le \|(R_k^\varepsilon)\widehat{\ }\|_{1,\mathbf{R}}.$$

On the other hand we have

$$(R_k^\varepsilon)\widehat{\ }(y) = \sum_{a\in A}(m(y) - m(a))\tilde{f}(a)\widehat{H}_k\left(\frac{y-a}{\varepsilon}\right)\varepsilon^{-n} \tag{7}$$

(by linearity it is enough to verify (7) for a single exponent which is trivial). Thus using the substitutions $z = \frac{y-a}{\varepsilon}$ for $a \in A$ we get

$$\begin{aligned}\|(R_k^\varepsilon)\widehat{\ }\|_{1,\mathbf{R}} &= \int_{\mathbf{R}^n}\left|\sum_{a\in A}(m(y) - m(a))\tilde{f}(a)\widehat{H}_k\left(\frac{y-a}{\varepsilon}\right)\varepsilon^{-n}\right| dy \\ &\le \sum_{a\in A}|\tilde{f}(a)|\int_{\mathbf{R}^n}\left|\widehat{H}_k(z)\right| |m(a+\varepsilon z) - m(a)|\, dz.\end{aligned}$$

Note that $\widehat{H}_k \in L_1(\mathbf{R}^n)$ because $H_k \in \mathcal{D}$. So $R_k^\varepsilon \in L_\infty(\mathbf{R}^n)$ and as the set A is finite applying Lebesgue Dominated Convergence Theorem we get (6). □

Now we can easily get the proof of Proposition 1.

Proof. Fix $c > 0$, $t > 1$ and $k = 1, 2, \ldots$. It follows from Proposition 2 that for sufficiently small $\varepsilon > 0$ one has $\|R_k^\varepsilon\| \le (1 - \frac{1}{t})c$. Thus

$$\left\{|H_k^\varepsilon G| > c\right\} \subset \left\{|M_{\mathbf{R}}(H_k^\varepsilon F)| > \frac{c}{t}\right\}.$$

So by (1),

$$c\lambda_n\left(\left\{|H_k^\varepsilon G| > c\right\}\right) \le t\frac{c}{t}\lambda_n\left(\left\{|M_{\mathbf{R}} H_k^\varepsilon F| > \frac{c}{t}\right\}\right) \le t\|H_k^\varepsilon F\|_1. \tag{8}$$

Let $[z]$ denote the greatest integer less than or equal to z and let 1_Ω denote the indicator function of a set Ω. Taking (4) into account we get

$$\begin{aligned} \{|H_k^\varepsilon G| > c\} &\supset \{|G| > c \;\&\; |\varepsilon x|_\infty \le k\pi\} \\ &\supset \{|G| > c \;\&\; |x|_\infty \le [\tfrac{k}{\varepsilon}]\pi\}. \end{aligned}$$

Thus

$$\begin{aligned} \lambda_n(\{|H_k^\varepsilon G| > c\}) &\ge \lambda_n(\{|G| > c \;\&\; |x|_\infty \le [\tfrac{k}{\varepsilon}]\pi\} \\ &= [\tfrac{k}{\varepsilon}]^n \mu_n(\{|g| > c\}) \end{aligned} \tag{9}$$

On the other hand, (5) yields

$$\|H_k^\varepsilon F\|_{1,\mathbf{R}} \le \|F \cdot 1_{\{|x|_\infty \le ([\frac{k+1}{\varepsilon}]+1)\pi\}}\|_{1,\mathbf{R}} = ([\tfrac{k+1}{\varepsilon}] + 1)^n \|f\|_{1,\mathbf{T}}. \tag{10}$$

Combining (8), (9) and (10) we get

$$[\tfrac{k}{\varepsilon}]^n c\mu_n(\{g > c\}) \le t([\tfrac{k+1}{\varepsilon}]^n \|f\|_{1,\mathbf{T}}. \tag{11}$$

Multiplying both sides of (11) by ε^n and letting $\varepsilon \to 0$ we get $k^n c\mu_n(\{g > c\}) \le t(k+1)^n \|f\|_{1,\mathbf{T}}$. Now dividing by k, letting k go to infinity and then setting t to 1 we get (2). □

An immediate consequence of Proposition 1 is:

THEOREM 3. *Let m, $M_{\mathbf{R}}$, $M_{\mathbf{T}}$ be as in Proposition 1. In the case $m \in C(\mathbf{R}^n)$ for $\varepsilon > 0$ and $f \in Trig(\mathbf{T}^n)$ put $M_{\mathbf{T}}^\varepsilon = \sum m(\varepsilon a)\hat{f}(a)e^{i\langle a, \cdot\rangle}$. Then if $M_{\mathbf{R}}$ extends to a bounded operator $\overline{M}_{\mathbf{R}}$: $L_1(\mathbf{R}^n) \to L_1^*(\mathbf{R}^n)$ then $M_{\mathbf{T}}$ extends to a bounded operator $\overline{M}_{\mathbf{T}}$: $L_1(\mathbf{T}^n) \to L_1^*(\mathbf{T}^n)$ and $\|\overline{M}_{\mathbf{T}}\|^* \le \|\overline{M}_{\mathbf{R}}\|^*$. If $m \in C(\mathbf{R}^n)$ the same holds for $M_{\mathbf{T}}^\varepsilon$. Moreover in this case*

$$\|\overline{M}_{\mathbf{R}}\|^* = \sup_{\varepsilon > 0} \|\overline{M_{\mathbf{T}}^\varepsilon}\|^* \tag{12}$$

Proof. The first part of Theorem 3 is essentially a reformulation of Proposition 1. As norms of multiplier operators induced by $m(x)$ and $m(\varepsilon x)$ are the same we get the claim for $M^{\varepsilon}_{\mathbf{T}}$ when $m \in C(\mathbf{R}^n)$. Formula (12) is also trivial and our argument here is a modification of that used in [5, Theorem VII.3.18] where the classical L_p-case was considered: for $f \in \mathcal{D}$, $x \in \mathbf{R}^n$ let $f_\varepsilon(x) = \sum_{a\in\mathbf{Z}^n} \hat{f}(\varepsilon a)e^{i\langle a,x\rangle} = \varepsilon^{-n}\sum_{a\in\mathbf{Z}^n} f(\frac{x-2\pi a}{\varepsilon})$ and $g_\varepsilon(x) = \varepsilon^n \sum_{a\in\mathbf{Z}^n} m(\varepsilon a)\hat{f}(\varepsilon a)e^{i\langle\varepsilon a,x\rangle}$. Then $\lim_{\varepsilon\to 0} g_\varepsilon(x) = \overline{M}_{\mathbf{T}} f(x)$ as a Riemann integral of the function $m(y)\hat{f}(y)e^{i\langle y,x\rangle}$. So for any $K > 0$,

$$(13)\quad c\lambda_n(\{|\overline{M}_{\mathbf{R}} f| > c \;\&\; |x|_\infty \le K\}) \le \lim_{\varepsilon\to 0} c\lambda_n(\{|g_\varepsilon(x)| > c \;\&\; |x|_\infty \le K\}).$$

But

$$(14)\quad \begin{aligned} c\lambda_n(\{|g_\varepsilon(x)| > c \;\&\; |x|_\infty \le K\}) &= c\varepsilon^{-n}\lambda_n(\{|g_\varepsilon(\tfrac{x}{\varepsilon})| > c \;\&\; |x|_\infty \le \varepsilon K\}) \\ &\le c\varepsilon^{-n}\lambda_n(\{|g_\varepsilon(\tfrac{x}{\varepsilon})| > c \;\&\; |x|_\infty \le \pi\}). \end{aligned}$$

for a small ε. On the other hand we may look at $f_\varepsilon(x)$ as defined on $\mathbf{T}^n$ and, in this case, $g_\varepsilon(\frac{x}{\varepsilon}) = \varepsilon^n \sum_{a\in\mathbf{Z}^n} m(\varepsilon a)\hat{f}(\varepsilon a)e^{i\langle a,x\rangle}$ can be seen as $\varepsilon^n \overline{M}^{\varepsilon}_{\mathbf{T}} f_\varepsilon(x)$ so

$$(15)\quad \begin{aligned} c\lambda_n(\{|g_\varepsilon(\tfrac{x}{\varepsilon})| > c \;\&\; |x|_\infty \le \pi\}) &\le \varepsilon^n \|\overline{M}^{\varepsilon}_{\mathbf{T}}\|^* \|f_\varepsilon(x)\|_{1,\mathbf{T}} \\ &\le \varepsilon^n \|\overline{M}^{\varepsilon}_{\mathbf{T}}\|^* \|f(x)\|_{1,\mathbf{R}} \end{aligned}$$

Combining (13), (14), (15) and letting K go to infinity we get the claim. □

Remark 4. One can easily observe that the statement of Theorem 3 remains true if we consider the operators from the Lorentz spaces $L(r, p)$ $p \ge 1$, $0 < r < \infty$ ($1 \le r$ for $p = 1$) into $L(s, p)$ $0 < s < \infty$.

REFERENCES

1. N. Asmar, E. Berkson and J. Bourgain, *Restrictions from $\mathbf{R}^n$ to $\mathbf{Z}^n$ of weak type multipliers*, Studia Math. **108** (1994), 291–299.
2. A. P. Calderón, *Ergodic theory and translation-invariant operators*, Proc. Nat. Acad. Sci. (U.S.A.) **59** (1968), 349–353.
3. K. de Leeuw *On L_p multipliers* Ann. of Math. **81** (1965), 346–347.
4. A. Pelczyński *Boundness of canonical projections for Sobolew spaces generated by finite families of linear differential operators*, Proceedings of the Special Year in Modern Analysis at the University of Illinois, 1986–87, London Math. Soc. Lecture Notes Series 137, Cambridge University Press, Cambridge, England, 1989, pp. 395–415.
5. E. M. Stein, G. Weiss *Introduction to Fourier analysis on Euclidean spaces*, Princeton University Press, Princeton, New Jersey, 1971.

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