ILLINOIS JOURNAL OF MATHEMATICS
Volume 41, Number 3, Fall 1997

GROUP ACTIONS AND THE TOPOLOGY OF NONNEGATIVELY CURVED 4-MANIFOLDS

ANDREW HICKS

Abstract

We consider nonnegatively curved 4-manifolds that admit effective isometric actions by finite groups and from that draw topological conclusions about the manifold. Our first theorem shows that if the manifolds admits an isometric $Z_p \times Z_p$ for p large enough that the manifold has Euler characteristic less than or equal to five. Our second theorem requires no hypothesis on the structure of the group other then that it be large but it does require the manifold to be $\delta-$pinched, in which case we can then again conclude that the Euler characteristic is less than or equal to five.

1. Introduction

Little is known about the topology of compact positively or nonnegatively curved 4-manifolds. The only known simply-connected examples with positive curvature are S^4 and CP^2. For nonnegative curvature we have in addition $S^2 \times S^2$ and $\pm CP^2 \# CP^2$, this last example due to Cheeger [Ch]. Also, few topological obstructions to having positive or nonnegative sectional curvature are known (we will discuss this in more detail later). Our starting point is a theorem of Hsiang and Kleiner, which states that a positively curved 4-manifold admitting an effective isometric S^1 action must be homeomorphic to S^4 or CP^2. In this paper we examine what could be said if there is a finite group of isometries acting on a nonnegatively curved manifold, but not necessarily an entire circle acting on the manifold.

THEOREM 1. *Let M be a compact 4-dimensional manifold with nonnegative sectional curvature. Then there exists a positive integer N such that if p is a prime with $p > N$ and $Z_p \times Z_p$ acts on M isometrically and effectively, then $\chi(M) \leq 5$.*

The number N can in principle be calculated from the methods of the proof. For the proof of this theorem to work the number p must be a prime, although the theorem might be true without this assumption.

If we do not make any assumption about the structure of the group, we are able to prove a general theorem under the assumption of a fixed pinching:

Received July 1, 1996.
1991 Mathematics Subject Classification. Primary 53C20; Secondary 53C21.

THEOREM 2. *Let M be a compact 4-dimensional manifold with nonnegative sectional curvature. Then for every $\delta > 0$ there exists a positive integer $N = N(\delta)$ such if $1 \geq sec(M) \geq \delta > 0$ and $|Isom(M)| > N$, then $\chi(M) \leq 3$.*

Again, it is conceivable that the theorem is true in general for $sec(M) \geq 0$, i.e., that N does not depend on δ, but the methods of our proof do not work in that situation. The main method in proving the above two theorems is to gain control over the number of fixed points of the group action. These methods hold for groups and manifolds more general than the ones above, but only have topological implications in the above situations.

THEOREM 3. *Let M be a 4-dimensional manifold with nonnegative sectional curvature and let G be a finite abelian group acting isometrically and effectively on M. Then there exists a positive integer N such that if for every prime p that divides the order of G it is the case that $p > N$, then the fixed point set of G can contain at most 5 isolated fixed points.*

THEOREM 4. *Let M be a 4-dimensional manifold with nonnegative sectional curvature and let G be a finite group acting isometrically and effectively on M. Then there is a positive integer N such that if $n > N$ and G contains a subgroup isomorphic to $Z_n \times Z_n$, then G can act with at most 5 fixed points. In particular, all of the fixed points must be isolated.*

Notice that in Theorem 3 we control only the number of isolated fixed points and say nothing about the non-isolated ones, which would have to be embedded S^2's or RP^2's. The smallest number of fixed points of the actions in Theorems 3 and 4 maybe 4. One can easily construct an action of $Z_n \times Z_n$ on $S^2 \times S^2$ with 4 fixed points. No examples are known with 5 fixed points.

Observe that it follows from the work of Friedman that if M is simply connected and $\chi(M) \leq 3$ then we may conclude that M is homeomorphic to one of the five manifolds

$$S^4, S^2 \times S^2, CP^2, \pm CP^2 \# CP^2,$$

and that if $\chi(M) \leq 5$ we must add to the list the two spaces

$$\pm CP^2 \# CP^2 \# CP^2.$$

The proof makes use of a new metric invariant, the q-extent, introduced in [GM], and an associated inequality. The motivation for this work comes from a long series of results that relate the geometry and topology of nonnegatively and positively curved manifolds. Recent activity began with the result of Hsiang and Kleiner [HK], which is a statement about 4-manifolds. Let M be a compact positively curved 4-manifold. Then if M admits an effective isometric S^1-action, M is homeomorphic to S^4, RP^4

or CP^2. Here it is important to notice that we are combining the assumptions about dimension and the amount of symmetry. It is also interesting in light of the fact that the only known examples of simply connected 4-dimensional compact positively curved manifolds are S^4 and CP^2. In particular it is unknown whether $S^2 \times S^2$ carries a metric of positive curvature. An analogous theorem for nonnegative curvature does exist and was proved independently by Kleiner [K], Yang and Searle [SY]: a compact simply connected, nonnegatively curved 4-manifold admitting an effective isometric S^1-action is homeomorphic to S^4, CP^2, $\pm CP^2 \# CP^2$ or $S^2 \times S^2$. These last two results rely on the understanding of 4-manifolds due the the work of Friedman [F], so the techniques do not work in higher dimensions. But a hypothesis common to both theorems is that the 4-manifold has some continuous symmetry. A natural question to ask at this point is what results if one assumes only that the symmetry group is finite. Working in that direction and using an idea from [GM] the author proved Theorems 3 and 4. In the meantime Yang [Y] showed that if M is a compact, positively curved 4-manifold admitting an effective isometric Z_p action for p sufficiently large, then $\chi(M) \leq 7$. Finally, using a trick from [Y], the author showed Theorem 1 to be true.

2. Background

To prove the theorems stated in the introduction we first need to develop some foundations. If V, W are subsets of a metric space Z, then the Hausdorff distance between them is defined as

$$d_H^Z(V, W) = \inf\{\epsilon \mid V \text{ is contained in an } \epsilon\text{-neighborhood of } W \text{ and } W \text{ is contained in an } \epsilon\text{-neighborhood of } V\}.$$

If X, Y are metric spaces we define the Gromov-Hausdorff distance between X and Y as

$$d_{GH}(X, Y) = \inf\{d_H^Z(X, Y) \mid X, Y \text{ are isometrically imbedded in } Z\}.$$

If G is a closed subgroup of $O(n+1)$ then G acts isometrically on S^n and $X = S^n/G$ may be viewed as a collection of orbits in S^n equipped with the orbital distance metric, d_O, which is defined as

$$d_O(Gx, Gy) = \min\{d(gx, hy) \mid g, h \in G\}.$$

for orbits Gx and Gy, where d denotes the usual spherical distance on $S^n(1)$. It is not hard to see that this distance is the same as the Hausdorff distance between the orbits; i.e., $d_O(Gx, Gy) = d_H^{S^n}(Gx, Gy)$. Let $K(S^n)$ be the collection of all compact subsets of S^n. Then $(K(S^n), d_H^{S^n})$ is a metric space and there is a natural inclusion of X into $K(S^n)$, since a point in X is an orbit in S^n, which is compact. Additionally, X can be considered to be isometrically imbedded in $K(S^n)$, because, as noted above, these two metrics will be equal on any two orbits of the given group.

In several places we will need to consider metric spaces which are quotients of S^3 by a finite group. In general these spaces will not be Riemannian manifolds but nevertheless we will need to consider their geometry and to do that we need the following definitions. Suppose that (X, d) is a metric space and let γ: $[a, b] \to X$ be a path in X. Then the arclength of γ is well defined (though it may be infinite). We then say that X is a *Length space* if for any two points x, y in the same path component the distance between x and y is equal to the infimum of the arclengths of all paths connecting x and y. X is an *Alexandrov space* if it is a locally compact length space and locally has curvature bounded below in the sense of Toponogov; i.e., there exists a number k such that X locally satisfies the conclusion of Toponogov's comparison theorem with k being the curvature of the comparison space.

The q-extent was introduced in [GM] and a number of applications to Alexandrov and Riemannian geometry are given. One application is a necessary condition for G to act isometrically and effectively on a positively curved n-dimensional Riemannian manifold such that the action has $q + 1$ fixed points.

If (X, d) is a compact metric space then we define xt_q: $X^q \to R$ as

$$xt_q(x_1, \ldots, x_q) = \binom{q}{2}^{-1} \sum_{i<j} d(x_i, x_j)$$

The q-extent of X, $xt_q(X)$, is then defined as the maximum of xt_q on X^q. It is easy to see that

$$\operatorname{diam}(X) = xt_2(X) \geq xt_3(X) \geq \cdots \geq \frac{1}{2}\operatorname{diam}(X).$$

Now we will consider a sample calculation which we will need later, namely, we will compute the q-extent of the interval $[a, b]$. Suppose that we have q points $x_1, \ldots, x_q$ that actually achieve the q-extent of $[a, b]$. Then we claim that some pair of these points must lie at opposite ends of $[a, b]$. If not then we could construct a configuration that had a greater q-extent as follows. Without loss of generality we may assume that for all i $x_i \neq a$. Then we can create a new configuration by replacing the smallest x_i with a point at a. Then clearly the q-extent of this new configuration is larger then the q-extent of the original configuration, which is a contradiction. Therefore let us assume that $x_1 = a$ and $x_2 = b$.

We next claim that $x_3, \ldots, x_q$ achieves the $(q - 2)$-extent of $[a, b]$. We have

$$\binom{q}{2} xt_q(x_1, \ldots, x_q) = \binom{n-2}{2} xt_{q-2}(x_3, \ldots, x_q) + (n-1)(b-a)$$

If $x_3, \ldots, x_q$ did not achieve the $(q - 2)$-extent we could replace it with $q - 2$ points that did, say $x_3', \ldots, x_q'$. But then $xt_{q-2}(x_3, \ldots, x_q) < xt_{q-2}(x_3', \ldots, x_q')$ and so it follows from the above equation that

$$xt_q(x_1, x_2, \ldots, x_q) < xt_q(x_1, x_2, x_3', \ldots, x_q')$$

which is a contradiction since $x_1, \dots, x_q$ achieves the q-extent of $[a, b]$. Therefore we conclude that $x_3, \dots, x_q$ achieves the $(q-2)$-extent of $[a, b]$. Continuing this procedure of removing endpoints will eventually leave no points or a single point placed at some point of the interval. Therefore we conclude that if q is even then $x_1, \dots, x_q$ consists of pairs of points placed at the endpoints of the interval, and if q is odd it consists of $q-1$ points paired off at the endpoints of the interval plus one other point, whose position is easily seen to be arbitrary. A simple calculation then gives that

$$xt_{2n}([a,b]) = \frac{n}{2n-1}(b-a)$$

and

$$xt_{2n+1}([a,b]) = \frac{n+1}{(2n+1)}(b-a).$$

Assume that G acts isometrically and effectively on an n-dimensional Riemannian manifold M^n. If p is a fixed point of the action then we have the *isotropy representation* of G in the isometry group of the tangent space at p, $G \to Isom(T_pM), g \mapsto dg_p$. Since this action is effective we may view G as acting isometrically on the unit sphere in the tangent space at p, S^n_p. If p is an isolated fixed point, then G acts on S^n_p without fixed points (otherwise G would fix an entire geodesic).

Next consider the situation where the fixed point set of G contains at least $q+1$ fixed points $p_0, \dots, p_q$ and M^n has nonnegative curvature. M/G is not necessarily a Riemannian manifold but it is an Alexandrov space of curvature ≥ 0. Let $\bar{p}_0, \dots, \bar{p}_q$ be the images of $p_0, \dots, p_q$ under the quotient map. Connect $\bar{p}_0, \dots, \bar{p}_q$ with geodesics and consider the sum of all the angles, $\sum \angle$, between the geodesics at the points $\bar{p}_0, \dots, \bar{p}_q$. We will estimate this sum in two ways to derive the desired inequality. On one hand we have $\binom{q+1}{3}$ triangles formed by the geodesics, each with the sum of its angles $\geq \pi$. Thus

$$\sum \angle \geq \binom{q+1}{3}\pi$$

On the other hand we may estimate the sum of the angles by estimating the sum at each point $\bar{p}_i$. At $\bar{p}_i$ we have a space of directions, $S^n_{p_i}/G$ which is an Alexandrov space of curvature ≥ 1. The angle between two geodesics at $\bar{p}_i$ is equal to the distance between their corresponding directions in the space of directions. Thus the sum of the angles at $\bar{p}_i$ is the sum of all distances between q points in $S^n_{p_i}/G$, which is $\leq \binom{q}{2} xt_q(S^n_{p_i}/G)$. Therefore

$$\binom{q}{2}\sum_{i=0}^{q} xt_q(S^n_{p_i}/G) \geq \sum \angle$$

Combining these two inequalities yields

$$\frac{1}{q+1}\sum_{i=0}^{q} xt_q(S^n_{p_i}/G) \geq \frac{\pi}{3} \tag{1}$$

The above argument is due to K. Grove and S. Markvorsen (see [GM] for the statement of the inequality) and is the main tool of this paper. Observe that if the curvature of M^n is positive then the sum of the angles in any triangle is greater than π and so the above argument gives

$$\frac{1}{q+1}\sum_{i=0}^{q} xt_q(S^n_{p_i}/G) > \frac{\pi}{3} \tag{2}$$

3. Lemmas

LEMMA 1. *If X and Y are compact metric spaces with $d_{GH}(X,Y) < \epsilon$ then $|xt_q(X) - xt_q(Y)| < 2\epsilon$.*

Proof. Assume that X and Y are isometrically embedded in a third space, Z, such that $d^Z_H(X,Y) < \epsilon$ and let d_X and d_Y be their respective metrics. Choose q points in X, $x_1, \dots x_q$, that achieve the q-extent of X. X and Y each lie in an ϵ-neighborhood of each other, so for each point x_i in the chosen collection choose a point y_i in Y that has distance less than ϵ from x_i. Then

$$\begin{aligned}\binom{q}{2} xt_q(X) &= \binom{q}{2} xt_q(x_1,\dots,x_q) = \sum_{i<j} d_X(x_i,x_j)\\ &\leq \sum_{i<j} d_Y(y_i,y_j) + 2\epsilon < \binom{q}{2} xt_q(Y) + \binom{q}{2}2\epsilon.\end{aligned}$$

Therefore $xt_q(X) - xt_q(Y) < 2\epsilon$ and so by symmetry the result follows.

In the next two lemmas we consider $Isom(S^n)$ as a subset of $R^{(n+1)^2}$ and we measure the distance between two isometries in $Isom(S^n)$ with the usual Euclidean metric on $R^{(n+1)^2}$.

LEMMA 2. *If $G, H \subset Isom(S^n)$ and $d_H^{R^{(n+1)^2}}(G,H) < \frac{\epsilon}{n^3}$ then $d_{GH}(S^n/G, S^n/H) < \epsilon$.*

Proof. In [M] it is shown that if $G, H \subset Isom(S^n)$ and $d_H^{R^{(n+1)^2}}(G,H) < \frac{\epsilon}{n^3}$ then $d_H^{K_n}(S^n/G, S^n/H) < \epsilon$. But, as noted above, an n-dimensional spherical space form lies isometrically in $K(S^n)$.

LEMMA 3. *Suppose that K is a compact Lie group with a bi-invariant metric. Then for every $\epsilon > 0$ there is an $N = N(\epsilon, K)$ such that if L is a subgroup of K with $|L| > N$, then L has an element $g \neq e$ with $d(g, e) < \epsilon$.*

Proof. Choose $N > \frac{\mathrm{Vol}(K)}{\mathrm{Vol}(B_\epsilon)}$ where B_ϵ is the metric ball of radius ϵ about the identity element, e, of G. If $|L| > N$ then L must have two elements, g, h within ϵ of each other, or else we could put an epsilon ball around every element of L such that any two would be disjoint. Then the total volume of all the balls would be greater than the volume of K, which is a contradiction. Therefore we have two elements g, h within ϵ of each other, and so gh^{-1} will be within ϵ of e.

LEMMA 4. *If S^1 acts isometrically on S^3 without fixed points then $xt_5(S^3/S^1) = \frac{3\pi}{10}$.*

Proof. Any isometric S^1 action on S^3 is orthogonally equivalent to the action

$$\phi_{k,l}\colon\ S^1 \times \mathbf{C}^2 \to \mathbf{C}^2$$

$$e^{i\theta}(z, w) = (e^{il\theta}z, e^{ik\theta}w).$$

where $k, l \geq 0$, $(k, l) = 1$. If the action has no fixed points then $k, l > 0$. If $k > 1$ then the isotropy group at every point of the orbit $z = 0$ will be Z_k. Similarly if $l > 1$ then the isotropy group of any point of the orbit $w = 0$ is Z_l. Let $X_{k,l}$ denote the quotient space. If $k, l > 0$ then this space will be homeomorphic to a 2-sphere and have diameter $\frac{\pi}{2}$. $X_{k,l}$ will not be smooth at the points corresponding to orbits with isotropy (there will be at most 2). If $k = l = 1$ then we have the Hopf fibration and so $X_{1,1}$ is isometric to a round sphere of diameter $\frac{\pi}{2}$. In [HK] a distance non-decreasing map is constructed from $X_{k,l}$ to $X_{1,1}$. This implies that $xt_q(X_{k,l}) \leq xt_q(X_{1,1}) = xt_q(S^2(\frac{1}{2}))$. But we know from [GM] that $xt_{2q+1}(S^n(r)) = \frac{q+1}{2q+1}r$ so that $xt_5(S^3/S^1) \leq \frac{3\pi}{10}$. This value can actually be attained by placing 3 points at one pole and 2 at the other. Therefore equality holds.

LEMMA 5. *Let $G \subset O(k) \subset R^{k^2}$ be a group of matrices and let $\langle\ ,\ \rangle_{k^2}$ be the usual inner product on R^{k^2}, which we restrict to G. Then $\langle\ ,\ \rangle_{k^2}$ is bi-invariant with respect to the multiplication of G; i.e.,*

$$\langle d(L_A)_B X, d(L_A)_B Y, \rangle_{k^2} = \langle X, Y\rangle_{k^2} = \langle d(R_A)_B X, d(R_A)_B Y, \rangle_{k^2}$$

where $A, B \in G$, $L_A(X) = AX$ and $R_A(X) = XA$.

Proof. Since L_A and R_A are linear on R^{k^2} we have $d(L_A)_B = L_A$ and $d(R_A)_B = R_A$ Thus we need to show that $\langle AX, AY\rangle_{k^2} = \langle X, Y\rangle_{k^2} = \langle XA, YA\rangle_{k^2}$. To see this write $X = (x_1 \ldots x_k)$, $Y = (y_1 \ldots y_k)$ where the x_i's and y_i's are column vectors. Then $\langle AX, AY\rangle_{k^2} = \langle (Ax_1 \ldots Ax_k), (Ay_1 \ldots Ay_k)\rangle_{k^2} = \langle Ax_1, Ay_1\rangle_k + \cdots + \langle Ax_k, Ay_k\rangle_k = \langle x_1, y_1\rangle_k + \cdots + \langle x_k, y_k\rangle_k = \langle X, Y\rangle_{k^2}$, where $\langle\ ,\ \rangle_k$ is the usual inner product on R^K. The proof for right multiplication is similar.

LEMMA 6. *Let* $p\colon \tilde{X} \to X$ *be a covering of Riemannian manifolds. If* $A \subset X$ *then* $d_H^X(A, X) = d_H^{\tilde{X}}(p^{-1}(A), \tilde{X})$.

Proof. We will first show that $d_H^X(A, X) \le d_H^{\tilde{X}}(p^{-1}(A), \tilde{X})$ and then show that $d_H^X(A, X) \ge d_H^{\tilde{X}}(p^{-1}(A), \tilde{X})$.

For the first part it suffices to show that a neighborhood of radius $d_H^{\tilde{X}}(p^{-1}(A), \tilde{X})$ of A covers all of X. Fix $x \in X$. We will exhibit an element $a \in A$ such that $d(x, a) < d_H^{\tilde{X}}(p^{-1}(A), \tilde{X})$. Choose any $\tilde{x} \in p^{-1}(x)$. By the definition of $d_H^{\tilde{X}}(p^{-1}(A), \tilde{X})$ there exists $\tilde{a} \in p^{-1}(A)$ such that $d(\tilde{x}, \tilde{a}) < d_H^{\tilde{X}}(p^{-1}(A), \tilde{X})$. Then $p(\tilde{a}) \in A$ and $d(x, p(\tilde{a})) < d_H^{\tilde{X}}(p^{-1}(A), \tilde{X})$ because p is a distance decreasing map. Therefore the first part is proved.

Next pick $\tilde{x} \in \tilde{X}$. We need to show that there is an element $\tilde{a}$ of $p^{-1}(A)$ such that $d(\tilde{a}, \tilde{x}) < d_H^X(A, X)$. Pick a path γ in X from $p(\tilde{x})$ to a point a in A. Let $\tilde{\gamma}$ be a lift of γ such that $\tilde{x}$ is one endpoint of $\tilde{\gamma}$ and let $\tilde{a}$ be the other endpoint, which lies in $p^{-1}(A)$. Since p preserves arclenths, the length of γ and $\tilde{\gamma}$ are the same. Consequently $d(\tilde{a}, \tilde{x}) < d_H^X(A, X)$, proving the lemma.

4. Proofs of Theorems 1, 3 and 4

Proof of Theorem 3. Suppose that the fixed point set of G contains 6 isolated fixed points $p_0, \ldots, p_5$. At each such point p_i let G_i be the image of G under the isotropy representation in $T_{p_i}M$ and let $S^3_{p_i}$ denote the unit sphere in $T_{p_i}M$. From (1) it follows that

$$\frac{1}{6}\sum_{i=0}^{5} xt_5(S^3_{p_i}/G_i) \ge \frac{\pi}{3}. \tag{3}$$

We will show that if G satisfies the hypothesis of the theorem with N sufficiently large then $xt_5(S^3_{p_i}/G_i) < \frac{\pi}{3}$, for each i, causing the violation of (3). This leads us to conclude that an action of the group described in the theorem cannot have 6 isolated fixed points.

Since G_i is abelian it lies in a maximal torus of $SO(4)$. We then consider $SO(4)$ as a group of 4×4 matrices lying in R^{16}. Equip $SO(4)$ with the Riemannian metric that it inherits from R^{16} as a submanifold. By Lemma 5 this metric is bi-invariant. Restricting this metric to the maximal torus containing G_i gives rise to a distance metric d_{T^2} on the torus. Observe that for $x, y \in T^2$

$$d_{T^2}(x, y) \ge d_{R^{16}}(x, y)$$

where $d_{R^{16}}$ is the usual Euclidean metric on R^{16}.

Apply Lemma 3 to T^2 with $\epsilon = \frac{1}{4}(\frac{\pi}{3} - \frac{3\pi}{10})(\frac{1}{3^3}) = \frac{\pi}{3240}$ and the bi-invariant metric described above to obtain N. Therefore if G satisfies the hypothesis of the theorem then there exists $g \in G_i$ with $g \ne e$ and $d_{T^2}(g, e) < \epsilon$. Consider the 1-parameter subgroup of T^2, $S^1_{\langle g\rangle}$, that contains g and whose length from e to g is $d_{T^2}(g,e)$. Observe that $d_H^{T^2}(S^1_{\langle g\rangle}, \langle g\rangle) < \frac{\epsilon}{2}$ and so $d_H^{R^{16}}(S^1_{\langle g\rangle}, \langle g\rangle) < \frac{\epsilon}{2}$. Now we consider two cases; one where $S^1_{\langle g\rangle}$ is a circle that does not fix points and the other, where it is a circle that does fix points.

If $S^1_{\langle g\rangle}$ acts without fixed points, then we know by Lemma 4 that $xt_5(S^3_{p_i}/S^1_{\langle g\rangle}) = \frac{3\pi}{10}$. By Lemma 2 we have

$$d_{GH}(S^3_{p_i}/S^1_{\langle g\rangle}, S^3_{p_i}/\langle g\rangle) < \frac{\pi}{120}$$

and so, by Lemma 1,

$$|xt_5(S^3_{p_i}/S^1_{\langle g\rangle}) - xt_5(S^3_{p_i}/\langle g\rangle)| < \frac{\pi}{60}.$$

Since $xt_5(S^3_{p_i}/S^1_{\langle g\rangle}) = \frac{3\pi}{10}$ it follows that $xt_5(S^3_{p_i}/\langle g\rangle) < \frac{\pi}{3}$ and so $xt_5(S^3_{p_i}/G_i) < \frac{\pi}{3}$. This violates the inequality (3). Therefore such an action cannot exist.

If $S^1_{\langle g\rangle}$ acts with fixed points then we will use a similar idea. G_i cannot be contained in $S^1_{\langle g\rangle}$ because if it were it would have fixed points, which is impossible because if it did then p_i would no longer be an isolated fixed point. Therefore $H = G_i \cap S^1_{\langle g\rangle}$ is not equal to all of G_i and so G_i/H is not trivial. Thus we can write $G_i/H = \{h_1H, \dots, h_rH\}$ where $r > N$ (recall that all of the prime divisors of G_i are $> N$, so $r > N$). Consider the open balls $B_\epsilon(h_1), \dots, B_\epsilon(h_r)$. These neighborhoods cannot all be disjoint because by our choice of N in Lemma 3, $Vol(T^2) < NVol(B_\epsilon(e))$. So suppose that $B_\epsilon(h_a) \cap B_\epsilon(h_b)$ is not empty. Then $d_{T^2}(h_a, h_b) < 2\epsilon$, so $d_{T^2}(h_a h_b^{-1}, e) < 2\epsilon$. $h_a h_b^{-1}$ does not lie in $S^1_{\langle g\rangle}$ because if it did then $h_a h_b^{-1}$ would lie in $H = G_i \cap S^1_{\langle g\rangle}$, which is impossible since h_a and h_b were chosen as two distinct coset representatives and so cannot be congruent modulo H. Let $L = \langle h_a h_b^{-1}, g\rangle$ and let $p\colon R^2 \to T^2$ be the universal cover of T^2 equipped with the lifted metric of T^2. Then $p^{-1}(L)$ is a lattice in R^2 with $d_H^{T^2}(p^{-1}(L), R^2) < \frac{\pi}{1620}$. Then it follows from Lemma 6 that $d_H^{T^2}(L, T^2) < \frac{\pi}{1620}$. We then have

$$d_{GH}(S_{p_i}/L, S_{p_i}/T^2) < \frac{\pi}{60}$$

so that by Lemma 1

$$|xt_5(S_{p_i}/L) - xt_5(S_{p_i}/T^2)| < \frac{\pi}{30}.$$

But S^3/T^2 is isometric to $[0, \frac{\pi}{2}]$ and $xt_5([0, \frac{\pi}{2}]) = \frac{3\pi}{10}$. Thus $xt_5(S_{p_i}/L) < \frac{\pi}{3}$. It follows that $xt_5(S_{p_i}/G_i) < \frac{\pi}{3}$. Again, (3) is violated, so such an action cannot exist. □

Remarks. The hypothesis in Theorem 1 concerning the primes dividing the order of the group is necessary. If we assume only that the order of the group is large then the group may not be Hausdorff close to a circle that acts without fixed points and it may not be Hausdorff close to the entire torus. To see this consider Z_{2^k} embedded as a subgroup of a circle that wraps once around the torus in one direction and 2^k times in the other direction. The quotient of S^3 by this group does not have small enough 5-extent to violate (1).

A variation on this, to show that difficulties arise for non-cyclic groups, is the group $Z_2 \times Z_k$ embedded in the torus in the obvious way. Again, the quotient of S^3 does not have small enough 5-extent to violate (1).

Proof of Theorem 4. As above, apply Lemma 3 to T^2 with its natural bi-invariant metric and take $\epsilon = \frac{\pi}{3240}$. Assume that the fixed point set of G contains 6 fixed points $p_0, \ldots, p_5$. At each such point p_i let G_i be the image of G under the isotropy representation in $T_{p_i}M$. G_i is abelian and therefore lies in a maximal torus of $SO(4)$. Next, suppose that $n > N$. Thus $Z_n \times Z_n$ has an element $g \neq e$ with $d(g, e) < \epsilon$. Let $S^1_{\langle g\rangle}$ be defined as it was in the proof of Theorem 3. If $S^1_{\langle g\rangle}$ does not have any fixed points, then we may argue as we did in the proof of Theorem 3 and we are done. So suppose that $S^1_{\langle g\rangle}$ does have fixed points. Let $H = G_i \cap S^1_{\langle g\rangle}$. Then H is cyclic since it is a finite subgroup of a circle group and also has order $\leq n$ because the order of every element of $Z_n \times Z_n$ is at most n (it is precisely this which allows us to omit the hypothesis that the fixed points be isolated). Therefore we have at least n distinct coset representatives for G_i/H. Write $G_i/H = \{h_1H, \ldots, h_rH\}$ where $r > N$ and conclude the proof exactly as in the end of the proof of Theorem 3. □

Proof of Theorem 1. To bound the Euler characteristic we use the fact that $\chi(M) = \chi(F) + \chi(M, F)$ where we take F to be the set of points of M that are fixed by G. Then we just apply Theorem 2 to see that for sufficiently large p, F consists of no more than 5 points, so that $\chi(F) \leq 5$. Next we show that $\chi(M, F) = \chi(M - F)$. By definition,

$$\chi(M, F) = \sum_{k=1}^{n} \operatorname{rank} H_k(M, F)(-1)^k$$

But the rank of $H_k(M, F)$ is equal to the rank of $\tilde{H}_k(M/F)$ where $\tilde{H}$ denotes the augmented homology groups. Thus

$$\chi(M, F) = \sum_{k=1}^{n} \operatorname{rank} \tilde{H}_k(M/F)(-1)^k = \chi(M/F) - 1.$$

If a triangulation of M is chosen so that F is a set of vertices then it is easy to see that $\chi(M - F) = \chi(M/F) - 1$, so that $\chi(M, F) = \chi(M - F)$.

Now we use an idea from [Y], that $\chi(M) = \chi(F) + \chi(M, F) \leq C_4$ where C_4 is Gromov's upper bound for the sum of the betti numbers of M. Thus if $\chi(M, F)$ is divisible by a prime greater than C_4 it must be 0. To show this and complete the proof we need the following lemma.

LEMMA 7. *If G is a p-group acting isometrically on an n-dimensional closed Riemannian manifold N fixed point freely, then $p \mid \chi(N)$.*

Proof. Choose a triangulation T of M that is preserved by the action of G. Let T_n be the collection of n-dimensional simplices of the triangulation. The action of G on M induces an action of G on T which in turn induces an action on T_n. This action decomposes T_n into equivalence classes, namely, the orbits of the action. Let $S = \{\sigma_1, \dots, \sigma_k\}$ be a set of representatives from this decomposition, consisting of exactly one simplex for each equivalence class. Then we can write

$$T_n = \bigcup_{i=1}^{k} \bigcup_{g \in G} g\sigma_i .$$

Thus

$$|T_n| = \sum_{i=1}^{k} \left| \bigcup_{g \in G} g\sigma_i \right| .$$

Since G is a p-group $|\bigcup_{g \in G} g\sigma_i| = p^a$ for some $a \geq 0$ and any given i. We claim that $a > 0$. If $a = 0$ for some i then G would stabilize a simplex and since G is finite this would imply that it fixed a point of the simplex, contradicting the hypothesis that G acts fixed point freely. Therefore p divides each term in the above sum and so divides the sum itself. Therefore, since $\chi(M) = \sum_{i=1}^{n} |T_n|$ it follows that $p \mid \chi(M)$.

To complete the proof of Theorem 1 it is tempting to let $N = X - F$ in Lemma 7, but then N would not be a closed manifold. To remedy this difficulty we choose N to be the manifold that results from the removal of small open balls about the points of F, that are stable under the action of G. Then N is closed and has the same homotopy type as $M - F$. Additionally, G acts fixed point freely on N, so we may apply Lemma 7 to conclude that if $p > C_4$ then $p \mid \chi(N) = \chi(M - F) = \chi(M, F)$ and so $\chi(M, F) = 0$. Therefore if $p > \max(N, C_4)$ then $\chi(M) \leq 5$. □

5. Proof of Theorem 2

In order to prove Theorem 2 we must introduce a few more definitions. Let X and Y be metric spaces and suppose that $f\colon X \to Y$ is a Lipschitz map. Then $\operatorname{dil} f = \sup_{x \neq y} \frac{d(f(x), f(y))}{d(x,y)}$ is called the dilatation of f.

$$d_L(X, Y) = \inf\{|\ln(\operatorname{dil} f)| + |\ln(\operatorname{dil} f^{-1})| \mid \text{f is a bi-Lipschitz homeomorphism}\}$$

is the Lipschitz distance between X and Y, if a bi-Lipschitz homeomorphism exists. If one doesn't exist we set $d_L(X, Y) = \infty$. With this as a distance function the collection of compact metric spaces becomes a metric space itself. But it is not this space but a subspace that we are concerned with. Let $C(n, d, \Lambda, V)$ be the class of compact n-dimensional Riemannian manifolds M with diameter diam$(M) \leq d$,volume vol$(M) \geq V$, and sectional curvature $|K_M| \leq \Lambda^2$. Then the topology on this space induced by the above distance function is known as the *Lipschitz topology*. Note that the hypothesis that vol$(M) \geq V$ may be replaced by $i(M) \geq i_0$, where $i(M)$ denotes the injectivity radius of M [CE]. The important fact that we need is the following theorem of Peters [P]:

THEOREM 5. *Let $0 \leq \alpha \leq 1$. Then any sequence in $C(n, d, \Lambda, V)$ contains a subsequence converging with respect to the Lipschitz topology to an n-dimensional differentiable manifold M with metric g of Holder class $C^{1+\alpha}$.*

A useful reformulation of the above theorem (also from [P]) is the following.

THEOREM 6. *Let $\{(M_k, g_k\}$ be a sequence of manifolds in $C(n, d, \Lambda, V)$ and $0 \leq \alpha \leq 1$. There exists a subsequence $\{(M_l, g_l)\}$ with the following properties*:

(i) *Each M_l is diffeomorphic to a single fixed manifold M.*
(ii) *There exist diffeomorphisms F_l: $M \to M_l$ such that $\{(F_l)^* g_l\}$ converges in C^1 to a $C^{1+\alpha}$ metric g on M.*
(iii) diam(M_l) *converges to* diam(M).
(iv) *For the injectivity radii we have that* lim sup $i(M_l) \leq i(M)$.
(v) *If* $\exp^l$ *denotes the exponential map of M_l,* exp *that of (M, g), and $e\tilde{x}p^l = (F_l)^*\exp^l$, then $e\tilde{x}p^l$ converges to* $\exp_p$ *uniformly on compact subsets of T_pM and exp_p is Lipschitz.*

We now begin the proof the Theorem 2, which is by contradiction. If the theorem were not true then there would exist a number δ such that for every N there would be a manifold (M_N, g_N) with $1 \geq sec(M_N) \geq \delta > 0$, $|Isom(g_N)| > N$ and $\chi(M_N) > 3$. Therefore we have a sequence of manifolds $\{(M_i, g_i)\}$. We claim that there exists d, Λ, V such that $(M_i, g_i) \in C(n, d, \Lambda, V)$ for all i. It is clear that d and Λ can be found because of the curvature bounds. Rather than find V we look for a lower bound on the injectivity radius. For this we use the theorem, due to Klingenberg, that if the sectional curvature of a compact orientable even dimensional Riemannian manifold M^n satisfies $1 \geq K > 0$, then $i(M) \geq \pi$. Applying this theorem to the orientable double cover if necessary, we obtain that $i(M_i) \geq \frac{\pi}{2}$ under our assumptions. Therefore we may apply Theorem 6 to extract a convergent subsequence of $\{(M_i, g_i)\}$ converging to a manifold (M, g) which is $C^{1,\alpha}$, has $1 \geq sec(g) \geq \delta > 0$ and by hypothesis $\chi(M) > 3$. By (ii) of Theorem 6 we may assume that we have a single differentiable manifold M with a sequence of metrics on it $\{g_i\}$ converging to g. Our plan is as follows: we will show that since the order of the isometry groups of the $\{g_i\}$ are going to infinity, that the isometry group of g is actually infinite. This will imply the existence of an isometric S^1-action on (M, g) and we will then use that action to show that $\chi(M) \leq 3$, a contradiction. Therefore the N in the statement of the theorem must exist.

For a Riemannian manifold N with metric h denote the corresponding distance metric by d^h. We put a metric on $Isom(h)$ by letting

$$d^h_{\sup}(f, g) = \sup_{x \in X} d^h(f(x), g(x)).$$

Suppose $f_i \in Isom(g_i)$. As i approaches infinity df_i approaches being an orthogonal map with respect to the metric g. Thus for any v and i sufficiently large, $|df_i(v)| \sim |v|$. As a consequence $|df_i| < L$ for some $L \sim 1$. Next, choose a path γ between two points in M, p and q. The the length of $f_i \circ \gamma = \int_0^1 (f_i \circ \gamma)' = \int_0^1 |df_i \circ \gamma'| \leq L \int_0^1 |\gamma'| = L \cdot$ (length of γ). Thus $d^g(f_i(p), f_i(q)) \leq L d^g(p, q)$, so the f_i are an equicontinous collection. Therefore we may extract a convergent subsequence. Denote this subsequence by f_i and say that it converges to f. By construction f preserves distances. It follows that (see [KN], p. 169) that f is also C^1 and preserves the metric. Our next step is to show that there are infinitely many such isometries of (M, g) and then we may conclude that the isometry group of (M, g) contains a circle.

LEMMA 8. *For every $f_i \in Isom(g_i)$ there exists n such that $d^{g_i}_{\sup}(f_i^n, e) \geq \frac{inj(g_i)}{2} \geq \frac{\pi}{4}$.*

Proof. There are two cases to be considered: one where f_i has a fixed point and the other where it doesn't. First consider the case where f_i doesn't have a fixed point.

Note that we saw above that $inj(g_i) \geq \frac{\pi}{2}$. Choose $p \in M$ such that $d^{g_i}(p, f_i(p))$ is a minimum. If $d^{g_i}(p, f_i(p)) \geq \frac{inj(g_i)}{2}$ then we are done. Otherwise choose a geodesic, γ, connecting p to $f_i(p)$ and observe that because $d^{g_i}(p, f_i(p))$ is a minimum, $f_i(\gamma) \subset \gamma$. Therefore, since the effect of f_i on γ is translation by a distance equal to $d^{g_i}(p, f_i(p))$, the distance between p and $f_i^n(p)$ will increase with n as long as $f_i^n(p)$ lies no further than p then the injectivity radius. Therefore for some n, $d^{g_i}(p, f_i^n(p)) \geq \frac{inj(g_i)}{2}$ and so $d^{g_i}_{\sup}(f_i^n, e) \geq \frac{inj(g_i)}{2}$.

If f_i does have a fixed point, say p, consider df_{i_p}. Then df_{i_p} is a rotation of $S^3_p(\frac{inj(g_i)}{2}) \subset T_pM$ and we can choose n so that $df_{i_p}{}^n$ moves some point v of $S^3_p(\frac{inj(g_i)}{2})$ into the hemisphere opposite to it. Then the normal ball about p of radius $\frac{inj(g_i)}{2}$ is convex (see [CE]) and so we may apply Rauch's theorem to show that

$$d^{g_i}(\exp_p(v), \exp_p(f_i^n(v)) \geq \frac{inj(g_i)}{2}.$$

We have a hinge with both sides of length $\frac{inj(g_i)}{2} \geq \frac{\pi}{4}$ and an angle $\geq \frac{\pi}{2}$. As a comparison space we use a round sphere of radius 1. A calculation shows that a hinge on this sphere with angle $\frac{\pi}{2}$ and sides of length $\frac{\pi}{4}$ has endpoints whose distance from each other is $\frac{\pi}{3}$ (to see this convert the coordinates of the points into rectangular coordinates and calculate their dot product, which will be $\frac{1}{2}$). Therefore $d^{g_i}(\exp_p(v), \exp_p(f_i^n(v)) \geq \frac{\pi}{4}$ and so again we have that $d^{g_i}_{\text{sup}}(f_i^n, e) \geq \frac{inj(g_i)}{2}$.

LEMMA 9. *Isom(g) is infinite.*

Proof. Consider the following claim (which is not necessarily true):

CLAIM 1. *For every $\epsilon > 0$ there exist infinitely many natural numbers such that for each such i there exists an element f_i of $Isom(g_i)$, which is different from the identity element e of $Isom(g_i)$ and has $d^{g_i}_{\text{sup}}(f_i, e) < \epsilon$.*

If this statement were not true then there would exist an $\epsilon > 0$ such that for all but finitely many i $Isom(g_i)$ has the property that $d^{g_i}_{\text{sup}}(f, h) \geq \epsilon$ for $f \neq h$. Then for any positive integer n we will construct n sequences $\{h_i^1\}, \dots, \{h_i^n\}$ that each converges to a different element of $Isom(g)$, showing that $Isom(g)$ contains at least n elements. Define the first $n-1$ terms of the sequences $\{h_i^1\}, \dots, \{h_i^n\}$ to be any elements from the groups $Isom(g_i)$. $Isom(g_i)$ has at least i elements (recall that $|Isom(g_i)| > i$ by hypothesis), so for $i > n-1$ we let $h_i^1, \dots, h_i^n$ be any n distinct elements of $Isom(g_i)$. Then for any two sequences $\{h_i^\alpha\}, \{h_i^\beta\}$ with $\alpha \neq \beta$ we can extract two convergent subsequences which do not converge to the same element of $Isom(g)$ as follows: first choose a convergent subsequence $\{h_{i_k}^\alpha\}$ of $\{h_i^\alpha\}$ that converges to $f \in Isom(g)$. Then consider the corresponding subsequence of $\{h_i^\beta\}$, $\{h_{i_n}^\beta\}$. This subsequence may not converge but it has a convergent subsequence $\{h_{i_{n_k}}^\beta\}$ (since it is an equicontinous collection) that converges to an isometry $g \in Isom(g)$. The sequence $\{h_{i_{n_k}}^\alpha\}$ then converges to f, and $f \neq g$ since for all but finitely many i we have that $d^{g_i}_{\text{sup}}(h_i^\alpha, h_i^\beta) \geq \epsilon$. This implies that $Isom(g)$ must have at least n elements and since n was arbitrary it implies that $Isom(g)$ is infinite. Therefore assume that the claim is true.

Let n be a positive integer and take $\epsilon = \frac{1}{4n}$. We are going to show that $Isom(g)$ contains at least n elements. From the above statement there exists a sequence $\{i_k\}$ with $f_{i_k} \epsilon Isom(g_{i_k})$ such that $d^{g_{i_k}}_{\text{sup}}(f_{i_k}, e) < \epsilon$. We then decompose the isometry groups into annuli: let $A_j^{i_k} = \{f \in Isom(g_{i_k}) \mid (j-1)\epsilon \leq d^{g_{i_k}}_{\text{sup}}(f, e) \leq j\epsilon\}$. We know from lemma 8 that for each f_{i_k} there exists an integer $r = r(k)$ such that $d^{g_{i_k}}_{\text{sup}}(f_{i_k}^r, e) \geq \frac{inj(g_{i_k})}{2} \geq \frac{\pi}{4}$. Therefore for some l with $l\epsilon \geq \frac{\pi}{4}$ we have that $A_l^{i_k} \neq \phi$. Thus $l \geq n\pi$, so in particular $l \geq 2n$. This and the fact that $d^{g_{i_k}}_{\text{sup}}(f_{i_k}^m, f_{i_k}^{m+1}) < \epsilon$ implies that $A_j^{i_k} \neq \phi$ for $j = 1, \dots, 2n$. We then may let $h_{i_k}^{2j}$ be any element of $A_{2j}^{i_k}$, so that we have n sequences $h_{i_k}^2, \dots, h_{i_k}^{2n}$, all of which are bounded away from each other since any two have at least one whole annuli between them. Arguing as above, there is a subseqence of $\{i_k\}$ for which the corresponding n subsequences all converge. Since these subsequences are all bounded away from each other they must converge to n different elements of $Isom(g)$. Since n was arbitrary this shows that $Isom(g)$ is an infinite group.

It follows from G being infinite that G must be a Lie group of dimension greater than 0, since the isometry group of a compact Riemannian manifold is compact. Therefore the limit manifold admits an isometric S^1-action, which is C^1 by [KN], p. 169. The Euler characteristic of M is therefore the Euler characteristic of F, the fixed point set of the action, since on $M - F$ there is a free action of Z_p for every p and so every p divides the Euler characteristic of $M - F$. F is a disjoint union of totally geodesic submanifolds and since each component of F must have even codimension we conclude that F is a union of a finite number of isolated points and surfaces. It follows from [HK] or [GM] that the number of isolated points is at most three. (In [GM] the hypothesis is that the manifold is smooth but the proof carries over to our case anyway.) Next we claim that F has at most one 2-dimensional component. This follows from a generalized version of Frankel's theorem proved in [Pn]: if H and G are two totally quasigeodesic subsets of an Alexandrov space Σ with curvature ≥ 1 and $\dim H + \dim G \geq \dim \Sigma$, then $H \cap G \neq \emptyset$. Therefore F can have at most one two-dimensional component. If F has a two-dimensional component it is either an RP^2 or S^2 since it is positively curved. Therefore the contribution to $\chi(F)$ from the 2-dimensional component is at most 2. The number of isolated fixed points is at most three. We next argue that if the fixed point set contained a 2-dimensional component that then the number of isolated fixed points could be only one. Our argument is essentially that given in [HK], but we give it here for completeness.

Assume that there were two isolated fixed points, p and q. Let γ be a distance minimizing geodesic connecting p and q and let $S^1\gamma$ denote the image of γ under the action. Let η be a distance minimizing geodesic connecting N and $S^1\gamma$. With the notation in Lemma 4, the isotropy represention at p (or q) is orthogonally equivalent to the action of $\phi_{k,l}$. But if k or l were greater then 1 then the circle would contain a cyclic subgroup Z_k (or Z_l) whose represention at p would be reducible implying that p would not be an isolated fixed point of the Z_k action. Therefore, since we know the action is effective, the cyclic group would have a two-dimensional component containing p. But it would also fix N, contradicting the generalized version of Fraenkel's theorem. Thus the representation at p or q is orthogonally equivalent to $\phi_{1,1}$. It then follows that $S^1\gamma$ is in fact a smooth submanifold which is totally geodesic at p and q. Therefore if the endpoint of η in γ is p or q we may make a second variation argument to arrive at a contradiction, again by the generalized version of Frankel's theorem proved in [Pn]. Therefore assume that the endpoint is at a point of $S^1\gamma$ other than p or q. Since the is action is orthogonally equivalent to $\phi_{1,1}$ follows that points nearby p or q lie in the principle orbit of the action, in particular, part of the interior of γ near p does. If the isotropy group along γ was non-trivial anywhere, say at a point r then the action of that group would fix p and r and move the segment of γ connecting them, and therefore there would be a broken geodesic connecting r and p, which is a contradiction. Therefore all of γ, hence all of $S^1\gamma$ lies in the principle orbit. A similiar argument shows that η lies in the principle orbit. Let X denote the union of the principle orbit and N and when passing to the quotient space X/S^1 we will put a bar over the name of a set. In [HK] it is show that $\bar{N}$ is a totally geodesic component of the boundary of X/S^1. Additionally, since η and $S^1\gamma$ are perpendicular to orbits of the action, $\bar{\eta}$ and $\bar{\gamma}$ are geodesics in X/S^1, and the endpoint of $\bar{\eta}$ is again a point on the interior of $\bar{\gamma}$. A second variation argument then gives a contradiction. Therefore if the fixed point set contains a 2-dimensional component it must contain at most one isolated fixed point. Therefore $\chi(M) = \chi(F) \leq 3$ and a contradiction is achieved. □

Acknowledgments. The author would like to thank Karsten Grove, David Harbater and Wolfgang Ziller for many useful discussions.

References

[B] M. Berger, *Les varietes Riemaniennes homogenes simplement connexes a courbure strictement positive*, Ann. Scuola Norm. Sup. Pisa **15** (1961), 179–246.

[BB] L. Berard Bergery, *Les varietes Riemaniennes homogenes simplement connexes de dimension impair a courbure strictement positive*, J. Math. Pures Appl. **55** (1976), 47–68.

[BGP] Y. Burago, M. Gromov and G. Perelman, *A.D. Alexandrov's spaces with curvatures bounded from below*, Russian Math Surveys **47** (1992), 1–58.

[CE] J. Cheeger and D. Ebin, *Comparison theorems in Riemannian geometry*, North-Holland, Amsterdam, 1975.

[Ch] J. Cheeger,*Some examples of manifolds of nonnegative curvature*, J. Differential Geom. **8** (1978), 623–628.

[CG] J. Cheeger and D. Gromoll, *On the structure of complete manifolds of nonnegative curvature*, Ann. Math. **96** (1972), 413–443.

[G] M. Gromov, *Curvature, diameter and Betti numbers*, Comm. Math. Helv. **56** (1981), 179–195.

[GrM] D. Gromoll and W. Meyer, *On complete open manifolds of positive curvature*, Ann. of Math. (2) **90** (1969), 45–90.

[Gr] K. Grove, "Metric differential geometry" in *Differential geometry* (Proc. Nordic Summer School, Lyngby 1985), V. L. Hansen, ed., Lecture Notes in Math., vol. 1263, Springer-Verlag, 1987, pp. 171–227.

[GM] K. Grove and S. Markvorsen, *New extremal problems for the Riemannian recognition program via Alexandrov geometry*, preprint.

[H] R. Hamilton, *Three manifolds with positive Ricci curvature*, J. Differential Geom. **17** (1982), 255–306.

[HK] W. Y. Hsiang and B. Kleiner, *On the topology of positively curved* 4*-manifolds with symmetry*, J. Differential Geom. **30** (1989), 615–621.

[K] B. Kleiner, *Riemannian four-manifolds with nonnegative curvature and continous symmetry*, University of California at Berkeley Thesis, April 1990.

[KN] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, vol. 1, Wiley, N.Y., 1963.

[L] A. Lichnerowicz, *Spineurs harmoniques*, C.R. Acad. Sci. Paris **257** (1963), 7–9.

[P] S. Peters, *Convergence of Riemannian manifolds*, Comp. Math. **62** (1987), 3–16.

[Pn] A. Petrunin, *Parallel transportation and second variation*, preprint.

[M] J. McGowan, *The diameter function on the space of space forms*, University of Maryland at College Park Thesis, 1991.

[S] C. Searle, *Manifolds of positive curvature with large symmetry groups*, University of Maryland at College Park Thesis, 1992.

[SY] C. Searle and D. Yang, *On the topology of nonnegatively curved simply connected 4-manifolds with continuous symmetry*, Duke Math. J. **74** (1994), 547–556.

[W] N. R. Wallach, *Compact homogeneous Riemannian manifolds with strictly positive curvature*, Ann. Math. **96** (1972), 277–295.

[Y] D. Yang, *On the topology of nonnegatively curved simply connected 4-manifolds with discrete symmetry*, Duke Math. J. **74** (1994), 531–545.

Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104

hicks@math.upenn.edu