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NORMS AND LOWER BOUNDS OF OPERATORS ON THE LORENTZ SEQUENCE SPACE $d(w, 1)$

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ABSTRACT. Conditions are found under which the norm of an operator on a Banach sequence space is determined by its action on decreasing, positive sequences. For the space $d(w, 1)$, the norm and "lower bound" of such operators can be equated to the supremum and infimum of a certain sequence. These quantities are evaluated for the averaging, Copson and Hilbert operators, with the weighting sequence given either by $w_n = 1/n^\alpha$ or by the corresponding integral.

1. Introduction

The classical inequalities of Hardy, Copson and Hilbert [10] describe the norms of certain matrix operators on the sequence space ℓ_p. Numerous generalizations, together with results on the companion problem of "lower bounds", have been given by Bennett [2], [3], [4], [5], [6], Lyons [14] and others. There is also an extensive literature on analogous results for the continuous case (e.g., [1], [15], [16]).

In the present paper, we address the problem of finding the norms and lower bounds of these operators when ℓ_p is replaced by the Lorentz sequence space $d(w, 1)$ determined by a weighting sequence (w_n). The "lower bound" notion is particularly natural for such spaces, since it is defined in terms of decreasing sequences.

Under fairly general conditions, which we identify, the norm of an operator on any symmetric Banach sequence space is determined by decreasing, non-negative sequences. For such operators, our problem can be reformulated without reference to Lorentz sequence spaces: the norm on $d(w, 1)$ equates to the supremum of $\|Ax\|/\|x\|$ for *decreasing* sequences in the weighted ℓ_1-space $\ell_1(w)$. Also, a pleasantly simple characterization is available. In fact, both problems reduce to the study of a certain sequence, as follows. Let the operator have matrix $(a_{i,j})$, and let $v_j = \sum_{i=1}^{\infty} a_{i,j} w_i$. Write $W_n = w_1 + \cdots + w_n$ (and V_n similarly). Then the norm and lower bound of A are the supremum and infimum of V_n/W_n. This amounts to saying that both are determined by elements of the form $(1, \ldots, 1, 0, \ldots)$.

However, evaluating these quantities in particular cases can be far from trivial, and we turn to the problem of doing so for the classical operators mentioned above, with a view to finding exact answers where possible. We consider two natural choices of weighting sequence, defined respectively by $w_n = 1/n^\alpha$ and $W_n = n^{1-\alpha}$ (the second example is equally "natural" in the context of Lorentz spaces). These are two

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alternative analogues of $1/x^\alpha$ in the continuous case, and our results show strikingly how little information about the discrete case can be derived from the continuous case. Indeed, for each of the operators considered, the continuous analogue of our problem is trivial, because there is an isometric relationship of the form $\|A(f)\| = C(\alpha)\|f\|$ for all positive functions f. Nothing of the sort happens in the discrete case. Although the norms arising from our two choices of (w_n) are actually equivalent, they repeatedly lead to quite different problems of evaluation. In each case, we have to evaluate specific suprema and infima, usually concerning partial sums or tails of series. The resulting problems can be anything from straightforward to intractable; some are of real interest purely as questions on inequalities. They are quite similar to problems encountered in determining the q-concavity constants of Lorentz spaces [11]. In most cases, the required bounds are found by showing that (V_n/W_n) is monotonic, though even then it can be substantially easier simply to show that the infimum is the first term and the supremum is the limit (or conversely). Also, if (v_n/w_n) is monotonic, then its bounds coincide with those of (V_n/W_n); sometimes this eases the passage when $w_n = 1/n^\alpha$. There is both an interplay and a contrast between the results for the two choices of (w_n).

Specimen results are as follows. For the averaging (alias Cesaro) operator, the above $C(\alpha)$ is $1/\alpha$. We find that for $w_n = 1/n^\alpha$, this operator has norm $\zeta(1+\alpha)$ and lower bound $1/\alpha$. By contrast, for $W_n = n^{1-\alpha}$, the norm is $1/\alpha$, while the lower bound is $\sum_{n=1}^\infty 1/n^\alpha(n+1)$ (denote this by S). Furthermore, the slight change to (w_n) is enough to change the sequence under investigation from a decreasing one to an increasing one. We consider two versions of the Hilbert operator, denoted by H_1 and H_0. For H_1, with $w_n = 1/n^\alpha$, the norm is $\pi/\sin\alpha\pi$ (which is the $C(\alpha)$ in the continuous case), while the lower bound is above quantity S. Exactly the same computations solve the problem for H_0 with $W_n = n^{1-\alpha}$. Meanwhile, for H_0 with $w_n = 1/n^\alpha$, the sequence (v_n/w_n) is no longer monotonic, and we are unable to give an exact solution.

We finish with an example showing that it is possible to have (V_n/W_n) bounded while (v_n/w_n) is unbounded.

There are several ways in which this investigation suggests further problems. Firstly, there is the question of extending the results to $d(w,p)$ with $p > 1$. It turns out that the "V_n/W_n" characterization can be generalized for lower bounds, but not for norms, as a result of which quite different methods are needed. Some results for this case are presented in [12] and [13].

Also, one can consider other specific choices of operator or of (w_n), or else attempt to extend the results to wider classes of operators, such as summability or Hankel matrices, or to more general weighting sequences. However, the differences mentioned above between closely related particular cases suggest that serious difficulties will be encountered. At best one can hope for inequalities, rather than exact evaluations.

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2. General matrix operators

We start by describing our problem in the notation of weighted ℓ_1-spaces rather than Lorentz sequence spaces. Let $w = (w_n)$ be a decreasing, non-negative sequence with $\lim_{n\to\infty} w_n = 0$ and $\sum_{n=1}^{\infty} w_n$ divergent. Write $W_n = w_1 + \cdots + w_n$. Then $\ell_1(w)$ is the space of sequences $x = (x_n)$ with

$$\|x\|_{\ell_1(w)} = \sum_{n=1}^{\infty} w_n |x_n|$$

convergent.

Now consider the operator A defined by $Ax = y$, where $y_i = \sum_{j=1}^{\infty} a_{i,j} x_j$. We shall write $\|A\|_E$ for the norm of A when regarded as an operator from a space E to itself. We assume throughout that

$$a_{i,j} \geq 0 \text{ for all } i, j. \tag{1}$$

We denote by e_j the sequence having 1 in place j and 0 elsewhere. We assume further that each $A(e_j)$ is in $\ell_1(w)$, that is:

$$\sum_{i=1}^{\infty} a_{i,j} w_i \text{ is convergent for each } j. \tag{2}$$

We define

$$v_j = v_j(A, w) = \sum_{i=1}^{\infty} a_{i,j} w_i.$$

Formally, (v_j) is the sequence $A^*(w)$.

By condition (1), $\|A(x)\|_{\ell_1(w)} \leq \|A(|x|)\|_{\ell_1(w)}$, and hence non-negative sequences x are sufficient to determine $\|A\|_{\ell_1(w)}$. However, $\ell_1(w)$ is of limited interest as a Banach space, being just an isometric copy of ℓ_1 itself. As already stated, our real objective is to evaluate the norm of A as an operator on the Lorentz sequence space $d(w, 1)$. Under conditions established below (Proposition 4), this coincides with

$$\Delta_{w,1}(A) = \sup\{\|Ax\|_{\ell_1(w)} : x \in \delta_1(w), \|x\|_{\ell_1(w)} = 1\},$$

where $\delta_1(w)$ is the set of *decreasing*, non-negative sequences in $\ell_1(w)$.

At the same time, we shall consider the *lower bound* of A, defined (following Lyons [14] and Bennett [2]) as

$$m_{w,1}(A) = \inf\{\|Ax\|_{\ell_1(w)} : x \in \delta_1(w), \|x\|_{\ell_1(w)} = 1\}.$$

(The corresponding definition without the restriction to decreasing sequences leads to nothing of interest: for all the operators considered below, it would equate trivially to 0, both in ℓ_p and in Lorentz spaces.)

Since $\sum_{n=1}^{\infty} w_n$ is divergent, we have $x_n \to 0$ for all x in $\delta_1(w)$. We repeatedly use the following lemma on Abel summation (we omit the well-known proof).

LEMMA 1. *Suppose that* (a_n), (b_n) *are non-negative sequences and that* (b_n) *decreases and tends to* 0. *Write* $A_n = \sum_{j=1}^{n} a_j$. *If* $\sum_{n=1}^{\infty} a_n b_n$ *is convergent (say to* S*), then* $A_n b_n \to 0$ *as* $n \to \infty$ *and*

$$\sum_{n=1}^{\infty} A_n(b_n - b_{n+1}) = S.$$

PROPOSITION 1. *Suppose that* A *satisfies conditions* (1) *and* (2). *Let* $v_j = \sum_{i=1}^{\infty} a_{i,j} w_i$ *and* $V_n = v_1 + \cdots v_n$. *Then*

$$\begin{aligned} \|A\|_{\ell_1(w)} &= \sup_{n\geq 1} \frac{v_n}{w_n}, \\ \Delta_{w,1}(A) &= \sup_{n\geq 1} \frac{V_n}{W_n}, \\ m_{w,1}(A) &= \inf_{n\geq 1} \frac{V_n}{W_n} \end{aligned}$$

(*with the convention that if either side is finite, then so is the other*). *Both* $\Delta_{w,1}(A)$ *and* $m_{w,1}(A)$ *can be evaluated by considering only elements of the form* $e_1 + \cdots + e_n$.

Proof. Write $\|x\|$ for $\|x\|_{\ell_1(w)}$. Let $\sup_{n\geq 1}(v_j/w_j) = B$. If $x = e_j$, then $\|x\| = w_j$, while $\|Ax\| = v_j$. Hence $\|A\| \geq B$ (also when $B = \infty$). Now suppose $B < \infty$, and let (x_j) be any non-negative sequence in $\ell_1(w)$. Then $\sum_{j=1}^{\infty} v_j x_j$ is convergent, and we have

$$\begin{aligned} \|Ax\| &= \sum_{i=1}^{\infty} w_i \sum_{j=1}^{\infty} a_{i,j} x_j \\ &= \sum_{j=1}^{\infty} v_j x_j. \end{aligned}$$

Hence $\|A\| = B$.

Now let the supremum and infimum of V_n/W_n be C, c respectively. Let (x_j) be a decreasing, non-negative sequence. By the above and Lemma 1,

$$\|Ax\| = \sum_{j=1}^{\infty} V_j(x_j - x_{j+1}),$$

while

$$\|x\| = \sum_{j=1}^{\infty} W_j(x_j - x_{j+1}).$$

Hence, clearly, $c\|x\| \leq \|Ax\| \leq C\|x\|$. Further, if $x = e_1 + \cdots + e_n$, then $\|x\| = W_n$ and $\|y\| = V_n$, so such elements suffice to show that $\Delta_{w,1}(A) = C$ and $m_{w,1}(A) = c$.

Remark 1. Obviously, if A is regarded as an operator from $\ell_1(w)$ to $\ell_1(w')$, the same statements hold with $v_j = \sum_{i=1}^{\infty} a_{i,j} w'_i$.

Remark 2. Clearly, $\inf_{n\geq 1} v_n/w_n$ equals the infimum of $\|Ax\|_{\ell_1(w)}$ for positive elements x with $\|x\|_{\ell_1(w)} = 1$. However, as with the supremum, this is not the target of our investigation.

Remark 3. Though our objective in this paper is limited to the discrete case, we note that the above proof adapts without difficulty to the continuous case for sufficiently well-behaved functions. The corresponding formulae are

$$(Af)(x) = \int_0^{\infty} a(x, y) f(y)\, dy, \qquad v(y) = \int_0^{\infty} a(x, y) w(x)\, dx,$$

Integration by parts replaces Abel summation, and we conclude that $c\|f\|_{L_1(w)} \leq \|A(f)\|_{L_1(w)} \leq C\|f\|_{L_1(w)}$ for decreasing, positive functions f, where c and C are the infimum and supremum of $V(x)/W(x)$. As already mentioned, the continuous analogues of the specific operators considered here are actually isometric on the positive part of $L_1(w)$.

In certain cases, the supremum and infimum of (v_n/w_n) coincide with those of (V_n/W_n), because of the well-known facts listed in the following lemma (we omit the proofs).

LEMMA 2. (i) *If* $c \leq \frac{v_n}{w_n} \leq C$ *for all* n, *then* $c \leq \frac{V_n}{W_n} \leq C$ *for all* n.
(ii) *If* (v_n/w_n) *is increasing* (*or decreasing*), *then so is* (V_n/W_n).
(iii) *If* $\frac{v_n}{w_n} \to L$ *and* $W_n \to \infty$ *as* $n \to \infty$, *then* $\frac{V_n}{W_n} \to L$ *as* $n \to \infty$ (*this also holds with* $L = \infty$).

Hence, for example, if (v_n/w_n) is increasing and tends to the limit L, then $\sup(V_n/W_n) = L$ and $\inf(V_n/W_n) = v_1/w_1$. The same conclusion holds provided that we can show that $v_1/w_1 \leq v_n/w_n \leq L$ for all n; in some cases, this is much easier than showing that the sequence is increasing.

We now translate the above into the language of Lorentz sequence spaces. (Though this provides motivation for our study, it is not logically essential: our later theorems can simply be regarded as statements about $\Delta_{w,1}(A)$ and $m_{w,1}(A)$ without reference to Lorentz spaces.) Given a null sequence $x = (x_n)$, let (x_n^*) be the decreasing rearrangement of $|x_n|$.

The Lorentz sequence space $d(w, 1)$ is the space of null sequences x for which x^* is in $\ell_1(w)$, with norm $\|x\|_{w,1} = \|x^*\|_{\ell_1(w)}$. Clearly $\|x\|_{w,1} = \|x\|_{\ell_1(w)}$ for decreasing, non-negative sequences x. By Abel summation, for x in $d(w, 1)$ we have

$$\|x\|_{w,1} = \sum_{n=1}^{\infty} W_n(x_n^* - x_{n-1}^*)$$

and also

$$\|x\|_{w,1} = \sum_{n=1}^{\infty}(w_n - w_{n+1})X_n^*$$

where $X_n^* = x_1^* + \cdots + x_n^*$ (this is where we need the condition $w_n \to 0$). Since $|X_n| \le X_n^*$ for all n, this shows that $d(w,1)$ is contained in $\ell_1(w)$. It also shows that $d(w,1)$ has the following property: we say that a Banach lattice $(E, \|\ \|_E)$ of real null sequences has *property* (*PS*) if it satisfies:

if $x \in E$ and $Y_n^* \le X_n^*$ for all n, then $y \in E$ and $\|y\|_E \le \|x\|_E$.

Note that the same conclusion follows if x is non-negative and $Y_n^* \le X_n$ for all n, since $X_n \le X_n^*$. By Ky Fan's lemma [9, III.3.1], property (PS) holds in any *symmetric Banach sequence space*, i.e., a Banach lattice of null sequences with symmetric norm such that $\|x\|_E = \lim_{n\to\infty} \|P_n x\|_E$ for all x, where P_n is the projection onto the first n terms.

Our next result may be of independent interest. It describes conditions under which the norm of an operator on such a space is determined by its action on decreasing sequences. However, for the particular operators considered below, this property is very easily seen directly.

THEOREM 2. *Let $(E, \|\ \|_E)$ be a Banach lattice of sequences with property (PS). Let A be an operator from E to itself, given by $Ax = y$, where $y_i = \sum_{j=1}^{\infty} a_{i,j}x_j$, where $a_{i,j} \ge 0$ for all i, j. Suppose further that A satisfies:*

(3) *for all subsets M, N of $\mathbb{N}$ having m, n elements respectively, we have*

$$\sum_{i\in M}\sum_{j\in N} a_{i,j} \le \sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j}.$$

Then $\|A(x^*)\|_E \ge \|A(x)\|_E$ for all non-negative elements x of E. Hence decreasing, non-negative elements x are sufficient to determine $\|A\|_E$.

Proof. Let $y = A(x)$, $z = A(x^*)$. We show that

$$(*) \qquad Y_m^* \le Z_m \quad \text{for all } m.$$

Let $y_i^* = y_{\sigma(i)}$, and let $M = \{\sigma(i) : 1 \le i \le m\}$. Also, let $x_j^* = x_{\tau(j)}$. Then

$$\sum_{i=1}^{m} y_i^* = \sum_{i\in M} y_i = \sum_{i\in M}\sum_{j=1}^{\infty} a_{i,\tau(j)}x_j^* = \sum_{j=1}^{\infty}\left(\sum_{i\in M} a_{i,\tau(j)}\right)x_j^*.$$

By Abel summation (since $x_n^* \to 0$), this equals

$$\sum_{n=1}^{\infty}\left(\sum_{i\in M}\sum_{j\in N(n)} a_{i,j}\right)(x_n^* - x_{n+1}^*),$$

where $N(n) = \{\tau(j) : 1 \le j \le n\}$. Meanwhile,

$$\sum_{i=1}^{m} z_i = \sum_{i=1}^{m} \sum_{j=1}^{\infty} a_{i,j} x_j^* = \sum_{n=1}^{\infty} \left(\sum_{i=1}^{m} \sum_{j=1}^{n} a_{i,j} \right) (x_n^* - x_{n+1}^*).$$

The required inequality follows from (3).

Note. The converse is "nearly" true. More exactly, statement $(*)$ (for all x) implies (so is equivalent to) condition (3), as one sees easily by considering $x = \sum_{j \in N} e_j$.

Matrices satisfying condition (3) are by no means instantly recognisable. The next result provides sufficient conditions that are transparently satisfied in many cases of interest, including those considered below. Write

$$r_{i,n} = \sum_{j=1}^{n} a_{i,j}, \quad c_{m,j} = \sum_{i=1}^{m} a_{i,j},$$

the partial sums along row i and column j respectively. Consider the following conditions:

(4) $r_{i,n}$ decreases with i for each n.
(4*) $a_{i,j}$ decreases with i for each j.
(5) $c_{m,j}$ decreases with j for each m.
(5*) $a_{i,j}$ decreases with j for each i.

Clearly, (4*) is stronger than (4), and (5*) is stronger than (5).

PROPOSITION 3. *Condition* (3) *implies* (4) *and* (5). *Conversely,* (4) *and* (5*), *or* (5) *and* (4*), *imply* (3).

Proof. (i) Suppose that (4) is false, so that $r_{m,n} < r_{m+1,n}$ for some m, n. Let $M = \{1, 2, \dots m-1, m+1\}$, $N = \{1, 2, \dots, n\}$. Then

$$\sum_{i \in M} \sum_{j \in N} a_{i,j} = \sum_{i \in M} r_{i,n} > \sum_{i=1}^{m} r_{i,n} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{i,j},$$

so (3) fails. Similarly for (5).

(ii) Assume that (4) and (5*) hold, and consider M, N as in (3). For fixed i, the largest n terms $a_{i,j}$ are the first n terms, so

$$\sum_{j \in N} a_{i,j} \le \sum_{j=1}^{n} a_{i,j} = r_{i,n}.$$

In the same way, by (4),

$$\sum_{i\in M} r_{i,n} \le \sum_{i=1}^{m} r_{i,n} = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j}.$$

Note. A diagonal matrix, decreasing along the diagonal, satisfies (3) but not (4*) or (5*). A matrix that satisfies (4) and (5), but not (3), is

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

For this matrix, if $x = e_3$, then $x^* = e_1$, and (with above notation) $y_1^* = 2$ while $z_1 = 1$.

Condition (4*) is clearly equivalent to the statement that $A(x)$ is decreasing for any non-negative x. Condition (4) is equivalent to the statement that $A(x)$ is decreasing for decreasing, non-negative x, since, firstly, if $x = e_1 + \cdots + e_n$, then $y_i = r_{i,n}$, and secondly, by Abel summation again,

$$y_i = \sum_{j=1}^{\infty} r_{i,j}(x_j - x_{j+1}).$$

Also, under condition (4), our V_n has a simple interpretation:

$$V_n = \sum_{i=1}^{\infty} r_{i,n} w_i = \|A(e_1 + \cdots + e_n)\|_{w,1}.$$

Note however that v_j only equals $\|A(e_j)\|_{w,1}$ if (4*) holds.

Denote by $\|A\|_{w,1}$ the norm of A as an operator on $d(w, 1)$. We have now completed the identification of $\|A\|_{w,1}$ stated earlier.

PROPOSITION 4. *If A satisfies conditions* (1), (2) *and* (3), *then* $\|A\|_{w,1} = \Delta_{w,1}(A)$, *and hence is given by the expression* $\sup_n(V_n/W_n)$ *in Proposition* 1.

Proof. By Theorem 2, $\|A\|_{w,1}$ is determined by decreasing, non-negative sequences x. Since A satisfies condition (4), if x is decreasing and non-negative, then so is Ax, so that $\|Ax\|_{w,1} = \|Ax\|_{\ell_1(w)}$.

We finish this section with two further remarks.

(i) The quantity $\sup_{n\ge 1} V_n/W_n$ equates to the norm of the sequence v $(= A^*(w))$ in the dual space to $d(w, 1)$. However, this fact will not make any difference to our computations.

(ii) If condition (5) holds, then condition (2) reduces to convergence of $\sum_{i=1}^{\infty} a_{i,1} w_i$, since this series can be rewritten $\sum_{m=1}^{\infty} c_{m,1}(w_m - w_{m+1})$, and condition (5) says that $c_{m,j} \le c_{m,1}$ for all j.

3. Partial sums and tails of $\sum \frac{1}{n^\alpha}$

The following mostly well-known facts will be used repeatedly in evaluating the suprema and infima arising in our chosen particular cases. Let $\alpha > 0$, and write

$$a_n = \frac{1}{n^\alpha},$$
$$b_n = \int_{n-1}^{n} \frac{1}{t^\alpha}\,dt,$$

and (as usual) $A_n = a_1 + \cdots a_n$, etc. For $\alpha < 1$, the usual integral comparison gives $b_2 + \cdots + b_n \le A_n \le B_n$, or

$$\frac{1}{1-\alpha}(n^{1-\alpha} - 1) \le A_n \le \frac{n^{1-\alpha}}{1-\alpha},$$

hence $A_n/B_n \to 1$ as $n \to \infty$. We need to know also that A_n/B_n is increasing. The following is the key lemma.

LEMMA 3. *Let $b_n = \int_{n-1}^{n} t^{-\alpha}\,dt$. If $\alpha > 0$, then $n^\alpha b_n$ decreases with n and $n^\alpha b_{n+1}$ increases with n. The opposite conclusions apply when $\alpha < 0$.*

Proof. Write $c_n = n^\alpha b_n$. Then

$$c_{n+1} = (n+1)^\alpha \int_{n}^{n+1} \frac{1}{s^\alpha}\,ds = (n+1)^\alpha \int_{n-1}^{n} \frac{1}{(t+1)^\alpha}\,dt.$$

For $n-1 \le t \le n$, we have $(n+1)/n \le (t+1)/t$, hence if $\alpha > 0$, then $(n+1)^\alpha/(t+1)^\alpha \le n^\alpha/t^\alpha$ and $c_{n+1} \le c_n$ (with the reverse inequality when $\alpha < 0$). Similarly for the second statement.

PROPOSITION 5. *Let $0 < \alpha < 1$ and let $A_n = \sum_{j=1}^{n} 1/j^\alpha$. Then $A_n/n^{1-\alpha}$ increases and tends to $1/(1-\alpha)$.*

Proof. By Lemma 3, a_n/b_n increases. Hence, by Lemma 2(ii), A_n/B_n increases. The limit follows from the inequalities above.

We now consider the tail of the series for $\zeta(1+\alpha)$. For the tail of a series, the analogous result to Lemma 2(ii) is the following.

LEMMA 4. *Suppose that $a_n > 0$, $b_n > 0$ for all n and that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent. Let $A_{(n)} = \sum_{j=n}^{\infty} a_j$, similarly $B_{(n)}$. If (a_n/b_n) is increasing (or decreasing), then so is $(A_{(n)}/B_{(n)})$.*

Proof. Elementary.

PROPOSITION 6. *Let $\alpha > 0$ and let $A_{(n)} = \sum_{j=n}^{\infty} 1/j^{1+\alpha}$. Then $n^\alpha A_{(n)}$ is decreasing, $(n-1)^\alpha A_{(n)}$ increasing. Both tend to $1/\alpha$ as $n \to \infty$.*

Proof. Let $a_n = 1/n^{1+\alpha}$ and

$$b_n = \int_{n-1}^{n} \frac{1}{t^{1+\alpha}}\, dt.$$

Then $B_{(n+1)} = 1/\alpha n^\alpha$. By the usual integral comparison,

$$\frac{1}{\alpha n^\alpha} \leq A_{(n)} \leq \frac{1}{\alpha(n-1)^\alpha},$$

which implies the stated limits. By Lemma 3, (a_n/b_{n+1}) is decreasing, So by Lemma 4, $A_{(n)}/B_{(n+1)} = \alpha n^\alpha A_{(n)}$ is decreasing. Similarly, $A_{(n)}/B_{(n)}$ is increasing.

Remark. This is stated without proof in [6], Remark 4.10.

4. The averaging operator

The averaging operator A is defined by: $y_n = \frac{1}{n}(x_1 + \cdots + x_n)$. It is given by the Cesaro matrix

$$a_{i,j} = \begin{cases} 1/i & \text{for } j \leq i \\ 0 & \text{for } j > i. \end{cases}$$

This is a lower triangular "summability" matrix. In our terms, it satisfies conditions (4) and (5*). When A is regarded as an operator on ℓ_p (where $p > 1$), Hardy's inequality ([10], Section 9.8) states that $\|A\| = p^*$, and the lower bound $m(A)$ is $\zeta(p)^{1/p}$ [2]. (The element e_1 is enough to show that A does not map ℓ_1 into ℓ_1.)

The problems considered here are better illustrated by comparison with the following analogous problem in the continuous case. For a function f, let $(Af)(x) = \frac{1}{x}\int_0^x f(t)\, dt$. Let $w(x) = x^{-\alpha}$ and

$$\|f\|_{L_1(w)} = \int_0^\infty w(x)|f(x)|\, dx.$$

By simply reversing the order of integration, we see that for all non-negative f (not necessarily decreasing), we have

$$\|A(f)\|_{L_1(w)} = \frac{1}{\alpha}\|f\|_{L_1(w)}.$$

This is an isometric relationship, showing trivially that both the norm and the lower bound are $1/\alpha$. In the discrete case, we have two candidates for weighting sequences analogous to $x^{-\alpha}$, and the relationship is far from isometric in either case.

Condition (2) requires convergence of $\sum_{i=1}^{\infty} w_i/i$, and v_n is given by

$$v_n = \sum_{i=n}^{\infty} \frac{w_i}{i}.$$

For the weighting sequence $w_n = 1/n^\alpha$, our earlier results provide an immediate solution to both problems.

THEOREM 7. *Let A be the averaging operator, and let $w_n = 1/n^\alpha$, where $0 < \alpha \leq 1$. Then*

$$\|A\|_{w,1} = \Delta_{w,1}(A) = \|A\|_{\ell_1(w)} = \zeta(1+\alpha),$$

$$m_{w,1}(A) = \frac{1}{\alpha}.$$

Proof. We now have $w_i/i = 1/i^{1+\alpha}$, so $v_n = A_{(n)}$ in the notation of Proposition 6, which tells us that $n^\alpha v_n$ ($= v_n/w_n$) is decreasing and tends to $1/\alpha$. By Lemma 2, it follows that $m_{w,1}(A) = 1/\alpha$ and

$$\sup_{n\geq 1} \frac{V_n}{W_n} = \sup_{n\geq 1} \frac{v_n}{w_n} = \frac{v_1}{w_1} = \zeta(1+\alpha).$$

We now consider our second choice of weighting sequence, defined by $W_n = n^{1-\alpha}$ (where $0 < \alpha < 1$), so that

$$w_n = n^{1-\alpha} - (n-1)^{1-\alpha} = \int_{n-1}^{n} \frac{1-\alpha}{t^\alpha}\, dt.$$

The slight change to w_n is enough to change our problem completely. Since W_n is now simpler than w_n, we work with V_n/W_n instead of v_n/w_n. In contrast to the previous case, we will show that this sequence is *increasing*. Directly from the expression for v_n, we have

$$\begin{aligned} V_n &= w_1 + \cdots + w_n + n \sum_{j\geq n+1} \frac{w_j}{j} \\ &= W_n + n v_{n+1}, \end{aligned}$$

so

$$\frac{V_n}{W_n} = 1 + \frac{n v_{n+1}}{W_n} = 1 + n^\alpha v_{n+1}.$$

Write $c_n = 1/[n^\alpha(n+1)]$ and (as before) $C_{(n)} = \sum_{j\geq n} c_j$.

LEMMA 5. *With this notation, we have* $1 + n^\alpha v_{n+1} = n^\alpha C_{(n)}$ *for* $n \geq 1$. *Also,* $v_1 = C_{(1)}$.

Proof. By Abel summation,

$$\begin{aligned} v_{n+1} &= \sum_{j \geq n+1} \frac{w_j}{j} \\ &= \sum_{j \geq n+1} \left(\frac{1}{j} - \frac{1}{j+1} \right) W_j - \frac{W_n}{n+1} \\ &= \sum_{j \geq n} \left(\frac{1}{j} - \frac{1}{j+1} \right) W_j - \frac{W_n}{n} \\ &= \sum_{j \geq n} \frac{1}{j^\alpha (j+1)} - \frac{1}{n^\alpha}. \end{aligned}$$

The first statement follows. Further, $v_1 = 1 + v_2 = C_{(1)}$.

THEOREM 8. *Let* $c_n = 1/[n^\alpha (n+1)]$, *where* $\alpha > 0$. *Then* $n^\alpha C_{(n)} \to 1/\alpha$ *as* $n \to \infty$. *Also,* $n^\alpha C_{(n)}$ *is increasing if* $0 < \alpha \leq 1$ *and decreasing if* $\alpha > 1$.

Hence if $W_n = n^{1-\alpha}$ *and* A *is the averaging operator, then*

$$\Delta_{w,1}(A) = \|A\|_{w,1} = 1/\alpha, \qquad m_{w,1}(A) = \sum_{n=1}^{\infty} c_n.$$

Proof. Clearly,

$$\frac{1}{(n+1)^{1+\alpha}} \leq c_n \leq \frac{1}{n^{1+\alpha}}.$$

The stated limit follows, by Proposition 6.

To prove monotonicity, we use Lemma 4. Let $d_n = 1/n^\alpha - 1/(n+1)^\alpha$, so that $D_{(n)} = 1/n^\alpha$. Then

$$\begin{aligned} \frac{d_n}{c_n} &= \frac{(n+1)^\alpha - n^\alpha}{n^\alpha (n+1)^\alpha} n^\alpha (n+1) \\ &= \frac{(n+1)^\alpha - n^\alpha}{(n+1)^{\alpha-1}} \\ &= \frac{\alpha}{(n+1)^{\alpha-1}} \int_n^{n+1} t^{\alpha-1}\, dt. \end{aligned}$$

By Lemma 3, this is decreasing if $0 < \alpha < 1$, increasing if $\alpha > 1$. By Lemma 4, the same is true of $D_{(n)}/C_{(n)}$.

Note. The same method shows that the opposite conclusions hold for $c'_n = 1/[n(n+1)^\alpha]$.

5. The Copson operator

The "Copson" operator C is defined by $y = Cx$, where

$$y_i = \sum_{j=i}^{\infty} \frac{x_j}{j}.$$

It is given by the transpose of the Cesaro matrix

$$a_{i,j} = \begin{cases} 1/j & \text{for } i \leq j \\ 0 & \text{for } i > j. \end{cases}$$

This is an upper triangular matrix satisfying (5) and (4*). By Hardy's inequality applied to the dual, $\|C\| = p$ as an operator on ℓ_p. Copson's original result [8] was in fact the reverse inequality for the case $0 < p < 1$.

The analogous operator in the continuous case is

$$(Cf)(x) = \int_x^{\infty} \frac{f(y)}{y}\, dy,$$

and with $w(x) = x^{-\alpha}$, one sees by reversing the integration that

$$\|C(f)\|_{L_1(w)} = \frac{1}{1-\alpha} \|f\|_{L_1(w)}$$

for all non-negative f.

All versions of our problem are much easier for C. In fact, the lower bound problem is almost trivial for general (w_n):

PROPOSITION 9. *Whenever (w_n) is such that C maps $d(w, 1)$ into $d(w, 1)$, we have $m_{w,1}(C) = 1$.*

Proof. Though this follows easily from our general formula, it is more instructive to argue directly, as follows. If $y = Cx$, it is easily checked that

$$Y_n = X_n + n \sum_{j \geq n+1} \frac{x_j}{j}.$$

Hence if x is non-negative, then $Y_n \geq X_n$ for all n, hence $\|y\| \geq \|x\|$ (this applies to any symmetric Banach sequence space). Further, we have $C(e_1) = e_1$.

Remark. The same is clearly true for any *quasi-summability matrix* (i.e., an upper triangular matrix with column sums equal to 1) that satisfies our other conditions.

A pleasantly simple statement can also be made about the norm of C for general (w_n). With the notation of Section 2,

$$v_n = \frac{1}{n}(w_1 + \cdots + w_n) = \frac{W_n}{n}.$$

Recall that (w_n) is said to be 1-*regular* if

$$r_1(w) = \sup_{n \geq 1} \frac{W_n}{n w_n}$$

is finite. By Proposition 1 and Lemma 2(i), we have at once:

PROPOSITION 10. *If (w_n) is 1-regular, then C maps $d(w, 1)$ into $d(w, 1)$, and*

$$\|C\|_{w,1} = \Delta_{w,1}(C) \leq \|C\|_{\ell_1(w)} \leq r_1(w).$$

THEOREM 11. *Let C be the Copson operator, and let $w_n = 1/n^\alpha$, where $0 < \alpha < 1$. Then*

$$\|C\|_{w,1} = \Delta_{w,1}(C) = \|C\|_{\ell_1(w)} = \frac{1}{1-\alpha}.$$

Proof. With our standing notation,

$$\frac{v_n}{w_n} = \frac{W_n}{n w_n} = \frac{W_n}{n^{1-\alpha}}.$$

Our W_n is the A_n of Proposition 5, which tells us that $W_n/n^{1-\alpha}$ increases and tends to $1/(1-\alpha)$. The statement follows by (ii) and (iii) of Lemma 2. (Of course, this also shows that $r_1(w) = 1/(1-\alpha)$).

Remark. When $\alpha = 1$, so that $w_n = 1/n$, we have

$$\frac{v_n}{w_n} = W_n \to \infty \quad \text{as } n \to \infty,$$

so C is not a bounded operator on $d(w, 1)$, although of course it satisfies condition (2).

THEOREM 12. *Let C be the Copson operator, and let w_n be defined by $W_n = n^{1-\alpha}$, where $0 < \alpha < 1$. Then (again)*

$$\|C\|_{w,1} = \Delta_{w,1}(C) = \frac{1}{1-\alpha}.$$

Proof. We now have

$$V_n = \sum_{j=1}^{n} \frac{W_j}{j} = \sum_{j=1}^{n} \frac{1}{j^\alpha},$$

so the new V_n/W_n is exactly the v_n/w_n of Theorem 11, and Proposition 5 again gives the statement.

6. The Hilbert operator

Two versions of the Hilbert operator, which we denote by H_1 and H_0 respectively, are given by the matrices

$$a_{i,j} = \frac{1}{i+j}, \qquad a_{i,j} = \frac{1}{i+j-1}.$$

These are Hankel matrices satisfying (4*) and (5*). Hilbert's inequality ([10], section 9.1) gives the norm of both operators on ℓ_p (for $p > 1$) as $\pi/\sin(\pi/p)$. It is shown in [2] that $m_p(H_0) = \zeta(p)^{1/p}$, and the same method shows that $m_p(H_1) = (\zeta(p)-1)^{1/p}$.

Nearly any study of the Hilbert operator depends on the well-known integral

$$\int_0^\infty \frac{1}{t^\alpha(t+c)}\,dt = \frac{\pi}{c^\alpha \sin\alpha\pi} \qquad (0 < \alpha < 1).$$

The analogous operator in the continuous case is

$$(Hf)(x) = \int_0^\infty \frac{f(y)}{x+y}\,dy.$$

With $w(x) = x^{-\alpha}$, one finds, using the integral just quoted, that

$$\|H(f)\|_{L_1(w)} = \frac{\pi}{\sin\alpha\pi}\|f\|_{L_1(w)}$$

for all non-negative f. With our two choices of both (w_n) and the operator, this isometric relationship is replaced by no fewer than eight distinct problems! Fortunately, they do not all need to be considered separately.

We start by considering H_1, with $w_n = 1/n^\alpha$. In our usual notation, we have

$$v_n = \sum_{i=1}^\infty \frac{1}{i^\alpha(i+n)}.$$

THEOREM 13. *With v_n defined in this way, we have $\sup_{n\geq 1} n^\alpha v_n = \pi/\sin\alpha\pi$ and $\inf_{n\geq 1} n^\alpha v_n = v_1$. Hence if $w_n = 1/n^\alpha$, where $0 < \alpha < 1$, then*

$$\|H_1\|_{w,1} = \Delta_{w,1}(H_1) = \|H_1\|_{\ell_1(w)} = \frac{\pi}{\sin\alpha\pi},$$

$$m_{w,1}(H_1) = v_1 = \sum_{i=1}^\infty \frac{1}{i^\alpha(i+1)}.$$

Proof. By comparison with the integral above, we have $v_n \leq \pi/(n^\alpha \sin\alpha\pi)$, hence

$$n^\alpha v_n \leq \frac{\pi}{\sin\alpha\pi}.$$

Also,

$$v_n \geq \int_1^\infty \frac{1}{t^\alpha(t+n)}\, dt,$$

and

$$\int_0^1 \frac{1}{t^\alpha(t+n)}\, dt \leq \int_0^1 \frac{1}{nt^\alpha}\, dt = \frac{1}{(1-\alpha)n}.$$

Hence

$$n^\alpha v_n \geq \frac{\pi}{\sin \alpha\pi} - \frac{1}{(1-\alpha)n^{1-\alpha}},$$

which proves the stated supremum, and hence the statement concerning norms of H_1.

We now turn to the lower bound. Note that

$$n^\alpha v_n = \sum_{i=1}^\infty \frac{1}{(i/n)^\alpha(i+n)}.$$

For $k \geq 1$, let $E_k = \{i \in \mathbb{Z} : (k-1)n < i \leq kn\}$. If $i \in E_k$, then $\frac{i}{n} \leq k$, so

$$\left(\frac{i}{n}\right)^\alpha (i+n) \leq k^\alpha(kn+n) = nk^\alpha(k+1).$$

Since E_k has n members,

$$\sum_{i \in E_k} \frac{1}{(i/n)^\alpha(i+n)} \geq \frac{n}{nk^\alpha(k+1)} = \frac{1}{k^\alpha(k+1)}.$$

Hence

$$n^\alpha v_n \geq \sum_{k=1}^\infty \frac{1}{k^\alpha(k+1)} = v_1.$$

Remark 1. Recall that by Theorem 8, we have $v_1 \leq 1/\alpha$.

Remark 2. It can in fact be shown that $n^\alpha v_n$ increases with n (which of course implies both statements): see [7].

Remark 3. When $\alpha = 1$, we have

$$v_n = \sum_{i=1}^\infty \frac{1}{i(i+n)} = \frac{1}{n}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right),$$

hence $nv_n \to \infty$ as $n \to \infty$, and H_1 is not a bounded operator on $d(w, 1)$.

The operator H_0 (with $w_n = 1/n^\alpha$) is much harder to deal with. Clearly, $v_n(H_0, w) = v_{n-1}(H_1, w)$ for $n \geq 2$, and $v_1(H_0, w) = \zeta(1+\alpha)$. The limit of $n^\alpha v_n$ is still $\pi/\sin\alpha\pi$, but this is less than $\zeta(1+\alpha)$ when α is less than approximately 0.32. It is quite easy to show that $n^\alpha v_n \leq \pi/\sin\alpha\pi$ for large enough n. Computations by an associate, R. Lashkaripour, indicate that $n^\alpha v_n$ either increases throughout, or decreases for a certain number of terms and then increases. This, if proved, would imply that $\|H_0\|_{w,1}$ is the greater of $\zeta(1+\alpha)$ and $\pi/\sin\alpha\pi$. For sequences of this type, one can hardly hope for an exact expression for the lower bound. However, we can easily give a lower estimate:

PROPOSITION 14. *If $w_n = 1/n^\alpha$, then $m_{w,1}(H_0) \geq \dfrac{1}{\alpha}$.*

Proof. We now have

$$v_n = \sum_{i=1}^{\infty} \frac{1}{i^\alpha(i+n-1)} = \sum_{j=n}^{\infty} \frac{1}{j(j-n+1)^\alpha} \geq \sum_{j=n}^{\infty} \frac{1}{j^{1+\alpha}}.$$

As noted in Proposition 6, this is not less than $1/(\alpha n^\alpha)$, so $(v_n/w_n) \geq 1/\alpha$ for all n.

The contrast between H_0 and H_1 is enough to show how remote the possibility is of finding any kind of solution to these problems for Hankel operators in general.

We turn to the case where w_n is defined by $W_n = n^{1-\alpha}$. Note first that, with the notation of Section 2,

$$V_n = \sum_{i=1}^{\infty} r_{i,n} w_i = \sum_{i=1}^{\infty} W_i(r_{i,n} - r_{i+1,n}).$$

This time, we consider H_0 first, since it turns out (in the same way as in Theorem 12) that we have solved the problem for this operator already ! For H_0, we have

$$r_{i,n} = \frac{1}{i} + \cdots + \frac{1}{i+n-1},$$

hence

$$r_{i,n} - r_{i+1,n} = \frac{1}{i} - \frac{1}{i+n} = \frac{n}{i(i+n)},$$

and by the above

$$\frac{V_n}{W_n} = \frac{1}{n^{1-\alpha}} \sum_{i=1}^{\infty} i^{1-\alpha} \frac{n}{i(i+n)} = n^\alpha \sum_{i=1}^{\infty} \frac{1}{i^\alpha(i+n)}.$$

This is precisely the v_n/w_n of Theorem 13, so we have:

THEOREM 15. *With w_n defined by $W_n = n^{1-\alpha}$, we have*

$$\|H_0\|_{w,1} = \Delta_{w,1}(H_0) = \frac{\pi}{\sin \alpha\pi}, \qquad m_{w,1}(H_0) = \sum_{i=1}^{\infty} \frac{1}{i^\alpha (i+1)}.$$

For H_1, we have instead

$$r_{i,n} - r_{i+1,n} = \frac{1}{i+1} - \frac{1}{i+n+1} = \frac{n}{(i+1)(i+n+1)},$$

so that

$$\frac{V_n}{W_n} = n^\alpha \sum_{i=1}^{\infty} \frac{i^{1-\alpha}}{(i+1)(i+n+1)}.$$

THEOREM 16. *With w_n defined by $W_n = n^{1-\alpha}$, we have*

$$\|H_1\|_{w,1} = \Delta_{w,1}(H_1) = \frac{\pi}{\sin \alpha\pi}, \qquad m_{w,1}(H_1) = V_1 = \sum_{i=1}^{\infty} \frac{i^{1-\alpha}}{(i+1)(i+2)}.$$

Proof. The norm estimation only requires slight adaptations to the proof of Theorem 13. Clearly,

$$\frac{V_n}{W_n} \le n^\alpha \sum_{i=1}^{\infty} \frac{1}{i^\alpha (i+n)},$$

As seen in Theorem 13, this is not greater than $\pi / \sin \alpha\pi$. For any $N \ge 2$,

$$\begin{aligned} \sum_{i=N-1}^{\infty} \frac{i^{1-\alpha}}{(i+1)(i+n+1)} &\ge \left(\frac{N-1}{N}\right)^{1-\alpha} \sum_{i=N-1}^{\infty} \frac{1}{(i+1)^\alpha (i+n+1)} \\ &= \left(\frac{N-1}{N}\right)^{1-\alpha} \sum_{i=N}^{\infty} \frac{1}{i^\alpha (i+n)}. \end{aligned}$$

As in Theorem 13, we see that

$$n^\alpha \int_N^\infty \frac{1}{t^\alpha (t+n)}\, dt \to \frac{\pi}{\sin \alpha\pi} \quad \text{as } n \to \infty,$$

from which it follows that $\sup_{n \ge 1}(V_n / W_n) = \pi / \sin \alpha\pi$.

For the lower bound, we again follow the method of Theorem 13, but the details are a bit more awkward. We have to show that $V_n / W_n \ge V_1$ for all $n \ge 2$. Define the sets E_k as before. For $i \in E_k$, we have $\frac{i}{n} \le k$, so

$$n^\alpha \frac{i^{1-\alpha}}{(i+1)(i+n+1)} \ge \frac{d_i}{k^\alpha},$$

where

$$d_i = \frac{i}{(i+1)(i+n+1)}.$$

Our statement will follow in the same way as before if we can show that

$$(*) \qquad \sum_{i \in E_k} d_i \geq \frac{k}{(k+1)(k+2)}$$

for each k. We only sketch the details. Clearly, for $i \in E_k$,

$$d_i \geq \frac{i-1}{i(i+n+1)} \geq \frac{i-1}{kn(i+n+1)}.$$

From this one shows (separately for $i < kn$ and $i = kn$) that $d_i \geq (k-1)/kn(k+1)$ for $i \in E_k$, so that

$$\sum_{i \in E_k} d_i \geq \frac{k-1}{k(k+1)}.$$

This implies $(*)$ when $k \geq 2$, and one then shows directly that $\sum_{i \in E_1} d_i \geq \frac{1}{6}$.

7. An example

We now give an example to show that it is possible to have (V_n/W_n) bounded, while (v_n/w_n) is not. Clearly, $\sup(v_n/w_n)$ is sensitive to variations in each individual w_n, and its equivalence to $\sup(V_n/W_n)$ in the above results only occurred because our choices of (w_n) were very "smooth".

Example. Consider again the averaging operator. Recall that for this operator we have

$$v_n = \sum_{i \geq n} \frac{w_i}{i},$$

$$\frac{V_n}{W_n} = 1 + \frac{nv_{n+1}}{W_n}.$$

We shall choose w_n so that (v_n/w_n) is unbounded, while $nv_n/W_n \to 0$ as $n \to \infty$ (hence $V_n/W_n \to 1$).

Let $E_k = \{i \in \mathbb{Z} : k! \leq i < (k+1)!\}$, and let $w_i = 1/(k!k)$ for $i \in E_k$. Clearly, $\sum_{i \in E_k} w_i = 1$ and by integral estimation

$$\log(k+1) \leq \sum_{i \in E_k} \frac{1}{i} \leq \log(k+2) \qquad \text{for } k \geq 2.$$

So if $n = k!$, then $w_n = 1/(k!k)$ while

$$v_n \geq \frac{1}{k!k} \sum_{i \in E_k} \frac{1}{i} \geq \frac{\log k}{k!k}.$$

Hence (v_n/w_n) is unbounded. For any $n \in E_k$ (where $k \geq 3$), we also have

$$W_n \geq W_{k!} = k,$$

while

$$\begin{aligned} v_n &\leq \sum_{j=k}^{\infty} \frac{\log(j+2)}{j!j} \\ &\leq \sum_{j=k}^{\infty} \frac{1}{j!j^{1/2}} \qquad \text{since } \log(j+2) \leq j^{1/2} \\ &\leq \frac{1}{k^{1/2}} \sum_{j=k}^{\infty} \frac{1}{j!} \\ &\leq \frac{1}{k^{1/2}} \frac{1}{(k-1)!(k-1)}. \end{aligned}$$

Since $n \leq (k+1)!$, we obtain

$$\frac{nv_n}{W_n} \leq \frac{1}{k^{1/2}} \frac{k+1}{k-1},$$

and hence the required statement.

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