A NOTE ON ABSTRACT (M)-SPACES

BY

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The following result is a consequence of the theorem that is proved in this note: Every Banach lattice with a strong order unit can be renormed so that the resulting space is an abstract (M)-space with a unit element. As will be seen from the proof, this rather unexpected result is a simple consequence of several known theorems to be found in various places in [2], [3], and [4].

A *locally convex lattice* $E(\mathfrak{T})$ is a vector lattice E over the real field equipped with a Hausdorff locally convex topology $\mathfrak{T}$ which has a generating family $\{p_\alpha\}_{\alpha \in A}$ of semi-norms satisfying

(1) If $|x| \leqq |y|$, then $p_\alpha(x) \leqq p_\alpha(y)$ for all $\alpha \in A$.

A real vector lattice which is a Banach space whose norm satisfies (1) is called a *Banach lattice*. An *abstract* (M)*-space* is a Banach lattice whose norm also satisfies[1]

(2) If $x \geqq \theta$, $y \geqq \theta$, then $\| \sup(x, y) \| = \max\{\| x \|, \| y \|\}$.

A subset H of the positive cone $K = \{x \in E : x \geqq \theta\}$ in a vector lattice E is an *exhausting subset of* K if for each $x \in K$ there are an $h \in H$ and a positive number λ such that $x \leqq \lambda h$. An element $e \in K$ is called a *strong order unit* if $\{e\}$ is an exhausting subset of K. An element $u \in K$ of a Banach lattice E is called a *unit element* if $\| u \| = 1$ and $\| x \| \leqq 1$ implies that $x \leqq u$. More information as well as further references concerning all of the notions defined above, with the exception of that of (M)-space, can be found in [2] and [3]; an account of the basic theory of (M)-spaces is given, for example, in [1].

The properties of the order topology $\mathfrak{T}_0$, introduced independently by Namioka[2] [2] and Schaefer [3], will play a central role in the considerations that follow. $\mathfrak{T}_0$ can be defined as the finest locally convex topology on the vector lattice E for which each order interval

$$[-x, x] = \{z \in E : -x \leqq z \leqq x\} \qquad (x \in K)$$

is a topologically bounded set. Thus a neighborhood basis of the zero element θ is provided by the class of all convex circled sets that absorb each order interval in E. If $E(\mathfrak{T})$ is a locally convex lattice, and if $\{p_\alpha\}_{\alpha \in A}$ is a generating system of semi-norms for $\mathfrak{T}$ satisfying (1), then each p_α is

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[1] θ denotes the additive identity in E.

[2] Namioka calls $\mathfrak{T}_0$ the "order bound topology $\mathfrak{T}_b$".

clearly bounded on each order bounded set; hence $\mathfrak{T}_0$ is finer than $\mathfrak{T}$ on E. The following result, showing that $\mathfrak{T}_0$ actually coincides with the given topology on a large class of locally convex lattices, appears in the unpublished notes [4] but does not seem to be a part of the available literature.

LEMMA. *If $E(\mathfrak{T})$ is a complete metrizable locally convex lattice, then $\mathfrak{T} = \mathfrak{T}_0$.*

Proof. We have already noted that $\mathfrak{T}_0$ is finer than $\mathfrak{T}$. On the other hand, [2, 5.5 Corollary] shows that if $E(\mathfrak{T})$ is a complete metrizable locally convex lattice, then every positive linear form on $E(\mathfrak{T})$ is continuous. Consequently the topological dual E' of $E(\mathfrak{T})$ is contained in $E^+ = K^* - K^*$ (where K^* denotes the cone of positive linear forms on E). The opposite inclusion follows from [3, (1.3)]; hence $E' = E^+$. Since $E(\mathfrak{T}_0)$ also has E^+ as its topological dual, $\mathfrak{T}$ and $\mathfrak{T}_0$ must both coincide with the Mackey topology $\tau(E, E^+)$, E being bornological for $\mathfrak{T}_0$ and metrizable for $\mathfrak{T}$ (see [2, 4.10]).

We shall now prove our main result:

THEOREM. *If $E(\mathfrak{T})$ is a metrizable complete locally convex lattice, then $E(\mathfrak{T})$ is the inductive limit of a family of linear subspaces that are abstract (M)-spaces with unit elements. If, in addition, E contains a strong order unit, then $\mathfrak{T}$ can be generated by a norm for which $E(\mathfrak{T})$ is an (M)-space with unit element.*

Proof. By the lemma preceding the theorem, the given topology on E coincides with the order topology $\mathfrak{T}_0$. Suppose that H is an exhausting subset of K, and form the subspace $E_h =$ Linear Hull $[-h, h]$ for each $h \in H$. E_h is a lattice ideal in E which is archimedean since the cone in E is closed. Thus the Minkowski functional p_h of $[-h, h]$ is a norm on E_h which generates the order topology $\mathfrak{T}_0$ on E_h (see [3, 4.1]). Moreover p_h satisfies (2) for each $h \in H$. For if $x, y \in K \cap E_h = K_h$, then it is clear that

$$\max\{p_h(x), p_h(y)\} \leqq p_h(\sup(x, y)).$$

On the other hand, $x \leqq p_h(x)h$, $y \leqq p_h(y)h$ since K_h is closed, so that

$$\sup(x, y) \leqq \max\{p_h(x), p_h(y)\}h.$$

Therefore

$$p_h(\sup(x, y)) \leqq \max\{p_h(x), p_h(y)\}$$

which completes the verification of (2) for p_h. It is clear that h is a unit element in E_h. The first assertion of the theorem now follows from the fact that $\mathfrak{T}_0$ (and hence $\mathfrak{T}$) has been shown to be the inductive limit topology with respect to the family of subspaces $\{E_h(\mathfrak{T}_0)\}_{h \in H}$ (see [3, 4.4]). If E contains a strong order unit e, then $E_e = E$, so that E equipped with the norm p_e is an (M)-space with unit element e.

References

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