ON CONNECTED NORMED ALGEBRAS WHOSE NORMS SATISFY A REALITY CONDITION

BY

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1. Introduction

Certain Banach algebras can be represented as the algebra of all continuous real-valued functions on a suitable compact space, provided that they satisfy certain types of "reality conditions." (Compact spaces and locally compact spaces are always required to be Hausdorff spaces in this note, in addition to the usual requirements which such spaces must satisfy.) For example, Segal showed in [9] that if a commutative Banach algebra A with unit has a norm N such that $N(x^2) = N(x)^2$ and $N(x^2 - y^2) \leqq \max(N(x^2), N(y^2))$ for all x and y in A, then A can be identified with the algebra of all continuous real-valued functions on some compact space. In [6], Kadison showed that if a Banach algebra A with unit e has a norm N such that

$$N(a^2 + \cdots + d^2) \geqq N(a)^2$$

for arbitrary $a, \cdots, d$ in A, and if $x^2 + e$ has an inverse in A for each x in A, then A can be identified with the algebra of all continuous real-valued functions on a suitable compact space.

These results of Segal and Kadison are characterized by the fact that the imposition of a condition on the norm of the algebra insures that only *real*-valued functions are obtained, although the extra algebraic condition of Kadison also tends to have this effect. In this note, we shall consider connected normed algebras whose norms satisfy a "reality condition" of the type employed by Segal or Kadison. The corollary of Theorem 5 will include as special cases both Segal's result and the commutative case of Kadison's result. Specifically, it is shown that if A is a nonzero, commutative, complete normed algebra, over the real field with some power of its ordinary absolute value used as an absolute value, and if the norm N for A has the property that

$$N(x^2 + y^2) \geqq N(x)^2$$

(or $N(x^2) = N(x)^2$ and $N(x^2 - y^2) \leqq \max(N(x^2), N(y^2))$) for all x and y in A, then A may be identified with the algebra of all continuous real-valued functions on a suitable compact space in case A has a unit element, and A may be identified with the algebra of all continuous real-valued functions which "vanish at infinity" on a suitable noncompact locally compact space if A is without unit. In the corollary of Theorem 6, a similar conclusion is obtained in case the hypotheses concerning the scalar field are replaced by the assumptions that the scalar field is some field with absolute value and that the

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algebra is connected. (In this case the given algebra is only identified with the *ring* of continuous functions, since the algebra is not given originally as a *real* algebra.) The theorems to which these corollaries correspond are somewhat similar to their corollaries, but they do not assume commutativity and they do assume slightly stronger forms for the "reality conditions" on the norms.

The material contained in this paper depends upon Ostrowski's results in [8], whereas Mazur's Theorem [7] has usually been employed by others in comparable situations. (See Gelfand's methods in [5], for instance.) Some of the results which stem from those of Ostrowski have some intrinsic interest, in addition to their usefulness in proving the results mentioned in the preceding paragraph. Thus, Theorem 1 shows that if a connected, complete ring R with absolute value has a unit element, then R is algebraically and topologically isomorphic to the real field, the complex field, or the division ring of real quaternions. A corollary of this theorem then shows that if a connected ring R with absolute value has a nonzero center, then R is algebraically and topologically isomorphic to a subring of the division ring of all real quaternions. The case in which R is a field is well known and is due essentially to Ostrowski; in this case R would be algebraically and topologically isomorphic to a subfield of the complex field.

2. Preliminaries

It is assumed that the reader is familiar with the contents of [2]. Since we shall deal with both rings and algebras, the terms *homomorphism* and *isomorphism* will be reserved for use when reference is intended only to the ring structure of the systems under consideration; when these two words are to have a meaning different from this, the meaning will be specified. A norm-preserving isomorphism of a metric ring R into a metric ring R' will be called an *isometry* of R into R'. If there exists an isometry of a metric ring R onto a metric ring R', then R is said to be *isometric* to R'.

If ρ is a real number such that $0 < \rho \leqq 1$, then the symbols $\mathfrak{R}^{(\rho)}$, $\mathfrak{C}^{(\rho)}$, and $\mathfrak{Q}^{(\rho)}$ will denote respectively the field of all real numbers, the field of all complex numbers, and the division ring of all real quaternions, each provided with the ρ^{th} power of its ordinary absolute value as a norm. Each of these systems is then a connected, complete division ring with absolute value. Conversely, a connected, complete division ring with absolute value is necessarily isometric to $\mathfrak{R}^{(\rho)}$, to $\mathfrak{C}^{(\rho)}$, or to $\mathfrak{Q}^{(\rho)}$, where ρ is a suitable real number such that $0 < \rho \leqq 1$. This result is given essentially in Theorem 11 of [1]. However, it will be useful to obtain a more general result than this, and we shall therefore obtain the same conclusion when the assumption that a division ring is involved is replaced by the weaker hypothesis that the ring has a unit element.

THEOREM 1. *Let R be a connected, complete ring with absolute value, such that R has a unit element e. Then there exists a real number ρ, with $0 < \rho \leqq 1$, such that R is isometric to $\mathfrak{R}^{(\rho)}$, to $\mathfrak{C}^{(\rho)}$, or to $\mathfrak{Q}^{(\rho)}$.*

Proof. *Case* 1. *Let the complement of* 0 *be connected in R.* We note that R contains no nonzero generalized divisors of zero in the sense of [1], for the norm of R is an absolute value. Then Theorem 1 of [1] shows that R is a division ring; the use of Lemma 15 and Theorem 11 of [1] then yields the desired result.

Case 2. *Let the complement of* 0 *fail to be connected in R.* Then [4; Chap. V, §3, Ex. 4] outlines a method of showing that the additive group of R is isomorphic as a topological group to the additive group of real numbers. If c is any nonzero element of R, the mapping $x \rightarrow cx$ is a continuous endomorphism of the additive group of R, and it carries the connected set R into the set cR. Thus, cR is connected and is also a subgroup of the additive group of R. Furthermore, cR contains the nonzero element $ce = c$. Since the additive group of real numbers has no connected subgroups other than itself and the zero subgroup, we may conclude that the subgroup cR coincides with R. It follows that e is in cR, so that c has a right inverse in R. But c was any nonzero element of R, and we therefore conclude that R is a division ring. The proof is completed in the same way as in Case 1.

COROLLARY. *Let R be a connected ring with absolute value, such that R has a nonzero center. Then there exists a real number* ρ, *with* $0 < \rho \leqq 1$, *such that R is isometric to a subring of* $\mathfrak{Q}^{(\rho)}$. *If, in addition, R is commutative, then R is also isometric to a subring of* $\mathfrak{C}^{(\rho)}$.

Proof. Let D be the set of nonzero central elements of R, and apply Lemma 17 of [1] in order to construct the metric ring R_D and an isometry of R into R_D. (Note that the existence of a unit element is not used in the proof of Lemma 17, although for convenience all rings in [1] were assumed to have units.) It is clear that R_D is a ring with absolute value and that R_D has a unit element.

For each fixed s in D, the mapping $x \rightarrow [x/s]$ is a continuous mapping of the connected space R onto a subset B_s of R_D. Then each B_s is connected, each B_s obviously contains 0, and R_D is the union of the B_s as s ranges over D. It follows from [4; Chap. I, §11, Prop. 2] that R_D is connected.

The completion of R_D must then satisfy the hypotheses of Theorem 1, and therefore there exists a real number ρ, with $0 < \rho \leqq 1$, such that the completion of R_D is isometric to $\mathfrak{R}^{(\rho)}$, to $\mathfrak{C}^{(\rho)}$, or to $\mathfrak{Q}^{(\rho)}$. That is, the completion of R_D is isometric to a subring of $\mathfrak{Q}^{(\rho)}$, so that R_D is isometric to a subring of $\mathfrak{Q}^{(\rho)}$. Thus, R is isometric to a subring of $\mathfrak{Q}^{(\rho)}$.

In case R is also commutative, then R is isometric to a commutative subring of $\mathfrak{Q}^{(\rho)}$. But every commutative subring of $\mathfrak{Q}^{(\rho)}$ is contained in a maximal commutative subring, and it is easily established that every maximal commutative subring of $\mathfrak{Q}^{(\rho)}$ is isometric to $\mathfrak{C}^{(\rho)}$. The second assertion in the statement of the corollary follows immediately.

There are two special cases of some interest in which the corollary applies:

(1) *If R is a nonzero, commutative, connected ring with absolute value, then the corollary applies to R.*

(2) *If R is a connected ring with absolute value such that R has a unit element, then the corollary applies to R.*

The case in which R is actually a field in this corollary was given essentially by Ostrowski in [8].

It should be noted that the isometries are not necessarily unique in either the theorem or its corollary, but the number ρ is uniquely determined by the condition $\| 2c \| = 2^{\rho} \cdot \| c \|$ if c is any given nonzero central element of the ring in question. (Pseudonorms for rings will be indicated generally by letters such as N and N', but in the case of norms we may revert to the notation $\| \quad \|$ in some instances.)

3. Pseudonorms which satisfy a reality condition

The results of the preceding section show that certain types of connected metric rings may be embedded in the division ring of all real quaternions. We shall now impose further conditions on the norms of such rings, in order to obtain an embedding in the field of all real numbers. This will be useful in the next section, where rings of continuous *real*-valued functions will be constructed. The conditions imposed in this section on norms (or pseudonorms) may therefore be described as "reality conditions," since they lead to real numbers and to real-valued functions.

DEFINITION. If N is a pseudonorm for a ring R, such that for every natural number n

$$N((x_1^2 \cdots x_n^2) + (y_1^2 \cdots y_n^2)) \geqq N(x_1 \cdots x_n)^2$$

whenever $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$ are ordered n-tuples of elements of R, then N will be called a *Kadison* pseudonorm. A pseudonorm N for a ring R will be called a *Segal* pseudonorm if $N(x^2) = N(x)^2$ for all x in R, and if for every natural number n

$$N((x_1^2 \cdots x_n^2) - (y_1^2 \cdots y_n^2)) \leqq \max (N(x_1^2 \cdots x_n^2), N(y_1^2 \cdots y_n^2))$$

whenever $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$ are ordered n-tuples of elements of R.

A Kadison pseudonorm N always satisfies the condition $N(x^2 + y^2) \geqq N(x)^2$ for all x and y, by definition of a Kadison pseudonorm. When $y = 0$, we have $N(x^2) \geqq N(x)^2$, so that $N(x^2) = N(x)^2$ for all x when N is a Kadison pseudonorm. Thus, Kadison pseudonorms and Segal pseudonorms are always *power multiplicative* since they satisfy the condition $N(x^2) = N(x)^2$ for all x.

In the case of a commutative ring the criteria for determining whether a given pseudonorm is a Kadison pseudonorm or Segal pseudonorm become somewhat simpler.

THEOREM 2. *Let N be a pseudonorm for a commutative ring R. Then N is a Kadison pseudonorm* (*Segal pseudonorm*) *if and only if*

$$N(x^2 + y^2) \geqq N(x)^2$$

$$(N(x^2) = N(x)^2 \quad \textit{and} \quad N(x^2 - y^2) \leqq \max(N(x^2), N(y^2)))$$

for all x and y in R.

The proof is routine and is left to the reader.

Segal considered in [9] commutative Banach algebras having a Segal norm. In [6], Kadison considered the case of a Banach algebra with a norm N such that $N(a^2 + \cdots + d^2) \geqq N(a)^2$ for arbitrary $a, \cdots, d$. Theorem 2 shows that for a commutative Banach algebra such an N is actually a Kadison norm in the sense of this note. The norms considered by Kadison are different from Kadison norms as defined here when a noncommutative algebra is involved. However, Kadison also employed a further assumption that the algebras were "strictly real," as defined in [6]. In this note, the only reality conditions which will be employed are reality conditions on the norm: *It will be assumed that the norms are Kadison norms or Segal norms.* It seems to be necessary that we use the fairly strong form of definition given above for Kadison pseudonorms and Segal pseudonorms.

It is now possible to obtain embeddings in the real field if a "reality condition" on the norm is added to the hypotheses of the results of Section 2.

THEOREM 3. *Let R be a connected, complete ring with Kadison absolute value* (*Segal absolute value*), *such that R has a unit element. Then there exists a real number ρ, with $0 < \rho \leqq 1$, such that there exists an isometry σ of R onto $\mathfrak{R}^{(\rho)}$.*

Proof. Theorem 1 is applied, and the other possible conclusions are eliminated because the absolute values for $\mathfrak{C}^{(\rho)}$ and $\mathfrak{Q}^{(\rho)}$ are not Kadison absolute values or Segal absolute values.

COROLLARY. *Let R be a connected ring with Kadison absolute value* (*Segal absolute value*), *such that R has a nonzero center. Then there exists a real number ρ, with $0 < \rho \leqq 1$, such that there exists an isometry σ of R into $\mathfrak{R}^{(\rho)}$.*

Proof. This corollary follows from Theorem 3 in the same way that the corollary of Theorem 1 follows from that theorem. It is only necessary to note that when R has a Kadison absolute value (Segal absolute value), then R_D and the completion of R_D also have Kadison absolute values (Segal absolute values).

It has already been observed that ρ is uniquely determined in such results. The isometry in Theorem 3 is clearly unique (this is seen more easily if the inverse of the isometry is considered), and it follows easily that σ is uniquely determined in the corollary of Theorem 3. Thus, ρ and σ are unique in both Theorem 3 and its corollary.

We shall be interested in algebras as well as rings in this note, and we therefore introduce the definitions which follow.

DEFINITION. Let A be an algebra, over a field K with absolute value. A pseudonorm N for A is said to be *K-admissible* if $N(kx) = \| k \| \cdot N(x)$ whenever k is in K and x is in A.

DEFINITION. When an al bra A, over a field K with absolute value, is given, together with a K-admissible norm for A, then A is called a *normed algebra* over K. (See [4; Chap. IX, §3, No 7], for example, for further information concerning normed algebras.)

DEFINITION. A metric ring R with a stable norm will be called a *Kadison ring* (*Segal ring*) if the norm is also a Kadison norm (Segal norm). If a normed algebra A is also a Kadison ring (Segal ring) then A will be called a *Kadison algebra* (*Segal algebra*).

In [2], the notion of *induced pseudonorms* of a given pseudonorm was considered, and pseudo absolute values subordinate to a given pseudonorm were also constructed in some cases. It is desirable to show that various special properties of the original pseudonorm may be transmitted to the new pseudonorms. This is done in the results which conclude this section.

LEMMA 1. *Let N be a stable, power multiplicative pseudonorm for a ring A, and let c be an element of A which is not N-null. Then N_c is a nonzero, stable, power multiplicative pseudonorm subordinate to N such that*

(i) $N_c(c) = N(c)$;

(ii) *if N is a Kadison pseudonorm, then N_c is a Kadison pseudonorm*;

(iii) *if N is a Segal pseudonorm, then N_c is a Segal pseudonorm*;

(iv) *whenever A is an algebra, over a field K with absolute value, such that N is K-admissible, then N_c is K-admissible*;

(v) $N_c(cx) = N_c(c) \cdot N_c(x)$ *for all x in A.*

The proof is routine and is left to the reader.

THEOREM 4. *Let N be a stable, power multiplicative pseudonorm for a ring A, and let c be an element of A which is not N-null. Then there exists a nonzero pseudo absolute value N' subordinate to N such that*

(i) $N'(c) = N(c)$;

(ii) *if N is a Kadison pseudonorm, then N' is a Kadison pseudonorm*;

(iii) *if N is a Segal pseudonorm, then N' is a Segal pseudonorm*;

(iv) *whenever A is an algebra, over a field K with absolute value, such that N is K-admissible, then N' is K-admissible.*

Proof. Let $\mathfrak{N}$ be the set of all stable, power multiplicative pseudonorms N' subordinate to N and such that

(i) $N'(c) = N(c)$;

(ii) if N is a Kadison pseudonorm, then N' is a Kadison pseudonorm;

(iii) if N is a Segal pseudonorm, then N' is a Segal pseudonorm;

(iv) whenever A is an algebra, over a field K with absolute value, such that N is K-admissible, then N' is K-admissible;

(v) $N'(cx) = N'(c)\cdot N'(x)$ for all x in A.

Then Lemma 1 shows that N_c belongs to $\mathfrak{N}$ so that $\mathfrak{N}$ is not empty. It is easily verified that $\mathfrak{N}$ is a *hereditary system* and that $\mathfrak{N}$ contains minimal elements. If N' is such a minimal element, then Lemma 2 of [2] shows that N' is a pseudo absolute value. The properties of N' stated in the theorem follow from the fact that N' belongs to $\mathfrak{N}$.

4. Rings of continuous functions

In this section it will be shown that certain types of Kadison algebras and Segal algebras can be represented as rings of continuous real-valued functions on suitable topological spaces. Previous results in representing rings or algebras as rings of continuous functions have generally dealt either with nonzero commutative rings, or with rings with unit element in case commutativity was not assumed specifically; in some cases, such as in Gelfand's paper [5], both commutativity and the existence of a unit have been used. In this note, the rings are subjected to a condition which always holds in nonzero commutative rings or in rings with unit element.

DEFINITION. A nonzero ring R is called a *centrist* ring if the two-sided ideal generated by the center of R coincides with R.

Every ring with unit element is clearly a centrist ring, and every nonzero commutative ring is also a centrist ring. We shall therefore prove the next results for centrist algebras; the corresponding results for algebras with unit element and for nonzero commutative algebras will then follow as special cases.

Our terminology follows [4] in the definition of compact spaces and locally compact spaces. Thus, a *compact space* is defined to be a Hausdorff space for which every open covering has a finite subcovering, and a *locally compact space* is defined to be a Hausdorff space in which each point is an interior point of at least one compact subset.

If ρ is a real number such that $0 < \rho \leqq 1$, and if Φ is a compact space, the symbol $C(\Phi; \mathfrak{R}; \rho)$ will denote the set of all continuous real-valued functions $x(\varphi)$ on Φ, with algebraic operations defined in the obvious way, and with the norm N defined by $N(x) = \sup\{| x(\varphi) |^{\rho}; \varphi \in \Phi\}$ for all x in $C(\Phi; \mathfrak{R}; \rho)$. In case ρ is a real number, such that $0 < \rho \leqq 1$, and Φ is a noncompact locally compact space, the symbol $C(\Phi; \mathfrak{R}; \rho)$ will denote the set of all continuous real-valued functions on Φ which "vanish at infinity," with the same definitions for the norm and the algebraic operations as in the case of a compact space Φ. Thus, whenever ρ is a real number, with $0 < \rho \leqq 1$, and Φ is a locally compact space, whether Φ is compact or not, the system $C(\Phi; \mathfrak{R}; \rho)$ is defined and is clearly a nonzero, commutative, connected, complete normed algebra over $\mathfrak{R}^{(\rho)}$. In fact, $C(\Phi; \mathfrak{R}; \rho)$ is a Kadison algebra and a Segal algebra; each $C(\Phi; \mathfrak{R}; \rho)$ is of course a centrist algebra. We shall show that, conversely, a normed algebra which possesses sufficiently many of these properties of a $C(\Phi; \mathfrak{R}; \rho)$ may be identified with a $C(\Phi; \mathfrak{R}; \rho)$. The first result of this type will show that a centrist, complete Kadison algebra or Segal algebra over an $\mathfrak{R}^{(\rho)}$ may be identified with $C(\Phi; \mathfrak{R}; \rho)$, for a suitable locally compact space Φ.

LEMMA 2. *Let A be a nonzero, complete normed algebra over $\mathfrak{R}^{(\rho)}$, where ρ is a real number such that $0 < \rho \leqq 1$. Suppose that for each nonzero c in A there is a homomorphism φ of A into $\mathfrak{R}^{(\rho)}$, as algebras over $\mathfrak{R}^{(\rho)}$, such that $\| \varphi(c) \| = N(c)$ and $\| \varphi(x) \| \leqq N(x)$ for all x in A, where N is the norm for A. Then there exist a locally compact space Φ and a norm-preserving isomorphism σ of A onto $C(\Phi; \mathfrak{R}; \rho)$, as algebras over $\mathfrak{R}^{(\rho)}$, such that A has a unit element if and only if Φ is compact.*

A detailed proof of this key result is given in [3]. The space Φ is taken as the set of all nonzero homomorphisms φ of A into $\mathfrak{R}^{(\rho)}$, as algebras over $\mathfrak{R}^{(\rho)}$, such that $\| \varphi(x) \| \leqq N(x)$ for all x in A. Alternatively, Φ may be described as the set of all nonzero homomorphisms φ of A into $\mathfrak{R}^{(\rho)}$ such that $\| \varphi(x) \| \leqq N(x)$ for all x in A; this is true because such homomorphisms are continuous and must therefore be homomorphisms of A into $\mathfrak{R}^{(\rho)}$, as algebras over $\mathfrak{R}^{(\rho)}$. The mapping σ is defined by the condition $[\sigma(x)](\varphi) = \varphi(x)$ for all φ in Φ, and Φ is given the coarsest topology which will make all of the functions $\sigma(x)$ continuous on Φ. (A "coarser" topology is one which is "moins fine" in the sense of [4].)

LEMMA 3. *Let A be a normed algebra over $\mathfrak{R}^{(\rho)}$, where ρ is a real number such that $0 < \rho \leqq 1$, and let A have a nonzero center. Suppose that the norm for A is a Kadison absolute value or Segal absolute value. Then there exists a norm-preserving isomorphism τ of A into $\mathfrak{R}^{(\rho)}$, as algebras over $\mathfrak{R}^{(\rho)}$.*

Proof. The corollary of Theorem 3 shows that there is an isometry τ of A into $\mathfrak{R}^{(\lambda)}$, where λ is a suitable, uniquely determined, real number such that $0 < \lambda \leqq 1$. If c is a nonzero central element of A, then the last paragraph of Section 2 indicates that $N(2c) = 2^{\lambda} \cdot N(c)$, where N is the norm for A. On the other hand, the fact that N is $\mathfrak{R}^{(\rho)}$-admissible shows that $N(2c) = 2^{\rho} \cdot N(c)$, so that $\lambda = \rho$. Thus, τ is an isometry of A into $\mathfrak{R}^{(\rho)}$. That τ is actually an algebra-isomorphism follows easily from the continuity of τ.

THEOREM 5. *Let A be a centrist, complete Kadison algebra (Segal algebra) over $\mathfrak{R}^{(\rho)}$, where ρ is a real number such that $0 < \rho \leqq 1$. Then there exist a locally compact space Φ and a norm-preserving isomorphism σ of A onto $C(\Phi; \mathfrak{R}; \rho)$, as algebras over $\mathfrak{R}^{(\rho)}$, such that A has a unit element if and only if Φ is compact.*

Proof. Let N be the norm for A, and let c be any nonzero element in A. If Theorem 4 is applied to N and to the element c, we can find a pseudo absolute value N' subordinate to N and such that $N'(c) = N(c) \neq 0$, N' is a Kadison pseudonorm (Segal pseudonorm), and N' is $\mathfrak{R}^{(\rho)}$-admissible.

The ideal $I(N')$ is then a subalgebra of A, so that $\bar{A} = A/I(N')$ is an algebra over $\mathfrak{R}^{(\rho)}$. Let $\bar{N}$ be the function on $\bar{A}$ defined by the condition $\bar{N}(\eta(x)) = N'(x)$ for all x in A, where η is the natural mapping of A onto $\bar{A}$. Then $\bar{N}$ is an $\mathfrak{R}^{(\rho)}$-admissible norm for $\bar{A}$, so that $\bar{A}$ becomes a normed algebra when it is given this norm. Since N' is a Kadison pseudo absolute value (Segal pseudo absolute value), $\bar{N}$ must be a Kadison absolute value (Segal absolute value) for $\bar{A}$.

The two-sided ideal $I(N')$ does not contain c and is therefore distinct from A. Then the center of the centrist algebra A can not be contained in $I(N')$, so that there is a central element b in A such that b is not in $I(N')$. It is clear that η is a homomorphism of A onto $\bar{A}$, as algebras over $\mathfrak{R}^{(\rho)}$, and that the kernel of η is $I(N')$. Thus, $\eta(b)$ is a central element of $\bar{A}$, and $\eta(b)$ is not zero, so that the center of $\bar{A}$ is not zero. Lemma 3 may now be applied to $\bar{A}$, and we obtain a norm-preserving isomorphism τ of $\bar{A}$ into $\mathfrak{R}^{(\rho)}$, as algebras over $\mathfrak{R}^{(\rho)}$.

Let φ be the mapping obtained by applying η and then τ. Clearly, φ is a homomorphism of A into $\mathfrak{R}^{(\rho)}$, as algebras over $\mathfrak{R}^{(\rho)}$. We have

$$\| \varphi(c) \| = \| \tau(\eta(c)) \| = \bar{N}(\eta(c)) = N'(c) = N(c),$$

and also

$$\| \varphi(x) \| = \| \tau(\eta(x)) \| = \bar{N}(\eta(x)) = N'(x) \leqq N(x)$$

for all x in A. Application of Lemma 2 completes the proof.

COROLLARY. *Let A be a nonzero, commutative, complete normed algebra over $\mathfrak{R}^{(\rho)}$, where ρ is a real number such that $0 < \rho \leqq 1$. Suppose that the norm N for A is such that*

$$N(x^2 + y^2) \geqq N(x)^2$$

$$(N(x^2) = N(x)^2 \quad \text{and} \quad N(x^2 - y^2) \leqq \max(N(x^2), N(y^2)))$$

for all x and y in A. Then there exist a locally compact space Φ and a norm-preserving isomorphism σ of A onto $C(\Phi; \mathfrak{R}; \rho)$, as algebras over $\mathfrak{R}^{(\rho)}$, such that A has a unit element if and only if Φ is compact.

This corollary, with $\rho = 1$, and using the alternative hypothesis (in parentheses), coincides with Segal's Theorem 1 in [9] if it is assumed here that

A has a unit element. With the additional assumption of commutativity in Kadison's Theorem 6.5 of [6], the latter result becomes a special case of this corollary.

In the last theorem and its corollary, it was assumed that A was a normed algebra over some $\mathfrak{R}^{(\rho)}$, and A was therefore necessarily connected. It is possible to obtain comparable results in case the assumption that A is a normed algebra over some $\mathfrak{R}^{(\rho)}$ is replaced by the assumption that A is a connected normed algebra over some field K with absolute value. This is accomplished by using the following lemmas in order to reduce the situation to the one which has just been considered.

LEMMA 4. *Let A be a nonzero Kadison algebra (Segal algebra) over a field K with absolute value. Then the absolute value for K is a Kadison absolute value (Segal absolute value).*

The proof is trivial and is left to the reader.

LEMMA 5. *Let A be a centrist, connected normed algebra over a field K with absolute value. If the norm for A is stable and power multiplicative, then K is archimedean.*

Proof. Let N be the norm for A, and let c be a nonzero element in A. Theorem 4 shows that there exists a K-admissible pseudo absolute value N' subordinate to N, such that $N'(c) = N(c)$. Form the algebra $\bar{A} = A/I(N')$ over K, as in the proof of Theorem 5, and let η be the natural mapping of A onto $\bar{A}$. The norm $\bar{N}$ for $\bar{A}$ is defined by the condition $\bar{N}(\eta(x)) = N'(x)$ for all x in A; with the norm $\bar{N}$, the algebra $\bar{A}$ becomes a normed algebra over K. Also, $\bar{N}$ is an absolute value for $\bar{A}$. Since A is a centrist algebra, it follows in the same way as in the proof of Theorem 5 that $\bar{A}$ has a nonzero center.

Let D be the set of nonzero central elements of $\bar{A}$. If $\bar{A}_D$ is formed, by using Lemma 17 of [1], then $\bar{A}_D$ is a metric ring with unit element $\bar{e}$, and the norm for $\bar{A}_D$ is an absolute value. (Note that the proof of that lemma does not require the existence of a unit element in the original ring.) It can also be shown that $\bar{A}_D$ is a normed algebra over K.

The mapping η is a homomorphism of A onto $\bar{A}$, as algebras over K, and it is easy to establish that η is also continuous. Then $\bar{A}$ is connected since it is the continuous image of the connected set A; connectedness of $\bar{A}_D$ follows from the connectedness of $\bar{A}$ by the use of a device already employed in the proof of the corollary of Theorem 1. If the corollary of Theorem 1 is applied to $\bar{A}_D$, an isometry is obtained of $\bar{A}_D$ into $\mathfrak{Q}^{(\rho)}$, for a suitable real number ρ such that $0 < \rho \leqq 1$. But the mapping $k \rightarrow k\bar{e}$ is clearly an isometry of K into $\bar{A}_D$, and it follows that there is an isometry of K into $\mathfrak{Q}^{(\rho)}$.

In $\mathfrak{Q}^{(\rho)}$, we have $\| 2 \| = 2^\rho > 1$. Thus, in K we also have $\| 2 \| > 1$, and Lemma 14 of [1] shows that K is archimedean.

LEMMA 6. *Let A be a centrist, connected Kadison algebra (Segal algebra), over a complete field F with absolute value. Then there exists a real number ρ, with $0 < \rho \leqq 1$, such that there exists an isometry σ of F onto $\mathfrak{R}^{(\rho)}$. The number ρ and the isometry σ are uniquely determined.*

Proof. Lemma 5 shows that F is archimedean, so that Theorem 11 of [1] implies that there exists a real number ρ, with $0 < \rho \leqq 1$, such that F is isometric to $\mathfrak{R}^{(\rho)}$, to $\mathfrak{C}^{(\rho)}$, or to $\mathfrak{Q}^{(\rho)}$. But Lemma 4 shows that the absolute value for F is a Kadison absolute value (Segal absolute value), and it is impossible for $\mathfrak{C}^{(\rho)}$ or $\mathfrak{Q}^{(\rho)}$ to have a Kadison absolute value or Segal absolute value; therefore, F is isometric to $\mathfrak{R}^{(\rho)}$. It is obvious that ρ must be uniquely determined; also, it is easily seen that the isometry of F onto $\mathfrak{R}^{(\rho)}$ is uniquely determined on the prime field of F, and therefore on F, which is the closure of its prime field.

THEOREM 6. *Let A be a centrist, connected, complete Kadison algebra (Segal algebra) over a field K with absolute value. Then there exist a locally compact space Φ, a real number ρ with $0 < \rho \leqq 1$, and an isometry σ of A onto $C(\Phi; \mathfrak{R}; \rho)$, such that A has a unit element if and only if Φ is compact.*

Proof. The completion of A can be considered as a normed algebra over the completion of K. (See [4; Chap. IX, §3, No 7].) Thus, if F is the completion of K, then A is a normed algebra over F. Lemma 6 shows that there is a unique real number ρ, with $0 < \rho \leqq 1$, such that F is isometric to $\mathfrak{R}^{(\rho)}$. Since the isometry of F onto $\mathfrak{R}^{(\rho)}$ is also unique, we shall identify F with $\mathfrak{R}^{(\rho)}$. Theorem 5 may then be applied, but afterward σ is regarded only as an isometry in the statement of Theorem 6 since A was not originally given as an algebra over $\mathfrak{R}^{(\rho)}$.

COROLLARY. *Let A be a nonzero, commutative, connected, complete normed algebra over a field K with absolute value. Suppose that the norm N of A is such that*

$$N(x^2 + y^2) \geqq N(x)^2$$

$$(N(x^2) = N(x)^2 \quad \textit{and} \quad N(x^2 - y^2) \leqq \max(N(x^2), N(y^2)))$$

for all x and y in A. Then there exist a locally compact space Φ, a real number ρ with $0 < \rho \leqq 1$, and an isometry σ of A onto $C(\Phi; \mathfrak{R}; \rho)$, such that A has a unit element if and only if Φ is compact.

Note 1. The assumption that the algebra is connected may be replaced, in Theorem 6 and its corollary, by the assumption that the field K of scalars is archimedean; for the completion of K would then be connected, and the algebra would also be connected since it could be considered a normed algebra over the completion of K.

Note 2. In the corollary of Theorem 3, the conclusion is actually that σ is an isometry of R *onto* $\mathfrak{R}^{(\rho)}$, for $\sigma(R)$ would be the continuous image of the connected set R and therefore connected, and $\sigma(R)$ would be the homomorphic image of the additive group of R and therefore a subgroup of the additive group of real numbers; thus, $\sigma(R)$ would be a nonzero, connected subgroup of the additive group of real numbers and would consequently coincide with $\mathfrak{R}^{(\rho)}$. Similarly, the conclusion of Lemma 3 could be strengthened so that the norm-preserving isomorphism τ is a mapping *onto* $\mathfrak{R}^{(\rho)}$, if the connectedness of the algebra A is noted and employed.

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