SOME GENERALIZATIONS OF FINITE PROJECTIVE DIMENSION

BY

J. P. JANS[1]

1. Introduction

In this paper we shall deal with various generalizations of the concept of finite projective dimension. Modules of finite projective dimension and rings of finite global dimension have been extensively studied by a number of authors; see, for instance, [1, 3, 4, 5, 6]. These studies have added significantly to our knowledge of rings with minimum condition. The chief drawback, however, to the concept of finite projective dimension is that not enough modules have it. It is for this reason that we initiated this investigation into various generalizations.

Throughout the paper, module will mean finitely generated module, and ring will mean ring with minimum condition on the kind of ideals that match the modules. Since we do not change rings, we will write the functors Hom and Ext without mention of the rings. Notation and terminology will follow that of [3].

In Section 2 we develop a theory on "how many" modules are necessary for a projective resolution. The results in that section rest heavily on the work of Eilenberg concerning minimal resolutions. A handy tool in the study of minimal resolutions is the use of $\mathrm{Ext}(A, C)$ as a module over the endomorphism ring of C where C is irreducible. In this case, $\mathrm{Ext}(A, C)$ becomes a vector space, and we can count modules in the projective resolution for A by computing the dimensions of certain of these vector spaces.

In Section 3 we prove two versions of Nunke's theorem [10] for the type of modules under consideration. There, the class of rings for which we prove the theorem is more general than the class of rings with minimum condition with finite global dimension. However, we only consider finitely generated modules. Actually the theorem can be proved from the work of Auslander [1] for semiprimary rings of finite global dimension without the assumption of finite generation of the modules involved. Our method of proof works only for finitely generated modules.

2. Projective types

In this section, we shall be concerned with "how many" projective modules appear in the minimal projective resolution of a module. Because of the restrictions that we have imposed on the rings and modules that we are considering, we have the following facts at our disposal:

Received July 18, 1960.

[1] The author gratefully acknowledges the support of the National Science Foundation.

1. Every module M has a minimal projective resolution

$$\cdots M_r \xrightarrow{\delta_r} \cdots M_0 \xrightarrow{\varepsilon} M \to 0$$

which is characterized either by the property that $\mathrm{Ext}^r(M, F) \cong \mathrm{Hom}(M_r, F)$ for all irreducible F, or by the fact that $\delta_r : M_r \to M_{r-1}$ induces an isomorphism of M_r/NM_r with $\mathrm{Im}\ \delta_r/N\ \mathrm{Im}\ \delta_r$ where N is the radical of R.

2. The projective modules are direct sums of indecomposable projective modules each of which is isomorphic to an idempotent-generated left ideal of R.

3. R has a finite number of distinct irreducible modules $F_1, \cdots, F_n$ and the same number of distinct indecomposable projective modules $P_1, \cdots, P_n$ such that $P_i/NP_i \cong F_i$. Moreover, each indecomposable projective module is isomorphic to one of the P_i.

The above properties of finitely generated modules over rings with minimum condition are proved in various places, among which are [1, 3, 4].

Let $\alpha = (a_n)$ be a sequence of nonnegative integers. We shall say that a module M has *projective type* α if for each $r \geqq 0$ the r^{th} projective module M_r in the minimal projective resolution for M decomposes into the direct sum of a_r indecomposable projective modules. That the projective type is well defined follows from the uniqueness of the minimal projective resolution and the uniqueness of the decomposition of a module into the direct sum of indecomposables.

Note that if M is of projective type α for α which is ultimately a zero sequence, then M has finite projective dimension (and conversely).

In order to compute the projective type of a module we recall, at this point, some of the properties of the functor Ext. If A is a module and X a projective resolution of A, then $\mathrm{Ext}(A, C)$ can be computed as the homology groups of $\mathrm{Hom}(X, C)$. If $\mathfrak{E}(C)$ is the (module) endomorphism ring of C, then C, $\mathrm{Hom}(X, C)$, and $\mathrm{Ext}(A, C)$ become $\mathfrak{E}(C)$-modules in a natural way. That $\mathrm{Ext}(A, C)$ is a $\mathfrak{E}(C)$-module comes from the fact that the differentiation in $\mathrm{Hom}(X, C)$ is a $\mathfrak{E}(C)$-homomorphism, so images, kernels, and their quotients are again $\mathfrak{E}(C)$-modules. In the case under consideration, C will be one of the irreducible modules F_j, and $\mathfrak{E}(F_j)$ is therefore a division ring. Thus we may compute the dimensions for each r of $\mathrm{Ext}^r(A, F_j)$ over $\mathfrak{E}(F_j)$.

Perhaps we might be allowed to digress for a moment to the "group of extensions". In [3] it is shown that there is a one-to-one correspondence between the elements of $\mathrm{Ext}^1(A, C)$ and equivalence classes of exact sequences of the form $0 \to C \to X \to A \to 0$, and in this way $\mathrm{Ext}^1(A, C)$ can be thought of as the group of extensions of C by A (or A by C). This might lead one to believe that there is a one-to-one correspondence between the elements of $\mathrm{Ext}^1(A, C)$ and the "set" of modules X which can be fitted into the above exact sequence. This, however, is not the case. There are apt to be far fewer solutions, X, to the above sequence than elements in $\mathrm{Ext}^1(A, C)$. The idea of considering $\mathrm{Ext}^1(A, C)$ as a module over $\mathfrak{C}(C)$ allows one to cut down the number of extensions. For instance, it is not hard to show that if u is a unit of $\mathfrak{C}(C)$, and if e is in $\mathrm{Ext}^1(A, C)$, then the middle modules in the exact sequences corresponding to e and ue are actually isomorphic. Thus, as a corollary, we see that if C is irreducible and $\mathrm{Ext}^1(A, C)$ has dimension one over $\mathfrak{C}(C)$, then there are *really* at most two extensions of C by A (counting isomorphic ones as the same). It should be noted that the whole thing works just as well over $\mathfrak{C}(A)$.

The following theorem gives a computation of the projective type of a module in terms of the dimensions of $\mathrm{Ext}^r(A, F_j)$ over $\mathfrak{C}(F_j)$.

THEOREM 2.1. *The projective type of M is $\alpha = (a_r)$ where $a_r = \sum_{j=1}^{n} [\mathrm{Ext}^r(M, F_j) : \mathfrak{C}(F_j)]$.*

Proof. As mentioned above, we know that

$$\mathrm{Ext}^r(M, F_j) \cong \mathrm{Hom}(M_r, F_j) \cong \mathrm{Hom}(M_r/NM_r, F_j)$$

($\cong$ as $\mathfrak{C}(F_j)$-modules), where M_r is the r^{th} projective in the minimal resolution for M. Then M_r/NM_r is the direct sum of irreducible modules, and the number of these which are isomorphic to F_j is also the number of copies of P_j which occur in the direct sum decomposition of M_r. Thus, computing the dimension of $\mathrm{Hom}(M_r/NM_r, F_j)$ over $\mathfrak{C}(F_j)$ we see that this is the number of copies of F_j in the direct sum decomposition of M_r/NM_r. Putting all these equalities together we see that $[\mathrm{Ext}^r(M, F_j) : \mathfrak{C}(F_j)]$ is the number of copies of P_j in the direct sum decomposition of M_r. The total number of indecomposable modules in the direct sum decomposition of M_r is obtained by summing over j.

We include at this point a corollary to the theorem.

COROLLARY 2.2. *The number of copies of P_j occurring in the direct sum decomposition of M_r is $[\mathrm{Ext}^r(M, F_j) : \mathfrak{C}(F_j)]$.*

We remark that someone might find it interesting to study how often the different P_j occur in a projective resolution. If so, the above corollary may be of some value. In this paper we shall restrict ourselves to merely counting the total number of projectives occurring at each point in the projective resolution.

For two sequences α and β, $\alpha + \beta$ is obtained by adding term by term. $\alpha \leqq \beta$ will mean that every term of α is less than or equal to the corresponding term of β. For an integer n and a sequence α, $n\alpha$ means α added to itself n times. It is notationally convenient to think of our sequences as doubly infinite with zeroes corresponding to the negative integers. Then we can define a shift operation without supplying new terms at the beginning. If $\alpha = (a_r)$, we define $\alpha' = (a'_r)$ by the equation $a'_r = a_{r+1}$. Using this notation and the exact sequence of homology for Ext we can prove the following theorem.

THEOREM 2.3. *If* $0 \to A \to B \to C \to 0$ *is an exact sequence of modules and* A, B, *and* C *are of projective types* α, β, *and* γ *respectively, then the following inequalities hold*:

$$\beta \leqq \alpha + \gamma. \tag{1}$$

$$\alpha \leqq \gamma' + \beta. \tag{2}$$

$$\gamma' \leqq \beta' + \alpha. \tag{3}$$

Moreover, if the sequence splits, then $\beta = \alpha + \gamma$.

Proof. From the exact sequence of homology for Ext we have the following exact sequence

$$\begin{aligned}\operatorname{Ext}^r(C, F_j) \to \operatorname{Ext}^r(B, F_j) \to \operatorname{Ext}^r(A, F_j)& \\ \to \operatorname{Ext}^{r+1}(C, F_j) \to \operatorname{Ext}^{r+1}(B, F_j)&.\end{aligned}$$

We note at this point that all the connecting homomorphisms are $\mathfrak{C}(F_j)$-homomorphisms. If we let $\operatorname{Ext}^r = 0$ for negative r, the above sequence can be thought of as a piece of a doubly infinite exact sequence. Computing dimensions over $\mathfrak{C}(F_j)$, we obtain the following set of inequalities holding for all r:

$$\begin{aligned}[\operatorname{Ext}^r(B, F_j) : \mathfrak{C}(F_j)] \leqq {}& [\operatorname{Ext}^r(A, F_j) : \mathfrak{C}(F_j)] \\ &+ [\operatorname{Ext}^r(C, F_j) : \mathfrak{C}(F_j)].\end{aligned} \tag{1'}$$

$$\begin{aligned}[\operatorname{Ext}^r(A, F_j) : \mathfrak{C}(F_j)] \leqq {}& [\operatorname{Ext}^{r+1}(C, F_j) : \mathfrak{C}(F_j)] \\ &+ [\operatorname{Ext}^r(B, F_j) : \mathfrak{C}(F_j)].\end{aligned} \tag{2'}$$

$$\begin{aligned}[\operatorname{Ext}^{r+1}(C, F_j) : \mathfrak{C}(F_j)] \leqq {}& [\operatorname{Ext}^{r+1}(B, F_j) : \mathfrak{C}(F_j)] \\ &+ [\operatorname{Ext}^r(A, F_j) : \mathfrak{C}(F_j)].\end{aligned} \tag{3'}$$

Since the inequalities still hold when summed over j, the first part of the theorem then follows from Theorem 2.1.

If the sequence $0 \to A \to B \to C \to 0$ splits, then

$$\operatorname{Ext}^r(B, F_j) \cong \operatorname{Ext}^r(A, F_j) \oplus \operatorname{Ext}^r(C, F_j)$$

($\mathfrak{C}(F_j)$-isomorphism and direct sum). Then computing dimensions and summing over j gives $\beta = \alpha + \gamma$.

The above theorem shows that in the exact sequence $0 \to A \to B \to C \to 0$ the projective types of any two of the modules limit the possibilities for the third. For example, if any two are of bounded projective type (i.e., the type of a bounded sequence), then so is the third. Of course, the analogous property holds for modules of finite projective dimension.

We note here a corollary which will be of some use later.

COROLLARY 2.4. *If $0 \to A \to B \to C \to 0$ is exact, the number of copies of P_j occurring in the direct sum decomposition of the r^{th} term in the minimal projective resolution for B is less than or equal to the sum of the corresponding numbers for A and C.*

Proof. The proof is shorter than the corollary and consists of equation (1′).

Now let $\alpha_1, \cdots, \alpha_n$ be the projective types of all the irreducible modules for the ring R, and define $\sup_{1 \leqq i \leqq n} \alpha_i = \rho$. In analogy to projective dimension we shall say that R has *global projective type* ρ. The following theorem shows the connection between the global projective type of the ring and the projective type of each of its modules.

THEOREM 2.5. *If R has global projective type ρ, and if M is an R-module of composition length t, then the projective type of M is less than or equal to $t\rho$. Moreover, ρ is the smallest sequence with this property.*

Proof. The proof is by induction on t. The definition of ρ insures that it is true for $t = 1$. Assume that it has been established for modules of composition length less than t, and let M have composition length t. We split off an irreducible factor to obtain the exact sequence $0 \to A \to M \to F \to 0$ where F has composition length 1 and A has composition length $t - 1$. The desired inequality now follows by applying the inequality (1) of Theorem 2.3 and the induction hypothesis. From the definition of ρ it is clear that no sequence smaller than ρ could satisfy the conditions of the theorem even for $t = 1$.

We would like, now, to give some estimates on the global projective type of a ring in terms of the internal structure of the ring. We can get one such estimate from the composition factors for the indecomposable projective modules. The indecomposable projective P_j has a unique maximal submodule NP_j, and, as mentioned before, P_j/NP_j is isomorphic to F_j. Looking farther down in P_j, we consider the factor NP_j/N^2P_j, a direct sum of irreducible modules. We define the $n \times n$ matrix B with integer entries by the condition that b_{ij} is the number of copies of F_i that appear in the direct sum decomposition of NP_j/N^2P_j. Finally we define the matrix D where d_{ij} is the number of composition factors of NP_j isomorphic to F_i. It should be noted that these matrices are related to the Cartan matrix of the ring [6]. The following theorem gives a rather crude bound on the global projective type in terms of these matrices. It is not hard to show that the estimate is good for rings with radical square zero, but it can be quite misleading for many other rings as we shall see later. Before we consider the theorem we prove a lemma which will be of some help.

LEMMA 2.6. *If M has t_j composition factors isomorphic to F_j, then the minimal projective mapping onto M is a direct summand of $\sum_{j=1}^{n} t_j P_j$. ($t_j P_j$ means the direct sum of t_j copies of P_j.)*

Proof. Induce on the composition length of M. Certainly it is true for one composition factor. If M has t composition factors, we can split off an irreducible to obtain the exact sequence $0 \to S \to M \to F \to 0$, and by the induction hypothesis the lemma holds for both ends of the sequence. The inequality (1′) with $r = 0$ then shows that the lemma holds for the module M.

THEOREM 2.7. *If M_r is the r^{th} module in the minimal projective resolution for F_j, then the number of indecomposable projectives in the direct sum decomposition of M_r isomorphic to P_i is less than or equal to the ij entry of the matrix $D^{r-1}B$ for $r \geqq 1$.*

Proof. As noted above we may take for M_0 the module P_j and map it onto F_j. The kernel of this map is NP_j. For M_1 we may use the direct sum $\sum_{i=1}^{n} b_{ij} P_i$ since this maps onto NP_j/N^2P_j and, by projectivity, onto NP_j. Thus the theorem is true for $r = 1$. Actually, we can claim equality at this stage. However, at this point we cannot identify the kernel of the map of M_1 into M_0 in terms of that part of the structure of the ring that we have singled out.

Assume now that the inequality has been established for integers less than r, and that the number t_i of copies of P_i in the direct sum decomposition of M_{r-1} is less than or equal to the ij entry in $D^{r-2}B$. Thus $M_{r-1} = \sum_{k=1}^{n} t_k P_k$, and by minimality we may assume that

$$\operatorname{Ker} \delta_{r-1} \subseteq \sum_{k=1}^{n} t_k NP_k .$$

We do not know the composition factors for $\operatorname{Ker} \delta_{r-1}$, but we do know that they appear among those of $\sum_{k=1}^{n} t_k NP_k$, so that the number of composition factors of $\operatorname{Ker} \delta_{r-1}$ isomorphic to F_i is less than or equal to $\sum_{k=1}^{n} d_{ik} t_k$. Thus by the lemma, we may conclude that the number of copies of P_i occurring in the direct sum decomposition of M_r is less than or equal to $\sum_{k=1}^{n} d_{ik} t_k$ which in turn is less than or equal to the ij entry of the matrix $D(D^{r-2}B) = D^{r-1}B$. This completes the proof of the theorem.

We can use the theorem to get an estimate of the global projective type of the ring.

COROLLARY 2.8. *The global projective type of the ring R is less than the sequence (t_r) where t_r is the supremum over j of the sum of the elements in the j^{th} column of $D^{r-1}B$.*

We remark that if the radical of R is square zero, then the matrices D and B are equal, and that the inequalities of the theorem and its corollary may be replaced by equalities. This follows from the fact that for such rings at every stage of the minimal projective resolution for F_i, $\operatorname{Ker} \delta_r$ is actually equal to NM_r rather than being merely contained in it.

We conclude this section with some examples. Let q be a fixed positive integer and consider the sequence (q^r) of positive powers of q. This is the global projective type of the ring of $q + 1$ by $q + 1$ matrices with coefficients in a field having the same entry down the diagonal, arbitrary entries in the bottom row, and zeroes everywhere else. One sees this by observing that the ring has only one irreducible and that at every stage of the minimal projective resolution of that irreducible it is necessary to have q times as many projectives in the direct sum decomposition as in the stage before.

The following example shows that our bound on the global projective type developed in Theorem 2.7 and Corollary 2.8 is sometimes a bit big. Let R be the ring of matrices of the form

$$\begin{pmatrix} x & 0 & 0 \\ y & z & 0 \\ w & u & x \end{pmatrix}$$

with coefficients in a field. It is not hard to show that this ring has global dimension 2. However, the matrices B and D of Theorem 2.7 and Corollary 2.8 are

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

A little matrix computation will show that the conclusion of Corollary 2.8 does not even indicate that this ring is of bounded global projective type.

3. Nunke's theorem

In this section, we shall prove two variations of Nunke's theorem. Nunke showed [10] that for the integers Z if $\operatorname{Hom}(A, Z) = 0 = \operatorname{Ext}^1(A, Z)$, then $A = 0$.

We were led to this study by the attempt to develop duality theories for modules over rings with minimum condition. Usually, a good place to start looking for a duality is the functor $\operatorname{Hom}(A, R)$ which converts left modules A to their duals (since R is two-sided). This idea has been studied for quasi-Frobenius rings in [9] and more recently generalized by Azumaya [2]. In the latter case, however, the dual of a module is a module over a different ring.

There are two objections to the use of $\operatorname{Hom}(A, R)$ for a duality. In the first place it is only half exact, and in the second place it can happen that $\operatorname{Hom}(A, R)$ is zero but that A is not. The assumption that R is quasi-Frobenius eliminates both of these objections by making R injective and putting enough different irreducible submodules into R.

In order to circumvent the above difficulties, we might try to use the whole of $\operatorname{Ext}(A, R)$ for the dual of A, and it is this which leads us to a version of Nunke's theorem. In this version we assume that $\operatorname{Ext}^n(A, R) = 0$ for all $n \geqq 0$ and try to show $A = 0$.

We shall make use of the following concept which is defined in terms of a projective resolution of a module. Let M be an R-module; we shall say that M has an *ultimately closed projective resolution*

$$\cdots M_n \xrightarrow{\delta_n} \cdots M_0 \xrightarrow{\varepsilon} M \to 0$$

if there exists n such that $\operatorname{Im} \delta_n$ can be decomposed into a direct sum of modules T_i such that each T_i is a direct summand of $\operatorname{Im} \delta_q$ for $q < n$.

If we adopt the convention that the zero module is a direct summand of every module, it is clear that modules of finite projective dimension have ultimately closed projective resolutions. There are a number of other examples; for instance, if R is a ring with minimum condition with the property that R has only a finite number of finitely generated indecomposable modules, then every finitely generated module has an ultimately closed projective resolution. In fact, for such a module any projective resolution (consisting of finitely generated projective modules) is ultimately closed. This can be shown by applying the Krull-Schmidt theorem to the modules $\operatorname{Im} \delta_n$ until new indecomposables fail to appear. See [7, 8] for conditions on finite-dimensional algebras which imply that they have only a finite number of finitely generated indecomposable modules.

If R is a ring with minimum condition and with radical square zero, then every finitely generated R-module has an ultimately closed resolution. This can be seen by taking the minimal projective resolution for the module and observing that $\operatorname{Im} \delta_n$ for this resolution is a direct sum of irreducible modules of which R has only a finite number.

It should be pointed out that the condition for a module to have an ultimately closed projective resolution is a rather strong condition on the kernels involved in the projective resolution. It is not a condition on the projective modules themselves. It is conceivable that there is a module with a projective resolution which consists of the same projective repeated over and over but that the resolution is not ultimately closed. We doubt whether this can happen for we do not feel that there is enough room inside a projective for that many submodules and factor modules.

We shall say that a ring R with minimum condition is *ultimately closed* if every finitely generated R-module has an ultimately closed projective resolution consisting of finitely generated projectives.

THEOREM 3.1. *If A is a finitely generated module over an ultimately closed ring R, and if* $\operatorname{Ext}^n(A, R) = 0$ *for all* $n \geqq 0$, *then* $A = (0)$.

Proof. Assume A is such an R-module, and form the ultimately closed projective resolution of A. Now take the homomorphism groups of it into R:

$$0 \to \operatorname{Hom}(M_0, R) \xrightarrow{\delta_1^*} \operatorname{Hom}(M_1, R) \xrightarrow{\delta_2^*} \cdots .$$

Since the groups $\operatorname{Ext}^n(A, R)$ are assumed to be zero, the above sequence is exact. Moreover, it is a sequence of R-modules (of the opposite hand from A and M_i) because R is a two-sided module.

Now we can make use of the fact that the functor $\operatorname{Hom}(\,\cdot\,, R)$ ("the dual") defined on the category of finitely generated projective R-modules is exact and contravariant to the same category. In addition, the functor composed with itself is naturally equivalent to the identity functor. The exactness comes from the fact that the modules are all projective. That the functor composed with itself is the identity functor is the familiar proof that "the dual of the dual is the thing itself". In order to show that the second dual is not too big, one observes first that it is not too big for indecomposable projectives. Such a projective is an idempotent-generated (say left) ideal Re, and its dual is eR. Clearly, the second dual is again Re. Now in the general case apply the Krull-Schmidt theorem, and induce on the number of indecomposable direct summands.

Now we note that

$$0 \to \operatorname{Hom}(M_0, R) \xrightarrow{\delta_1^*} \operatorname{Hom}(M_1, R) \xrightarrow{\delta_2^*} \cdots$$

implies that if for some $i > 1$, $\operatorname{Im} \delta_i^*$ is *projective*, then the sequence splits all the way back. That is, $\operatorname{Im} \delta_j^*$ is projective for $j \leqq i$. Then we have

$$0 \to \operatorname{Hom}(M_0, R) \xrightarrow{\delta_1^*} \operatorname{Hom}(M_1, R) \to \operatorname{Im} \delta_2^* \to 0$$

is exact with $\operatorname{Im} \delta_2^*$ a finitely generated projective.

Applying the remark about duals and forming the homomorphism of this sequence into R, we have

$$M_1 \xrightarrow{\delta_1} M_0 \to 0$$

exact. This implies $A = 0$.

That is, the theorem will be proved if we can show that for some $n > 1$, $\operatorname{Im} \delta_n^*$ is projective. In fact we need not even go that far. If we can show the projective dimension of $\operatorname{Im} \delta_n^*$ is less than $n - 1$ for some $n \geqq 2$, that will be enough. This follows from the fact that

$$0 \to \operatorname{Hom}(M_0, R) \to \cdots \to \operatorname{Hom}(M_{n-1}, R) \to \operatorname{Im} \delta_n^* \to 0$$

is a projective resolution for $\operatorname{Im} \delta_n^*$, and if the projective dimension of $\operatorname{Im} \delta_n^*$ is less than $n - 1$, then some $\operatorname{Im} \delta_j^*$ must be projective for $j > 1$.

It should be noted that we have only used so far that A is finitely generated over R with minimum condition. In fact, what we need to prove the conclusion that $A = 0$ is that for some $n \geqq 2$ the projective dimension of $\operatorname{Im} \delta_n^*$ is $\leqq n - 1$. In a corollary following the theorem we will assume a condition of this sort, and the above argument would give the same conclusion.

Let us return to the proof of the theorem. The module $\operatorname{Im} \delta_i^*$ can be identified with the module $\operatorname{Hom}(\operatorname{Im} \delta_i, R)$ for each $i \geqq 1$. This is because of the string of equivalences

$$\begin{aligned}\operatorname{Im} \delta_i^* &= \operatorname{Ker} \delta_{i+1}^* = \{f \mid f \cdot \delta_{i+1} = 0\} = \{f \mid \operatorname{Ker} f \supseteq \operatorname{Im} \delta_{i+1}\} \\ &= \{f \mid \operatorname{Ker} f \supseteq \operatorname{Ker} \delta_i\} = \{f \mid f \in \operatorname{Hom}(\operatorname{Im} \delta_i, R)\}.\end{aligned}$$

By the assumption on the nature of the resolution of A, there exists $n > 1$

such that $\operatorname{Im} \delta_n = \sum \oplus_{i=1}^{r} T_i$ and each T_i is a direct summand of some $\operatorname{Im} \delta_q$ for $q < n$.

Hence, we have

$$\operatorname{Im} \delta_n^* \cong \operatorname{Hom}(\operatorname{Im} \delta_n , R) \cong \sum \operatorname{Hom}(T_i , R)$$

(equivalences as R-modules).

Also, for each i there exists $q < n$ such that $T_i \oplus T_i' = \operatorname{Im} \delta_q$, and thus $\operatorname{Hom}(T_i, R) \oplus \operatorname{Hom}(T_i', R) = \operatorname{Im} \delta_q^*$. Then dim $\operatorname{Hom}(T_i, R)$ is less than or equal to $q - 1$ which is strictly less than $n - 1$. But since $\operatorname{Im} \delta_n^* = \sum \oplus \operatorname{Hom}(T_1 R)$, we see that dim $\operatorname{Im} \delta_n^* = \max_i \dim T_i$ is also strictly less than $n - 1$. Thus the theorem follows from the remarks above.

As we remarked in the proof of the theorem we can get a similar theorem from the proof if we can insure that the projective dimension of $\operatorname{Im} \delta_n^* < n - 1$ for some n. Recall that the right finitistic global dimension of a ring is the supremum of the right projective dimensions of the right modules with finite projective dimension.

COROLLARY 3.2. *If a ring R with minimum condition has right finitistic global dimension $< \infty$, and if A is a finitely generated left module such that* $\operatorname{Ext}^m(A, R) = 0$ *for all* $m \geqq 0$, *then* $A = 0$.

The proof follows from the preceding remarks.

Actually, what we need in the proof of the corollary is that there does not exist for the ring R an infinite exact sequence

$$0 \longrightarrow M_0 \longrightarrow M_1 \longrightarrow \cdots \longrightarrow M_m \xrightarrow{\delta}$$

of projective modules with no splitting at any point. This is a little less than finite finitistic global dimension. We remark that it may be true that the finitistic global dimension of rings with minimum condition is always finite. We do not even know of an example of a semiprimary ring with "infinite" finitistic global dimension.

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UNIVERSITY OF WASHINGTON
SEATTLE, WASHINGTON