SOLVABLE FACTORIZABLE GROUPS II

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Let $G = HK$ where H and K are subgroups of G. A number of authors have given sufficient conditions on H and K that G be solvable. In case H and K are both Abelian, Itô [1] showed that $G^{(2)} = 1$, i.e., that G is solvable in two steps. It will be proved here that if H is finite Abelian and K finite Hamiltonian, then $G^{(4)} = 1$ (see Corollary 1).

Most of the papers on the subject limit themselves to the case where H and K are both finite. An easy theorem (Theorem 2) permits one to allow either H or K, but not both, to be infinite in a great many of these theorems. In the present case, if H is Abelian and K Hamiltonian, with one of them finite, then $G^{(6)} = 1$. Actually generalizations of the results quoted above are proved here.

Let G be a group with identity element 1. Let $G^{(1)}$ denote the commutator subgroup of G, and let $G^{(n+1)} = G^{(n)(1)}$ for all natural numbers n. Let $H \subset G$ mean that H is a subgroup of G and $H \lhd G$ mean that H is a normal subgroup of G. Let $Z(G)$ denote the center of G. If $a, b \in G$, let $[a, b] = aba^{-1}b^{-1}$ and $a^b = bab^{-1}$. If $H \subset G$ and $K \subset G$, then $[H, K]$ means the subgroup generated by all commutators $[h, k]$ with $h \in H$ and $k \in K$. Let $o(G)$ denote the order of G. Let $a \sim b$ mean that a is conjugate to b.

Lemma. *If $a_1, \cdots, a_m, b_1, \cdots, b_n \in G$, then $[a_1 \cdots a_m, b_1 \cdots b_n]$ is in the subgroup normally generated by the set $\{[a_i, b_j] \mid 1 \leqq i \leqq m, 1 \leqq j \leqq n\}$.*

Proof. If $a, b, c \in G$, then

$$(1) \qquad [a, bc] = [a, b][a, c]^b, \qquad [ab, c] = [b, c]^a[a, c].$$

The lemma follows easily by induction.

Theorem 1. *If $G = HK$ is a finite group, H an Abelian subgroup, $o(K^{(1)}) = p$, a prime, and $K^{(1)} \subset Z(K)$, then $G^{(4)} = 1$.*

Proof. By a theorem of Itô [2], $[H, K] \lhd G$. Hence $L = [H, K]K^{(1)}$ is a normal subgroup of G. Let u generate $K^{(1)}$. Let M be the subgroup normally generated by u, and N the subgroup normally generated by the set $\{[a, u] \mid a \in H\}$. All conjugates of u are obtained by conjugating u by elements of H since $K^{(1)} \subset Z(K)$ and $G = HK$. But if $h \in H$, then $u^h = [h, u]u \in L$. Hence $L \supset M \supset N$.

We shall show that (i) $G^{(1)} \subset L$, (ii) $G^{(2)} \subset M$, (iii) $G^{(3)} \subset N$, and (iv) $G^{(4)} = 1$.

(i) Every commutator is of the form $[ax, by]$ with $a, b \in H$ and $x, y \in K$. Hence by the lemma, $G^{(1)} \subset L$.

Received December 3, 1959

(ii) Temporarily, let $r, s \in [H, K]$. Then, in order to prove that $[ru^i, su^j] \in M$, it suffices to show that $[r, s] \in M$ since $[r, u] \in N$ by (1) and the fact that $u \in K^{(1)} \subset Z(K)$.

Let $a, b \in H$, $x, y \in K$. There exist $c, d, c' \in H$ and $z, w, z' \in K$ such that $x^b = zc$, $a^y = dw$, $(x^y)^b = z'c'$. By (1)

$$[a, x]^b = [a^b, x^b] = [a, zc] = [a, z];$$

$$[a, x]^y = [a^y, x^y] = [dw, x^y] = [w, x^y]^d[d, x^y];$$

$$[a, x]^{by} = ([a, x]^y)^b = [w, x^y]^{bd}[d, x^y]^b = [w, x^y]^{bd}[d, z'];$$

$$[a, x]^{yb} = [a, z]^y = [dw, z^y] = [w, z^y]^d[d, z^y].$$

Then

$$(2) \qquad [[a, x], [y^{-1}, b^{-1}]] \sim [a, x]^{by}([a, x]^{yb})^{-1} = [w, x^y]^{bd}[d, z'][d, z^y]^{-1}([w, z^y]^d)^{-1}.$$

Since $[w, x^y] \in K^{(1)}$ and $[w, z^y] \in K^{(1)}$, to prove that $[[a, x], [y^{-1}, b^{-1}]] \in M$, it suffices to show that $[d, z'][d, z^y]^{-1} \in M$.

Now $x^y = u^r x$ for some integer r. Hence $z'c' = (x^y)^b = (u^r x)^b = (u^r)^b zc$, and $z' = (u^r)^b zc''$ where $c'' \in H$. By (1)

$$[d, z'][d, z^y]^{-1} \equiv [d, z][d, z^y]^{-1} \pmod{M}.$$

If $z^y = z$, then the right member of the last congruence is 1; if $z^y \neq z$, then for some integer s, the right member equals

$$[d, z][d, zu^s]^{-1} = [d, z]([d, u^s]^z)^{-1}[d, z]^{-1} \in M.$$

Therefore, if $a, b \in H$ and $x, y \in K$, then $[[a, x], [y, b]] \in M$. Now $[b, y] = [y, b]^{-1}$. Also

$$[v, w^{-1}] = w^{-1}[w, v]w = (w^{-1}[v, w]w)^{-1},$$

so that if $[v, w] \in M$, also $[v, w^{-1}] \in M$. Therefore, if $a, b \in H$, $x, y \in K$, then $[[a, x], [b, y]] \in M$.

Hence, by the lemma, $G^{(2)} \subset M$.

(iii) Recalling the definition of N and using the lemma on $[aua^{-1}, bub^{-1}]$, we get $G^{(3)} \subset N$.

(iv) Let $a, c \in H$, $x \in K$. Then $xa = by$ for some $b \in H$, $y \in K$. Thus by (1),

$$[a, u]^x = [xa, u][x, u]^{-1} = [by, u] = [b, u];$$

$$[a, u]^c = [ca, u][c, u]^{-1} = [ca, u][c, u]^r$$

for some natural number r, since G is finite. Hence N is the set of all products of elements of the form $[a, u]$, or alternatively of the form $[u, a]$, $a \in H$.

Now apply (2) with $x = y^{-1} = u$, $u^b = zc = z'c'$, $a^{u^{-1}} = dw$. We may take $z = z'$, $c = c'$. Then we get

$$[[a, u], [u, b^{-1}]] \sim [d, z][d, z]^{-1}([w, z]^d)^{-1} \sim [w, z]^{-1} = 1 \text{ or } u^r$$

for some integer r. If $[[a, u], [u, b^{-1}]] = 1$ for all $a, b \in H$, then by the lemma, $G^{(4)} = 1$. If there are $a, b \in H$ such that $[[a, u], [u, b^{-1}]] = u^r$ with $(r, p) = 1$, then $u \in G^{(1)}$. If, inductively, $u \in G^{(n)}$, then since $[a, u]$ and $[u, b^{-1}] \in G^{(n)}$ by normality of $G^{(n)}$, $[[a, u], [u, b^{-1}]] \in G^{(n+1)}$, so that $u \in G^{(n+1)}$. But that implies that $u \in G^{(n)}$ for all n, so that G is not solvable.

If q is a prime different from p, $x \in K$, $y \in K$, $o(y) = q^i$, and $[x, y] \neq 1$, then $xyx^{-1} = yu^s$ where $(s, p) = 1$. Hence

$$1 = (xyx^{-1})^{q^i} = (yu^s)^{q^i} = y^{q^i}u^{sq^i} = u^{sq^i},$$

a contradiction. Hence $y \in Z(K)$. It follows that any Sylow q-subgroup K_q of K is central in K, if $q \neq p$. Therefore any Sylow p-subgroup $K_p \lhd K$. Hence K is nilpotent. By a theorem of Itô [2], G is solvable, a contradiction. Therefore $G^{(4)} = 1$.

COROLLARY 1. *If $G = HK$ where H is finite Abelian and K is finite Hamiltonian, then $G^{(4)} = 1$.*

Proof. Since K is Hamiltonian, $o(K^{(1)}) = 2$.

DEFINITION. A class C of groups is *hereditary* if it is closed under the taking of subgroups and homomorphic images.

THEOREM 2. *Let C be a hereditary class of solvable groups, D a hereditary class of finite groups, such that if $L = UV$, $U \in C$, $V \in D$, and U finite, then L is solvable. If $G = HK$ with $H \in C$, $K \in D$, then G is solvable.*

Proof. The index $[G:H]$ is finite. Hence there is an $N \subset H$ such that $N \lhd G$ and G/N is finite. Then $G/N = (H/N)(KN/N)$. Now $H/N \in C$ and is finite; $KN/N \cong K/(K \cap N) \in D$ and is finite. Hence G/N is solvable. Since $N \in C$, N is solvable. Therefore G is solvable.

Theorem 2 has many corollaries. Just one example will be given here.

COROLLARY 2. *Let $G = HK$ with H nilpotent, K Abelian or Hamiltonian, and H or K finite. Then G is solvable.*

Proof. The class C of nilpotent groups is hereditary, as is the class D of Abelian or Hamiltonian groups, and either class remains hereditary if the adjective "finite" is added. The corollary now follows from Theorem 2 and Scott [3, Theorem 1].

THEOREM 3. *Let C and D be as in Theorem 2. Suppose further, that there are natural numbers, m, n, such that if $H \in C$ then $H^{(m)} = 1$, and if $L = UV$ with $U \in C$, $V \in D$ and U finite, then $L^{(n)} = 1$. If $G = HK$ with $H \in C$, $K \in D$, then $G^{(m+n)} = 1$.*

Proof. Let $N \lhd G$, $N \subset H$, and let G/N be finite. Then

$$G/N = (H/N)(KN/N);$$

hence $(G/N)^{(n)} = 1$. Since $N \in C$, $N^{(m)} = 1$. Hence $G^{(m+n)} = 1$.

COROLLARY 3. *If $G = HK$, H Abelian, $o(K^{(1)}) = p$, a prime, $K^{(1)} \subset Z(K)$, and K finite, then $G^{(5)} = 1$.*

Proof. Let C be the class of Abelian groups, and D the class of finite groups which are either Abelian or satisfy the hypotheses that K satisfies. Then C is hereditary with $m = 1$. Again D is hereditary. By Theorem 1, $n = 4$. Hence by Theorem 3, $G^{(5)} = 1$.

COROLLARY 4. *If $G = HK$ with H Abelian and K finite Hamiltonian, then $G^{(5)} = 1$.*

COROLLARY 5. *If $G = HK$ with H finite Abelian, $o(K^{(1)})$ prime, and $K^{(1)} \subset Z(K)$, then $G^{(6)} = 1$.*

Proof. The proof is similar to the proof of Corollary 3.

COROLLARY 6. *If $G = HK$ with H finite Abelian and K Hamiltonian, then $G^{(6)} = 1$.*

BIBLIOGRAPHY

1. NOBORU ITÔ, *Über das Produkt von zwei abelschen Gruppen*, Math. Zeitschrift, vol. 62 (1955), pp. 400–401.
2. ———, *Remarks on factorizable groups*, Acta Sci. Math. Szeged, vol. 14 (1951), pp. 83–84.
3. W. R. SCOTT, *Solvable factorizable groups*, Illinois J. Math., vol. 1 (1957), pp. 389–394.

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