SOME REMARKS ON A. C. SCHAEFFER'S PAPER ON DIRICHLET SERIES[1]

BY
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Schaeffer's conjecture that the $(p + 1)/2$ functions 1, $\psi(\tau, \chi)/\phi(\tau)$ are a complete set of linearly independent multiples of Q' can be proved for $p = 3$, 5, and 7 in an elementary way.

Since from (1) and (30) of Schaeffer's paper we know that

$$\frac{\psi(\tau, \chi)}{\phi(\tau)} = 2 \sum_{a=1}^{(p-1)/2} \chi(a) \frac{f_a(\tau, 0)}{\phi(\tau)},$$

it suffices to show that the $(p + 1)/2$ functions 1, σ_a $(= f_a(\tau, 0)/\phi(\tau))$ are a complete set of linearly independent multiples of Q'. For the matrix

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 2\chi_0(1) & 2\chi_0(2) & \cdots & 2\chi_0\left(\frac{p-1}{2}\right) \\ 0 & 2\chi_1(1) & 2\chi_1(2) & \cdots & 2\chi_1\left(\frac{p-1}{2}\right) \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 2\chi_{(p-3)/2}(1) & 2\chi_{(p-3)/2}(2) & \cdots & 2\chi_{(p-3)/2}\left(\frac{p-1}{2}\right) \end{bmatrix}$$

is nonsingular, since

$$\begin{bmatrix} \chi_0(1) & \chi_0(2) & \cdots & \chi_0\left(\frac{p-1}{2}\right) \\ \chi_1(1) & \chi_1(2) & \cdots & \chi_1\left(\frac{p-1}{2}\right) \\ \vdots & \vdots & \cdots & \vdots \\ \chi_{(p-3)/2}(1) & \chi_{(p-3)/2}(2) & \cdots & \chi_{(p-3)/2}\left(\frac{p-1}{2}\right) \end{bmatrix}$$

$$\cdot \begin{bmatrix} \bar{\chi}_0(1) & \bar{\chi}_1(1) & \cdots & \bar{\chi}_{(p-3)/2}(1) \\ \bar{\chi}_0(2) & \bar{\chi}_1(2) & \cdots & \bar{\chi}_{(p-3)/2}(2) \\ \vdots & \vdots & \cdots & \vdots \\ \bar{\chi}_0\left(\frac{p-1}{2}\right) & \bar{\chi}_1\left(\frac{p-1}{2}\right) & \cdots & \bar{\chi}_{(p-3)/2}\left(\frac{p-1}{2}\right) \end{bmatrix} = \frac{p-1}{2} I,$$

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where I is the $((p-1)/2)$-dimensional identity matrix. Here

$$\chi_0, \quad \chi_1, \quad \cdots, \quad \chi_{(p-3)/2}$$

are the residue-characters modulo p such that $\chi_i(-1) = 1$ for $i = 0, 1, \cdots, (p-3)/2$, and we have used the fact that

$$\begin{aligned}\sum_{a=1}^{(p-1)/2} \chi_i(a)\bar{\chi}_k(a) &= \tfrac{1}{2}\Big\{\sum_{a=1}^{(p-1)/2} \chi_i(a)\bar{\chi}_k(a) + \sum_{a=1}^{(p-1)/2} \chi_i(p-a)\bar{\chi}_k(p-a)\Big\} \\ &= \tfrac{1}{2}\sum_{a=1}^{p-1} \chi_i(a)\bar{\chi}_k(a) \\ &= \tfrac{1}{2}(p-1)\delta_{ik},\end{aligned}$$

where δ_{ik} is the Kronecker delta.

For the proof of the conjecture for $p = 3$, 5, and 7 we shall use only the following trivial lemma.

LEMMA. *Let $(z-\alpha)^{a_i}f_i(z)$, $i = 1, 2, \cdots, n$, be n functions meromorphic in a region containing α, where α is neither a zero nor a pole of $f_i(z)$ and a_i are integers such that $a_1 < a_j$, $j = 2, 3, \cdots, n$. Then $(z-\alpha)^{a_1}f_1(z)$ cannot be expressed as a linear combination of the other functions.*

Proof. Suppose $(z-\alpha)^{a_1}f_1(z) = \sum_{j=2}^{n} A_j(z-\alpha)^{a_j}f_j(z)$. Then

$$f_1(z) = \sum_{j=2}^{n} A_j(z-\alpha)^{a_j-a_1}f_j(z).$$

Hence $f_1(\alpha) = 0$. But this contradicts the hypothesis.

The application of this lemma is the following: Let $g_1(z), g_2(z), \cdots, g_n(z)$ be meromorphic functions. If the order of vanishing[3] of $g_1(z)$ at a point is strictly less than that of any of the other functions, then to prove they are linearly independent it suffices to prove $g_2(z), \cdots, g_n(z)$ are linearly independent after discarding $g_1(z)$. After discarding some functions, there may remain several functions whose orders at a certain point are the smallest. In such a case we investigate their orders at some other point for the possibility of further discarding. In case all the orders of $g_i(z)$ at a point are distinct they are clearly linearly independent.

For the reason stated in the last two paragraphs of Schaeffer's paper, it suffices to show now that

(1) the $(p^3-p)/8$ functions in (47) and (48) are linearly independent, and

(2) the $(p+1)/2$ functions $1, \sigma_a$, where $a = 1, 2, \cdots, (p-1)/2$, are the only multiples of Q' among these functions.

I. The case $p = 3$

(1) The $(p^3-p)/8 = 3$ functions in (47) and (48) are $1, \sigma_1, \sigma_1^2$.

Since the orders of vanishing of $(3\tau-2)^{1/2}f_1(\tau, 0)$ and $(3\tau-2)^{1/2}\phi(\tau)$ at

[3] If the point is a pole of $g_i(z)$, then the order becomes negative.

TABLE 1

Functions	1/3	2/3	3
1	0	0	0
σ_1	0	1	−1
σ_1^2	0	2	−2

TABLE 2

Functions	1/5	3/5	2/5	4/5	5/1	5/3	Order of discarding	Vertex used in discarding
1	0	0	0	0	0	0	1	2/5
σ_1	1	0	4	1	−3	−3	8	4/5
σ_2	0	1	1	4	−3	−3	9	2/5
F_2	−1	1	2	−2	0	0	4	4/5
$F_2\sigma_1$	0	1	6	−1	−3	−3	5	4/5
$F_2\sigma_2$	−1	2	3	2	−3	−3	11	2/5
F_2^2	−2	2	4	−4	0	0	2	4/5
$F_2^2\sigma_1$	−1	2	8	−3	−3	−3	3	4/5
$F_2^2\sigma_2$	−2	3	5	0	−3	−3	6	1/5
σ_1^2	2	0	8	2	−6	−6	15	2/5
$\sigma_1\sigma_2$	1	1	5	5	−6	−6	13	2/5
σ_2^2	0	2	2	8	−6	−6	10	2/5
$F_2\sigma_1^2$	1	1	10	0	−6	−6	7	4/5
$F_2\sigma_1\sigma_2$	0	2	7	3	−6	−6	14	2/5
$F_2\sigma_2^2$	−1	3	4	6	−6	−6	12	2/5

the vertex 2/3 in terms of the appropriate variable u (see (41)) are 1 and 0 respectively, the orders of vanishing of 1, $\sigma_1 = f_1(\tau, 0)/\phi(\tau)$ and σ_1^2 are 0, 1, and 2. Hence by our lemma they are linearly independent.

(2) Consider Table 1 which shows the orders of vanishing of 1, σ_1, σ_1^2 at points 1/3, 2/3, and 3.

Since the multiplicity of Q' is $(p^2 - 1)(p - 1)/16$, which is 1 for $p = 3$, the only multiples of the divisor Q' are 1 and σ_1.

II. The case $p = 5$

(1) Table 2 shows the orders of vanishing of the $(p^3 - p)/8 = 15$ functions in (47) and (48) at the vertices. Since we can discard them all, they are linearly independent.

(2) For $p = 5$ the total multiplicity of Q' at 5/1 and 5/3 is

$$(p^2 - 1)(p - 1)/16 = 6.$$

TABLE 3

Functions	1/7	3/7	5/7	2/7	4/7	6/7	7/1	7/3	7/5	Order of discarding	Vertex used in discarding
1	0	0	0	0	0	0	0	0	0	1	4/7
σ_1	3	0	1	4	9	1	−6	−6	−6	33	2/7
σ_2	1	3	0	9	1	4	−6	−6	−6	2	4/7
σ_3	0	1	3	1	4	9	−6	−6	−6	16	2/7
F_2	−2	3	−1	−2	6	−4	0	0	0	13	2/7
$F_2\sigma_1$	1	3	0	2	15	−3	−6	−6	−6	22	6/7
$F_2\sigma_2$	−1	6	−1	7	7	0	−6	−6	−6	30	6/7
$F_2\sigma_3$	−2	4	2	−1	10	5	−6	−6	−6	14	2/7
F_3	−3	1	2	4	2	−6	0	0	0	18	6/7
$F_3\sigma_1$	0	1	3	8	11	−5	−6	−6	−6	19	6/7
$F_3\sigma_2$	−2	4	2	13	3	−2	−6	−6	−6	24	4/7
$F_3\sigma_3$	−3	2	5	5	6	3	−6	−6	−6	27	4/7
F_2^2	−4	6	−2	−4	12	−8	0	0	0	11	2/7
$F_2^2\sigma_1$	−1	6	−1	0	21	−7	−6	−6	−6	8	6/7
$F_2^2\sigma_2$	−3	9	−2	5	13	−4	−6	−6	−6	20	5/7
$F_2^2\sigma_3$	−4	7	1	−3	16	1	−6	−6	−6	12	2/7
F_3^2	−6	2	4	8	4	−12	0	0	0	3	6/7
$F_3^2\sigma_1$	−3	2	5	12	13	−11	−6	−6	−6	4	6/7
$F_3^2\sigma_2$	−5	5	4	17	5	−8	−6	−6	−6	7	6/7
$F_3^2\sigma_3$	−6	3	7	9	8	−3	−6	−6	−6	9	1/7
F_2F_3	−5	4	1	2	8	−10	0	0	0	5	6/7
$F_2F_3\sigma_1$	−2	4	2	6	17	−9	−6	−6	−6	6	6/7
$F_2F_3\sigma_2$	−4	7	1	11	9	−6	−6	−6	−6	17	1/7
$F_2F_3\sigma_3$	−5	5	4	3	12	−1	−6	−6	−6	10	1/7
σ_1^2	6	0	2	8	18	2	−12	−12	−12	42	4/7
σ_2^2	2	6	0	18	2	8	−12	−12	−12	23	4/7
σ_3^2	0	2	6	2	8	18	−12	−12	−12	31	2/7
$\sigma_1\sigma_2$	4	3	1	13	10	5	−12	−12	−12	38	4/7
$\sigma_2\sigma_3$	1	4	3	10	5	13	−12	−12	−12	26	4/7
$\sigma_1\sigma_3$	3	1	4	5	13	10	−12	−12	−12	34	2/7
$F_2\sigma_1^2$	4	3	1	6	24	−2	−12	−12	−12	28	6/7
$F_2\sigma_2^2$	0	9	−1	16	8	4	−12	−12	−12	37	4/7
$F_2\sigma_3^2$	−2	5	7	0	14	14	−12	−12	−12	15	2/7
$F_2\sigma_1\sigma_2$	2	6	0	11	16	1	−12	−12	−12	41	4/7
$F_2\sigma_2\sigma_3$	−1	7	2	8	11	9	−12	−12	−12	39	4/7
$F_2\sigma_1\sigma_3$	1	4	3	3	19	6	−12	−12	−12	32	2/7
$F_3\sigma_1^2$	3	1	4	12	20	−4	−12	−12	−12	21	6/7
$F_3\sigma_2^2$	−1	7	2	22	4	2	−12	−12	−12	25	4/7
$F_3\sigma_3^2$	−3	3	8	6	10	12	−12	−12	−12	35	2/7
$F_3\sigma_1\sigma_2$	1	4	3	17	12	−1	−12	−12	−12	29	6/7
$F_3\sigma_2\sigma_3$	−2	5	5	14	7	7	−12	−12	−12	36	4/7
$F_3\sigma_1\sigma_3$	0	2	6	9	15	4	−12	−12	−12	40	4/7

TABLE 4
$p = 11$

Functions	1/11	10/11
σ_5	0	25
σ_4	1	16
σ_3	3	9
σ_2	6	4
σ_1	10	1
F_5	−10	−20
F_4	−9	−18
F_3	−7	−14
F_2	−4	−8

TABLE 5
$p = 13$

Functions	1/13	12/13
σ_6	0	36
σ_5	1	25
σ_4	3	16
σ_3	6	9
σ_2	10	4
σ_1	15	1
F_6	−15	−30
F_5	−14	−28
F_4	−12	−24
F_3	−9	−18
F_2	−5	−10

TABLE 6

Functions	1/17	16/17
σ_8	0	64
σ_7	1	49
σ_6	3	36
σ_5	6	25
σ_4	10	16
σ_3	15	9
σ_2	21	4
σ_1	28	1
F_8	−28	−56
F_7	−27	−54
F_6	−25	−50
F_5	−22	−44
F_4	−18	−36
F_3	−13	−26
F_2	−7	−14

Then by Table 2, σ_1^2, $\sigma_1 \sigma_2$, σ_2^2, $F_2 \sigma_1^2$, $F_2 \sigma_1 \sigma_2$, $F_2 \sigma_2^2$ cannot be multiples of Q'. Also F_2, $F_2 \sigma_1$, $F_2 \sigma_2$, F_2^2, $F_2^2 \sigma_1$, $F_2^2 \sigma_2$ cannot be multiples of Q' because each of them has at least one negative order at vertices other than 5/1 and 5/3. (See the columns of 1/5 or 4/5.) Therefore the $(p + 1)/2 = 3$ functions 1, σ_1, and σ_2 are the only multiples of Q', and we already know they are linearly independent.

III. The case $p = 7$

(1) Table 3 shows the orders of vanishing of the $(p^3 - p)/8 = 42$ functions in (47) and (48) at the vertices. Since we can discard them all, they are linearly independent.

(2) For $p = 7$ the total multiplicity of Q' at 7/1, 7/3 and 7/5 is

$$(p^2 - 1)(p - 1)/16 = 18.$$

Then by Table 3, the last 18 functions cannot be multiples of Q'. Also the next last 20 functions cannot be multiples of Q' because each of them has at least one negative order at vertices 1/7 and 6/7. Therefore the $(p + 1)/2 = 4$ functions 1, σ_1, σ_2, and σ_3 are the only multiples of Q', and we already know they are linearly independent.

IV. The case $p > 7$

(1) For $p = 11$ we were able to discard only 28 functions among the $(p^3 - p)/8 = 165$ functions. For $p = 13$ we were able to discard 36 functions among the $(p^3 - p)/8 = 273$ functions.

(2) Since the total multiplicity of functions of type $F_a \sigma_b \sigma_c$, where $b \geqq 1$ and $c \geqq 1$, at the vertices p/ν, $\nu = 1, 3, \cdots, p - 2$, is $(p^2 - 1)(p - 1)/8$, they cannot be multiples of Q', whose multiplicity is $(p^2 - 1)(p - 1)/16$. Functions of type $F_a F_b \sigma_c$, where $a \geqq 2$ or $b \geqq 2$ or both, cannot be multiples of Q' for $p = 11$ and $p = 13$. This can be verified easily by considering Tables 4 and 5 which show the orders of vanishing of the functions σ_a, $a = 1, 2, \cdots, (p - 1)/2$, and F_a, $a = 2, 3, \cdots, (p - 1)/2$.

For $p > 13$ we need more columns other than the columns for the vertices $1/p$ and $(p - 1)/p$.

For example, for $p = 17$, we have the situation shown in Table 6. Here $F_2 \sigma_4$ is a function of type $F_a F_b \sigma_c$ which has positive order at both $1/p$ and $(p - 1)/p$.

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