LOCAL A-SETS, B-SETS, AND RETRACTIONS[1]

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Introduction

In the papers [2; 3], L. Cesari introduced the concept of a fine-cyclic element of a mapping (T, J) from a closed finitely connected Jordan region J into the Euclidean space E_3. Fine-cyclic elements constitute a decomposition of proper cyclic elements, and, in case J is unicoherent, coincide with proper cyclic elements. In [5] Cesari's concept of a fine-cyclic element has been extended to a Peano space in the following manner. First, a B-set of a Peano space P has been introduced as a generalization of an A-set of P. Specifically, a B-set B of P is a nondegenerate (more than one point) continuum of P such that either $B = P$ or else each component of $P - B$ has a finite frontier. A fine-cyclic element of P is defined to be a B-set of P whose connection is not destroyed by removing any finite set. It has been shown in [5] that in Peano spaces of finite degree of multicoherence the properties of B-sets and fine-cyclic elements are suitable extensions of the corresponding properties of A-sets and proper cyclic elements.

The first part of this paper shows that fine-cyclic elements are proper cyclic elements relative to some decomposition of a Peano space into a finite number of B-sets.

The second part deals with questions of retractions onto B-sets of a Peano space P. For technical reasons, the concept of a *local A-set* of a Peano space is introduced. For this preliminary survey it suffices to consider a local A-set as a set B which is an A-set relative to some connected open set $G \supset B$. A natural retraction from G onto B suggests itself, namely the one that sends each component of $G - B$ into its frontier relative to G. This retraction is similar to the one used by L. Cesari in [2; 3]. One of the main results of this paper states that this retraction can be extended to P so as to map $P - G$ into a dendrite in B. The last theorem provides some useful information on the composition of two retractions.

It will be shown that every local A-set is a B-set, and in case the underlying Peano space is of finite degree of multicoherence, every B-set is a local A-set.

1. Notation

Let X be a metric space, and let E be a subset of X. The distance function in X will be denoted by ρ, and the diameter of E will be abbreviated by $\delta(E)$. The closure and frontier of E will be designated by $c(E)$ and $\mathrm{Fr}(E)$. If

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$E \subset A \subset X$, then the closure and frontier of E relative to A will be written as $c_A(E)$ and $\mathrm{Fr}_A(E)$. The degree of multicoherence of a Peano space P will be denoted by $r(P)$ ([8]).

2. Lemma

Let P be a Peano space. If $r(P) = 0$, then P is unicoherent, and consequently every B-set of P is an A-set of P. Thus if B is a B-set of a Peano space P which is not an A-set of P, we infer that $r(P) \geqq 1$.

LEMMA. *Let P be a Peano space with $r(P) = n < \infty$, and let B be a B-set of P which is not an A-set of P. Then there exists a decomposition of P into B-sets B_1, B_2 such that*

(1) $P = B_1 \cup B_2, \quad B \subset B_1$;

(2) $r(B_i) < n, \quad i = 1, 2$;

(3) *$B_1 \cap B_2$ reduces to at most $n + 1$ points.*

Proof. Since B is not an A-set of P, there is a component G of $P - B$ whose frontier consists of more than one point. By [5], $P - G = B_1$, $c(G) = B_2$ are B-sets of P satisfying (1) and (3). To prove (2) let us first observe that by [5], $r(B_i) \leqq n$, $i = 1, 2$. Assume now that $r(B_1) = n$. Then there exist two continua F_1, F_2 of B such that $F_1 \cup F_2 = B_1$ and $F_1 \cap F_2$ decomposes into $n + 1$ distinct components. Let now A_1, A_2 be two continua in $B_2 = c(G)$ such that $A_1 \cap A_2 = \emptyset$; $\mathrm{Fr}(G) \cap A_i \neq \emptyset$, $i = 1, 2$; $\mathrm{Fr}(G) \subset A_1 \cup A_2$. Let T be a closed set in B_2 separating A_1, A_2 in B_2. Since $\mathrm{Fr}(G) \subset A_1 \cup A_2$, there follows that $T \subset G$. We may assume that the number of components of T is minimal in the sense of [5; §3 (iii), (iv)]. Let now S_1 be the union of all components of $c(G) - T$ not containing A_2, and let S_2 be the component of $c(G) - T$ containing A_2. By [5; §3(iv)], $T \cup S_1$ is a continuum. We consider now two cases.

Case 1. $F_i \cap \mathrm{Fr}(G) \neq \emptyset$, $i = 1, 2$. Then, say $F_1 \cap A_1 \neq \emptyset$, $F_2 \cap A_2 \neq \emptyset$. The sets $F_1^* = F_1 \cup T \cup S_1$, $F_2^* = F_2 \cup c(S_2)$ are two continua whose union is P, and since $T \subset G$, $F_1^* \cap F_2^* = (F_1 \cap F_2) \cup (F_2 \cap S_1) \cup (F_1 \cap c(S_2)) \cup (T \cap c(S_2))$. Since $E = (F_2 \cap S_1) \cup (F_1 \cap c(S_2)) \subset \mathrm{Fr}(G)$ and since $\mathrm{Fr}(G)$ is finite, the set E is finite. Moreover, $T \cap c(S_2) \neq \emptyset$, and $T \cap c(S_2) \subset G$. Consequently, $F_1^* \cap F_2^*$ decomposes into at least $n + 2$ distinct components, contradicting $r(P) = n$.

Case 2. $F_2 \cap \mathrm{Fr}(G) = \emptyset$. Then $\mathrm{Fr}(G) \subset F_1$. Let α be a simple arc in B_1 joining a point $x \in F_2$ to a point $x_0 \in \mathrm{Fr}(G)$. Denote by x^* the first point on α from x_0 in $F_1 \cap F_2$. If β is the subarc of α with endpoints x_0, x^*, then $\beta - x^* \subset F_1 - F_2$. Define $F_1' = F_1$, $F_2' = F_2 \cup \beta$. Then $F_1' \cap F_2'$ decomposes into $n + 1$ distinct components. The continua F_1', F_2' satisfy the conditions of case 1.

Therefore we have proved that $r(B_1) < n$. An entirely similar argument yields $r(B_2) < n$. Thereby, the proof of the lemma is complete.

3. Decomposition theorem

In this paragraph we will prove the following theorem.

THEOREM. *Let P be a Peano space with $r(P) < \infty$. Then there exists a finite number of B-sets $B_1, \cdots, B_n$ of P satisfying the following conditions:*

(1) $P = B_1 \cup \cdots \cup B_n$;

(2) *$B_i \cap B_j$ is either empty or else finite, $i \neq j$, $i, j = 1, \cdots, n$;*

(3) *Each proper cyclic element of B_i, $1 \leqq i \leqq n$, is a fine-cyclic element of B_i and hence of P;*

(4) *Each fine-cyclic element of P is a proper cyclic element of a unique B_i, $1 \leqq i \leqq n$.*

Proof. We will first show that every decomposition of P satisfying (1), (2), and (3) also has the property (4). For, let Δ be a fine-cyclic element of P. Then by [5], Δ is a fine-cyclic element of a unique B_i, $1 \leqq i \leqq n$. Since Δ is nondegenerate and cyclic, there is a unique proper cyclic element C of B_i containing Δ. By (3), C is a fine-cyclic element of P and $C = \Delta$.

We will now prove that P possesses a decomposition with the properties (1), (2), and (3). We may assume that there is a proper cyclic element C of P which is not a fine-cyclic element of P. Then we have a finite set of points K in C such that $C - K$ is not connected. Let G be a component of $C - K$. By [5], $B = c(G)$ is a B-set of P, and since C is cyclic, B is not an A-set of P. We can now apply §2 and obtain two B-sets B_1, B_2 of P such that $P = B_1 \cup B_2$, $B \subset B_1$, $r(B_i) < r(P)$, $i = 1, 2$, $B_1 \cap B_2$ reduces to a finite number of points. If each proper cyclic element of B_1, B_2 is also a fine-cyclic element of P, we are finished. Otherwise, apply §2 to B_1 or B_2, and since $r(P) < \infty$, this process terminates after a finite number of steps. It should also be noted that in unicoherent Peano spaces fine-cyclic elements coincide with proper cyclic elements.

4. Dendrites

In the sequel we will have occasion to use some properties of *dendrites* i.e., Peano spaces possessing no proper cyclic elements. The proof of (i) and (ii) is left to the reader.

(i) A nondegenerate continuum of a dendrite D is an A-set of D and thus a subdendrite of D.

(ii) Let $E_1, \cdots, E_n$ be a finite collection of mutually disjoint subsets of a Peano space P such that E_i is either a dendrite or else a single point, $i = 1, \cdots, n$. Then there exists a dendrite $D \subset P$ such that $D \supset E_1 \cup \cdots \cup E_n$.

Let M be a connected metric space which can be written as the union of a continuum P and a dendrite D such that $P \cap D$ is finite, say $\{x_0, \cdots, x_n\}$.

(iii) LEMMA. *Under the above conditions there exists a continuous mapping t from P into D such that $t(x_i) = x_i$, $i = 0, 1, \cdots, n$.*

Proof. If $n = 0$, map P into x_0. We may therefore assume that $n \geqq 1$. Let F be the mapping on $\{x_0, \cdots, x_n\}$ taking x_i into i/n, $i = 0, 1, \cdots, n$. By [4], F can be extended continuously to P preserving bounds. Let F^* be the extended mapping, and observe that F^* maps P onto the closed unit interval I. For each i, let H_i be the homeomorphism from $I_i = [(i-1)/n, i/n]$ onto the simple arc γ_i of D joining x_{i-1}, x_i such that $H_i[(i-1)/n] = x_{i-1}$, $H_i(i/n) = x_i$. The mapping t defined by $t(x) = H_i F^*(x)$, if $F^*(x) \in I_i$, satisfies the desired properties.

5. Local A-sets

Let P be a Peano space.

DEFINITION. A nondegenerate closed subset B of P will be termed a *local A-set* of P provided either $P = B$ or else there exists a connected open set G of P containing B such that the collection of components $\{O\}$ of $G - B$ satisfies the following conditions:

(1) For $O \in \{O\}$, $\mathrm{Fr}_G(O)$ is a single point.

(2) If O', O'' are two components of $\{O\}$ with $\mathrm{Fr}_G(O') \neq \mathrm{Fr}_G(O'')$, then $c(O') \cap c(O'') = \emptyset$.

Remark. The closure in (2) is relative to P. It should also be noted that $\mathrm{Fr}_G(O)$ is in B for every $O \in \{O\}$. If B is a local A-set of P, we shall use the notation B is a (G, A)-set of P so as to display the connected open set G containing B. Every A-set of P is a (G, A)-set of P for any $G \supset A$.

(i) THEOREM. *Let B be a (G, A)-set of a Peano space P and let $\{O\}$ be the collection of components of $G - B$. Then $\{O\}$ forms a null collection, i.e., for any $\varepsilon > 0$ there exists at most a finite number of $O \in \{O\}$ such that $\delta(O) > \varepsilon$.*

The proof is essentially the same as the one in [8, p. 68].

(ii) COROLLARY. *Let B be a (G, A)-set of a Peano space P, and let for $x \in B$, S_x be the union of all components O of $G - B$ such that $\mathrm{Fr}_G(O) = x$. Then for $x' \neq x''$, $c(S_{x'}) \cap c(S_{x''}) = \emptyset$.*

Proof. Let $\{O'\}$, $\{O''\}$ be the components of $G - B$ such that $\mathrm{Fr}_G(O') = x'$, $\mathrm{Fr}_G(O'') = x''$, respectively. Then from (i), $c(S_{x'}) = \bigcup c(O')$, $c(S_{x''}) = \bigcup c(O'')$, where the unions are extended over the respective classes. Since $c(O') \cap c(O'') = \emptyset$, the result follows.

(iii) THEOREM. *Let B be a (G, A)-set of a Peano space P. Then B is arcwise connected and therefore B is a continuum.*

Proof. Let α be a simple arc joining two points x_1, x_2 of B such that $\alpha \subset G$. Such a simple arc exists since G is arcwise connected. If α were not contained in B, a simple argument would show that there is a component of $G - B$ whose frontier is nondegenerate. Thus $\alpha \subset B$ and the theorem follows.

Since a Peano space is locally connected, we have the following theorem and corollary.

(iv) THEOREM. *Let B be a (G, A)-set of a Peano space P. Then $P - B$ has only a finite number of components O with $O - G \neq \emptyset$.*

COROLLARY. *Under the conditions of* (iv), *$P - B$ has only a finite number of components with a nondegenerate frontier.*

6. Retractions onto local A-sets

Let B be a (G, A)-set of a Peano space P. Define a mapping t from G onto B by (1) $t(x) = x$, $x \in B$, (2) $t(O) = \mathrm{Fr}_G(O)$, where O is a component of $G - B$. The proof of the next two theorems offers no difficulty ([7, pp. 85–86]).

(i) THEOREM. *Let B be a (G, A)-set of a Peano space P. Then the mapping t defined above is continuous and monotone, and t is the unique continuous and monotone retraction from G onto B.*

(ii) THEOREM. *Let B be a (G, A)-set of a Peano space P. Then for any connected subset K of G, $K \cap B$ is connected (possibly empty).*

Applying Sierpinski's criterion for local connectedness we have in view of (ii) the following result.

(iii) COROLLARY. *A (G, A)-set of a Peano space P is a Peano subspace of P.*

(iv) THEOREM. *Let B' be a (G', A)-set of a Peano space P, and let B'' be a (G'', A)-set of B'. Then B'' is a (G, A)-set of P.*

Proof. Since G'' is a connected open set of B' containing B'', we have a set G^* open in P such that $G'' = B' \cap G^*$. Define G to be the component of $G^* \cap G'$ containing B''. Let O be a component of $G - B''$. If $O \subset G' - B'$, it follows that $\mathrm{Fr}_G(O)$ is a single point. If $O \subset B'$, then $O \subset B' \cap G^* = G''$, and $\mathrm{Fr}_G(O)$ is a single point. We may thus assume that $O \cap B' = O \cap G'' \neq \emptyset$ and $O \cap (G' - B') \neq \emptyset$. By (ii), $O \cap G''$ is connected and thus lies in a component O'' of $G'' - B''$. Since for every component O' of $G' - B'$ with $O' \cap O \neq \emptyset$ we have that $\mathrm{Fr}_{G'}(O') \in O \cap B' = O \cap G'' \subset O''$, it follows by §5(i) that $\mathrm{Fr}_{G''}(O'') = \mathrm{Fr}_G(O)$ is a single point.

Let now O_x, O_y be two components of $G - B''$ such that $\mathrm{Fr}_G(O_x) = x$, $\mathrm{Fr}_G(O_y) = y$ with $x \neq y$. Let K_x be the collection of all components O_x' of $G' - B'$ such that $O_x' \cap O_x \neq \emptyset$. Let $O_x'' = O_x \cap B' \subset B' \cap G^* = G''$. By (ii), O_x'' is a connected set in $G'' - B''$, and $O_x \subset O_x'' \cup (\bigcup O_x')$, where the last union is extended over all $O_x' \in K_x$. In case $O_x'' = \emptyset$ $[O_x'' \neq \emptyset]$, we see that $\mathrm{Fr}_{G'}(O_x') = x$ $[\mathrm{Fr}_{G'}(O_x') \in O_x'']$, and hence by §5(i), $c(O_x) \subset c(O_x'') \cup [\bigcup c(O_x')]$. Similarly $c(O_y) \subset c(O_y'') \cup [\bigcup c(O_y')]$. Since $c(O_x') \cap c(O_y') = \emptyset$, $c(O_x'') \cap c(O_y'') = \emptyset$,

$c(O''_x) \cap c(O'_y) = c(O''_x) \cap \mathrm{Fr}_{G'}(O'_y) = \emptyset$, and $c(O''_y) \cap c(O'_x) = \emptyset$, it follows that $c(O_x) \cap c(O'_y) = \emptyset$, and the proof is complete.

7. B-sets and local A-sets

In this section we will study the relationship between local A-sets and B-sets of a Peano space P. Since a B-set need not be a Peano subspace of P ([5]), it follows that a B-set need not be a local A-set.

(i) THEOREM. *Let B be a (G, A)-set of a Peano space P. Then B is a B-set of P.*

Proof. Deny, and assume that there is a component Q of $P - B$ with $\mathrm{Fr}(Q)$ infinite. Then there is an infinite number of components $O_1, \cdots, O_n, \cdots$ of $G - B$ such that $O_n \subset Q$, $n = 1, 2, \cdots$. Determine $\varepsilon > 0$ so that the set $E(\varepsilon) = \{x : \rho(B, x) < \varepsilon\}$ is contained in G. Since by §5(i), $\delta(O_n) \to 0$ as $n \to \infty$, we infer that for n large, $\delta(O_n) < \varepsilon$. Since Q is connected, $c(O_n) \cap (Q - O_n) \neq \emptyset$. Let x be a point in $c(O_n) \cap (Q - O_n)$. Then $x \notin G$; for, if $x \in G$, then $x \in O_n$, which is impossible. Since $O_n \subset G$, we conclude that $x \in \mathrm{Fr}(G)$ and hence $\rho(B, x) \geqq \varepsilon$. This contradicts $\delta(O_n) < \varepsilon$.

(ii) THEOREM. *Let B be a B-set of P such that $P - B$ decomposes into a finite number of components with a nondegenerate frontier. Then B is a (G, A)-set of P.*

Proof. Let $P^* = P - \bigcup Q$, where the union is extended over all components Q of $P - B$ with a single frontier point. Then P^* is an A-set of P, and every component of $P^* - B$ has a nondegenerate frontier. By hypothesis the number of components of $P^* - B$ is finite, say $G_1, \cdots, G_n$. Let now $F = \{x_1, \cdots, x_k\} = \mathrm{Fr}(G_1) \cup \cdots \cup \mathrm{Fr}(G_n)$, and let $2\eta = \min [\rho(x_i, x_j), i \neq j, i, j = 1, \cdots, k]$.

Let $\{G'_i\}$ be connected open sets of P^* such that $x_i \in G'_i$ and $\delta(G'_i) < \eta$, $i = 1, \cdots, k$. Define $G^* = B \cup (G'_1 \cup \cdots \cup G'_k)$. Then G^* is open in P^*. Set $G \cup G^* \cup (\bigcup Q)$, where the last union is extended over all components of $P - B$ with a single frontier point. Since $P - G = P^* - G^*$, the set $P - G$ is closed, and consequently G is a connected open set of P containing B.

Let now O be a component of $G - B$. We assert that $\mathrm{Fr}_G(O)$ is a single point. Since this is obvious if O is a component of $P - B$ with a single frontier point, we may assume that $O \subset G^* - B$. There is then a unique i, $1 \leqq i \leqq k$, such that $O \subset G'_i$. Consequently, $\mathrm{Fr}_G(O) = x_i$. Finally, if O', O'' are two components of $G - B$ for which $\mathrm{Fr}_G(O') \neq \mathrm{Fr}_G(O'')$, then it follows readily that $c(O') \cap c(O'') = \emptyset$. This completes the proof.

In view of [5, §4] we have the following corollary.

(iii) COROLLARY. *Let P be a Peano space with $r(P) < \infty$. Then a nondegenerate closed subset of P is a (G, A)-set of P if and only if it is a B-set of P.*

Let B be a (G, A)-set of a Peano space P. In §5 (iv) we have shown that $P - B$ decomposes into a finite number of components $G_1, \cdots, G_n$ with more than one frontier point. From (i) there follows that $\mathrm{Fr}(G_i)$ is finite, $i = 1, \cdots, n$. Let $C = \mathrm{Fr}(G_1) \cup \cdots \cup \mathrm{Fr}(G_n) = \{x_1, \cdots, x_j\}$. If $j \geqq 2$, let $k = j$, and if $C = \emptyset$, i.e., if B is an A-set of P, let $k = 1$.

(iv) THEOREM. *Under the above conditions, let K be a continuum of P. Then $B \cap K$ is either empty or else decomposes into at most k components.*

Proof. If $C = \emptyset$, the result is well-known. We assume then that $C \neq \emptyset$. Let P^* be the intersection of all A-sets of P containing B. Then P^* is an A-set of P and $K^* = P^* \cap K$ is connected. We may suppose that $B \cap K \neq \emptyset$, $(P^* - B) \cap K \neq \emptyset$, $P^* \neq B$. Then $B \cap K = B \cap K^*$. For Q a component of $K^* - C$ we have by [6, p. 84], $c(Q) \cap C \neq \emptyset$. Thus every component of $B \cap K^*$ intersects C, and since C consists of k points, the proof is complete.

(v) THEOREM. *Let P, P^* be Peano spaces, and let m be a continuous and monotone mapping from P onto P^*. If B is a (G, A)-set of P, then $m(B) = B^*$ is either a single point or else a (G^*, A)-set of P^*.*

Proof. Assume that B^* does not reduce to a single point. By (ii) it suffices to show that B^* is a B-set of P^* such that $P^* - B^*$ reduces to a finite number of components with a nondegenerate frontier. Let now G^* be a component of $P^* - B^*$ with $\mathrm{Fr}^*(G^*)$ nondegenerate, where Fr^* denotes the frontier operation relative to P^*. The set $G' = m^{-1}(G^*)$ is a connected open set in $P - B$ and hence G' lies in a component G of $P - B$. We assert that $m[\mathrm{Fr}(G)] \supset \mathrm{Fr}^*(G^*)$. It is readily seen that for $x \in \mathrm{Fr}^*(G^*)$ the set $m^{-1}(x)$ is a continuum intersecting B and $\mathrm{Fr}(G') \subset c(G)$. Hence $m^{-1}(x) \cap \mathrm{Fr}(G) \neq \emptyset$, from which the desired conclusion follows. Since by (i) $\mathrm{Fr}(G)$ is finite, we infer that $\mathrm{Fr}^*(G^*)$ is finite and B^* is a B-set of P^*. Let now $G_1, \cdots, G_n$ be the components of $P - B$ with a nondegenerate frontier, and let $F = \mathrm{Fr}(G_1) \cup \cdots \cup \mathrm{Fr}(G_n)$. Then F is finite, and for every component G^* of $P^* - B^*$ with a nondegenerate frontier we have $\mathrm{Fr}^*(G^*) \subset m(F)$. Since $m(F)$ is finite and since P^* is locally connected, the number of components of $P^* - B^*$ with a nondegenerate frontier is finite.

8. Extension of retractions

Let B be a local A-set of a Peano space P.

DEFINITION. A continuous mapping t from P onto B will be termed a *retraction* from P onto B provided there exists a connected open set G of P with $G \supset B$ such that

(1) B is a (G, A)-set of P;

(2) $t \mid G$, t restricted to G, is the continuous and monotone retraction from G onto B (§4);

(3) $t(P - G)$ is a subset of a dendrite $E \subset B$.

Remark. It is clear that a mapping t as above is in general not monotone. In the sequel an expression such as "t is a retraction from P onto a (G, A)-set B of P" means that t satisfies the conditions (2), (3) relative to G.

(i) LEMMA. *Let t be a retraction from a Peano space P onto a (G, A)-set B of P, and let E be the dendrite in B such that $t(P - G) \subset E$. If O is a component of $P - B$ for which $O - G \neq \emptyset$, then $t(O) \subset E$.*

Proof. If we deny $t(O) \subset E$, we have a point $y \in O \cap G$ such that $t(y) \notin E$. Let O' be the component of $G - B$ which contains y. Then $O' \subset O$ and $t(y) = t(O') = x$, $x = \mathrm{Fr}_G(O')$. Since O is connected and $O - O' \neq \emptyset$, we have a point $z \in c(O') \cap (O - O')$. It follows that $z \notin G$ and hence $t(z) \in E$. Since $z \in c(O')$ and since t is continuous, $t(z) = x$, and thus $t(y) = t(z)$, a contradiction.

(ii) LEMMA. *Under the conditions of* (i), *if O is a component of $P - B$ for which $t(O) \subset E$, then* $\mathrm{Fr}\,(O) \subset E$.

Proof. Since E is closed and t is continuous, we have that $t[\mathrm{Fr}\,(O)] \subset E$. From the fact that $\mathrm{Fr}\,(O) \subset B$ and $t(x) = x$, $x \in B$, the desired inclusion follows.

THEOREM. *Let B be a (G, A)-set of a Peano space P. Then there exists a retraction from P onto B.*

Proof. Let $\{O\}$ be the collection of all components of $P - B$ such that $O - G \neq \emptyset$. The collection $\{O\}$ is finite, and $C = \bigcup \mathrm{Fr}\,(O)$, where the union is extended over all $O \in \{O\}$, is a finite set (§7(i)), say $C = \{x_1, \cdots, x_k\}$. Let E be a dendrite in B containing C. For $i = 1, \cdots, k$ let S_i be the union of all components of $G - B$ whose frontier relative to G is x_i. If we set $K_i = c(S_i)$, we infer from §5(ii) that $\{K_i\}$ is a collection of disjoint continua. Let K be the collection $K_1, \cdots, K_k$ plus all the single points of $P - (K_1 \cup \cdots \cup K_k)$. Since K is an upper semicontinuous collection of continua of P, we have by well-known theorems a Peano space P', whose points are the elements of K, and we have a monotone mapping m from P onto P' such that for $x \in P$, $m(x)$ is the unique element in K containing x.

Since $m(B) = B'$ is nondegenerate, it follows from 7(v) that B' is a (G', A)-set of P'. Let P^* be the smallest A-set of P' containing B' (P^* is the intersection of all A-sets of P' containing B'). Denote by Fr^* the frontier operation relative to P^*. It is readily seen that for every component G^* of $P^* - B'$, $\mathrm{Fr}^*\,(G^*)$ is a nondegenerate subset of $m(K_1 \cup \cdots \cup K_k)$, and that the number of components of $P^* - B'$ is finite (proof of §7(v)). Denote the components of $P^* - B'$ by $G_1^*, \cdots, G_n^*$.

Let h be the mapping m restricted to B. Then h is a homeomorphism on B, and consequently $h(E) = E'$ is a dendrite in B' containing $\mathrm{Fr}^*\,(G_i^*)$, $i = 1, \cdots, n$. The continuum $E' \cup G_i^* \cup \mathrm{Fr}^*\,(G_i^*)$ satisfies the conditions of §4(iii), and hence we have a continuous mapping t_i^* from $G_i^* \cup \mathrm{Fr}\,(G_i^*)$ into E' leaving fixed the points in $\mathrm{Fr}^*\,(G_i^*)$. Define a mapping t^* from P^* onto B' by (1) $t^*(x) = x$, if $x \in B'$; (2) $t^*(x) = t_i^*(x)$, if $x \in G_i^*$, $1 \leqq i \leqq n$. Denote by r^* the monotone retraction from P' onto P^*. The mapping $t = h^{-1}t^*r^*m$ is the desired retraction from P onto B.

9. Composition of retractions

Let B' be a local A-set of a Peano space P, and let B'' be a local A-set of B'. If t' is a retraction from P onto B' and t'' is a retraction from B' onto B'' in the sense of §8, then the composite mapping $t = t''t'$ may fail to be a retraction from P onto B''. In fact, a component of $P - B''$ need not map under t into a dendrite.

THEOREM. *Let B' be a local A-set of a Peano space P, and let B'' be a local A-set of B'. If t' is a retraction from P onto B', then there exists a retraction t'' from B' onto B'' such that $t''t'$ is a retraction from P onto B''.*

Proof. There is a connected open set $G' \subset P$ with $G' \supset B'$ such that B' is a (G', A)-set, $t' \mid G'$ reduces to the monotone retraction from G' onto B', and $t'(P - G') \subset E'$, where E' is a dendrite in B'. By §7(iv), $B'' \cap E'$ decomposes into a finite number of components which are either nondegenerate dendrites or single points.

Since B'' is a local A-set of B' we have a connected open set G'' of B' such that $G'' \supset B''$ and B'' is a (G'', A)-set. Let G^* be an open subset of P such that $G'' = B' \cap G^*$, and let G be the component of $G^* \cap G'$ containing B''. It follows from §6(iv) that B'' is a (G, A)-set of P. Let $\{Q\}$ be the collection of components of $P - B''$ such that $Q - G \neq \emptyset$. Since the collection $\{Q\}$ is finite, we have in view of §7(i) that $C = \bigcup \mathrm{Fr}(Q)$ is a finite set of points, where the union is extended over all $Q \in \{Q\}$. By §4(ii) we have a dendrite $E'' \subset B''$ such that $E'' \supset C \cup (B'' \cap E')$.

If Q'' is a component of $B' - B''$ with $Q'' - G'' \neq \emptyset$, we assert that $\mathrm{Fr}'(Q'') \subset E''$, where Fr' is the frontier relative to B'. To prove this, let Q be the component of $P - B''$ containing Q''. Then $\mathrm{Fr}'(Q'') \subset \mathrm{Fr}(Q)$, and the assertion follows if we can show that $Q \in \{Q\}$. However, if $Q \subset G$, then $Q'' \subset Q \cap B' \subset G \cap B' \subset G^* \cap B' = G''$, contradicting $Q'' - G'' \neq \emptyset$. Proceeding as in the previous paragraph, we have a retraction t'' from B' onto the (G'', A)-set B'' such that $t''(B' - G'') \subset E''$.

We will now verify that $t = t''t'$ is a retraction from P onto the (G, A)-set B''. To prove that $t \mid G$ reduces to the monotone retraction from G onto B'', it suffices to verify that for O a component of $G - B''$, $t(O)$ is a single point. This is immediate in case $O \subset P - B'$ or $O \subset B'$. We may thus assume that $O \cap B' \neq \emptyset$, $O \cap (P - B') \neq \emptyset$. Since t' is continuous, $t'(O)$ is a connected subset of $G'' - B''$, and hence $t'(O)$ lies in a component of $G'' - B''$. Thus $t''t'(O)$ is a single point.

We will now prove that $t(P - G) \subset E''$. Let $x \in P - G$, and let Q', Q'' be the components of $P - B'$, $P - B''$, respectively, containing x. It follows that $\mathrm{Fr}(Q'') \subset E''$. If $t'(x) \in c(Q')$, then $t'(x) \in c(Q'')$ and $t''t'(x) \in E''$. We may thus assume that $t'(x) \in E'$ and $Q' - G' \neq \emptyset$. Moreover, we may suppose that $t'(x) \notin B''$. For, if $t'(x) \in B''$, then $t'(x) \in B'' \cap E' \subset E''$, and $t''t'(x) \in E''$. Let Q be the component of $B' - B''$ such that $t'(x) \in Q$. If

$Q - G'' \neq \emptyset$, then $\mathrm{Fr}'(Q) \subset E''$, and $t''t'(x) \in E''$. If $Q \subset G''$, then $\mathrm{Fr}'(Q)$ is a single point, and thus either $E' \supset \mathrm{Fr}'(Q)$ or else $E' \cap \mathrm{Fr}'(Q) = \emptyset$. In the first case, $t''t'(x) = \mathrm{Fr}'(Q) \subset E' \cap B'' \subset E''$. In the second case, $E' \subset Q$ and thus $\mathrm{Fr}(Q') \subset E' \subset Q$ ((i) and (ii) of §8) and hence $Q'' \supset Q$. Consequently, $\mathrm{Fr}'(Q) \subset E''$ and $t''t'(x) \in E''$. This completes the proof.

Remarks. (1) In view of §7(iii), the theory of retraction developed above applies to B-sets of a Peano space of finite degree of multicoherence. (2) Let B' be a local A-set of P, and let B'' be an A-set of B'. Let t be a retraction from P onto B', and let r be the monotone retraction from B' onto B''. Then G'' in the above theorem can be taken as B' and G as G'. Consequently, rt is a retraction from P onto B''.

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