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OPERATORS ON ASYMPTOTIC ℓ_p SPACES WHICH ARE NOT COMPACT PERTURBATIONS OF A MULTIPLE OF THE IDENTITY

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ABSTRACT. We give sufficient conditions on an asymptotic ℓ_p (for $1 < p < \infty$) Banach space to ensure the space admits an operator, which is not a compact perturbation of a multiple of the identity. These conditions imply the existence of strictly singular noncompact operators on the HI spaces constructed by G. Androulakis and the author and by Deliyanni and Manoussakis. Additionally, we show that under these same conditions on the space X, ℓ_∞ embeds isomorphically into the space of bounded linear operators on X.

1. Introduction

Is there an infinite dimensional Banach space X on which every bounded linear operator is a compact perturbation of a multiple of the identity? Mentioned by Lindenstrauss as question 1 in his 1976 list of problems in Banach space theory [22], this problem has become known as the scalar-plus-compact problem and is one of the most famous in functional analysis. In this note, we give sufficient conditions on a space, whereby the space of bounded linear operators does not have such a decomposition. Let us start by reviewing results which relate to this famed open problem.

Lindenstrauss' question is related to the result of Aronszajn and Smith [12] in 1954, which implies that if a space X satisfies the above condition and is a complex space, then every bounded linear operator on X must have a nontrivial invariant subspace. Thus, a complex space which is a positive solution to Lindenstrauss' problem also serves as a positive solution to the invariant subspace problem for Banach spaces.

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That being said, the possibility that for any Banach space, there is an operator on the space which is not a compact perturbation of a multiple of the identity, is still in play. In support of this possibility, sufficient conditions have been established on a space X which imply ℓ_∞ embeds isomorphically into $\mathcal{L}(X)$, the space of bounded linear operators on X (see [2], [14], [21]). If a space X serves as a positive solution to the scalar plus compact problem, it has a basis (or more generally the approximation property) and a separable dual space, then $\mathcal{L}(X)$ must be separable. Curiously, each of the results in support of a negative solution to the scalar plus compact problem, require the existence of an unconditional basic sequence in the space. The weaker problem of whether there is an operator which is not a compact perturbation of a multiple of the inclusion from a subspace of a Banach space to the whole space has also received attention (see [4], [5], [19], [24]).

In their successful effort to construct the first example of a space with no unconditional basic sequence, Gowers and Maurey [20] constructed a space, which as Johnson observed, possesses a stronger property called hereditarily indecomposable (HI). A Banach space is HI if no (closed) infinite dimensional subspace can be decomposed into a direct sum of two further infinite dimensional subspaces. This groundbreaking construction was a great leap forward in the progression towards a positive solution to the scalar-plus-compact problem. More precisely, it was shown that every operator on the space of Gowers–Maurey can be decomposed as a strictly singular perturbation of a multiple of the identity operator. Spaces which have this property are now aptly referred to as spaces admitting "few operators". An operator on a Banach space is called strictly singular if the restriction of it to any infinite dimensional subspace is not an isomorphism. The ideal of strictly singular operators on a space contains that of the compact operators, but in some cases (e.g., ℓ_p, $1 \leq p < \infty$) they coincide. The fact that Gowers–Maurey space admits few operators is related to the fact that it is HI. In fact, it was shown in [20] that every complex HI space admits few operators. In 1997, Ferenczi proved [15] that a complex space X is HI if and only if every operator from a subspace of X into X is a multiple of the inclusion plus a strictly singular operator. It is not the case, however, that admitting few operators implies that the space is HI. The most recent in a collection of counterexamples is the paper of Argyros and Manoussakis [10] in which they construct a reflexive space admitting few operators for which every Schauder basic sequence has an unconditional subsequence. The most comprehensive resource for HI spaces and spaces admitting few operators is [11].

The natural question then becomes: for any of these spaces which admit few operators does there exist a strictly singular noncompact operator, or do the strictly singular and compact ideals coincide? There have been results in this direction as well. In 2000, Argyros and Felouzis [7] constructed an HI space X with the property that for every infinite dimensional subspace of X there is a strictly singular noncompact operator on X with range contained in the subspace. In 2001, Androulakis and Schlumprecht [6] constructed a strictly singular noncompact operator on the space of Gowers–Maurey. In 2002, Gasparis [16] did the same for certain members of the class of totally incomparable asymptotic ℓ_1 HI spaces constructed in [17]. In 2006, Androulakis, the author [3], Manoussakis, and Deliyanni [13] independently constructed different asymptotic ℓ_p HI spaces (for $p=2$ in the former case and for all $1<p<\infty$ in the latter). In the following, by extending results in [16], sufficient conditions are established under which a strictly singular noncompact operators can be found on each of these spaces.

2. Definitions and notation

Our notation is standard and can be found in [23]. Let $(e_i)_{i=1}^{\infty}$ denote the unit vector basis of $c_{00}(\mathbb{N})=c_{00}$, and $(e_i^*)_{i=1}^{\infty}$ the biorthogonal functionals of $(e_i)_i$. Let span$\{(e_i)_i\}$ denote vectors finitely supported on $(e_i)_i$. For a Banach space X, let $Ba(X)=\{x\in X:\|x\|\le 1\}$ and $S(X)=\{x\in X:\|x\|=1\}$. If $E,F\subset\mathbb{N}$ then $E<F$ if $\max E<\min F$. If $x=\sum_{i=1}^{\infty}a_ie_i$ for scalars $(a_i)_i$, let $\operatorname{supp}(x)=\{i:a_i\neq 0\}$ and the range of x, denoted $r(x)$, be the smallest interval containing $\operatorname{supp}(x)$.

The notion of Schreier families [1] is used throughout. They are defined inductively as follows. Let $S_0=\{\{n\}:n\in\mathbb{N}\}\cup\{\emptyset\}$. After defining S_n, let

$$S_{n+1}=\left\{F\subset\mathbb{N}:F=\bigcup_{i=1}^{m}F_i \text{ for } F_i\in S_n \text{ and } m\in\mathbb{N}, m\le F_1<\cdots<F_m\right\}\cup\{\emptyset\}.$$

A few properties of the Schreier families we need are:

- (Hereditary) For $n\in\mathbb{N}$, $S_n\subset S_{n+1}$.
- (Spreading) If $(p_i)_{i=1}^N\in S_n$ and $p_i\le q_i$ for all $i\le N$, then $(q_i)_{i=1}^N\in S_n$.
- (Convolution) If $(F_i)_{i=1}^N$ is a collection of subsets of $\mathbb{N}$ such that $F_i\in S_n$ for all $i\le N$, $F_1<\cdots<F_N$ and $(\min F_i)_{i=1}^N\in S_m$ for some $n,m\in\mathbb{N}$, then $\bigcup_{i=1}^N F_i\in S_{n+m}$.

Let $(E_i)_{i=1}^k$ be a sequence of successive subsets of $\mathbb{N}$, we say that $(E_i)_{i=1}^k$ is S_n admissible if $(\min E_i)_{i=1}^k\in S_n$. For $E_1<\cdots<E_k\subset\mathbb{N}$ and $(a_j)_j\in c_{00}$ the sequence $(x_i)_{i=1}^k$ defined by $x_i=\sum_{j\in E_i}a_je_j$ is called a block sequence of $(e_j)_j$. For a block sequence $(x_i)_{i=1}^k$ of $(e_j)_j$, we say that $(x_i)_{i=i}^k$ is S_n admissible if $(\operatorname{supp} x_i)_{i=1}^k$ is S_n admissible.

Herein, we define a class of spaces in terms of the norming functionals of the space. We begin by recalling the notion of a norming set [16].

Definition 2.1. A set $\mathcal{N} \subset \operatorname{span}\{(e_i^*)_i\}$ is called norming if the following conditions hold:

- $(e_n^*)_n \subset \mathcal{N}$.
- If $x^* \in \mathcal{N}$, then $|x^*(e_n)| \leq 1$ for all $n \in \mathbb{N}$.
- If $x^* \in \mathcal{N}$, then $-x^* \in \mathcal{N}$ ($\mathcal{N}$ is symmetric).
- If $x^* \in \mathcal{N}$ and E is an interval in $\mathbb{N}$, then $Ex^* \in \mathcal{N}$ (where Ex^* denotes the restriction of x^* to the coordinates in E).

If $\mathcal{N}$ is a norming set, we can define a norm $\|\cdot\|_{\mathcal{N}}$ on c_{00} by

$$\Big\|\sum_i a_i e_i\Big\|_{\mathcal{N}} = \sup\Big\{x^*\Big(\sum_i a_i e_i\Big) : x^* \in \mathcal{N}\Big\}$$

for every $(a_i) \in c_{00}$. Now, define the Banach space $X_{\mathcal{N}}$ to be the completion of c_{00} under the above norm. By the definition of norming set, $(e_i)_i$ is a normalized bimonotone basis for $X_{\mathcal{N}}$.

For the following definitions and notation, we closely follow [16]. The following are conditions on two increasing sequences of positive integers, $(n_i)_{i=1}^\infty$ and $(m_i)_{i=1}^\infty$.

(i) $m_1 > 3$, there is an increasing sequence of positive integers $(s_i)_{i=1}^\infty$, such that $m_{2j} = \prod_{i=1}^{j-1} m_{2i}^{s_i}$, $m_{2j+1} = m_{2i}^5$ for $i \geq 1$ and $m_1^5 = m_2$.

(ii) For the sequence of integers, $(f_i)_{i=2}^\infty$ defined by

$$f_j = \max\Big\{\rho n_1 + \sum_{1 \leq i < j} \rho_i n_{2i} : \rho, \rho_i \in \mathbb{N} \cup \{0\}, m_1^\rho \prod_{1 \leq i < j} m_{2i}^{\rho_i} < m_{2j}\Big\}$$

require that $4f_j < n_{2j}$ for all $j \geq 2$ and $5n_1 < n_2$.

We now define a particular type of norming set. Our definition is slightly less general than that which would be considered analogous to (M,N)-Schreier in [16]. Our goal is to tailor the definition of (M,N,q)-Schreier so as to make it as apparent as possible that the spaces found in [3] and [13] are (M,N,q)-Schreier for specified q.

Definition 2.2. For sequences $M = (m_i)_{i=1}^\infty$ and $N = (n_i)_{i=1}^\infty$ satisfying (i) and (ii) we call a norming set $\mathcal{N}$, (M,N,q)-Schreier (for $1/q + 1/p = 1$ and $1 < p, q < \infty$) if for the following sets, with $k \in \mathbb{N}$,

$$\mathcal{N}_k = \Big\{\frac{1}{m_{2k}} \sum_i \gamma_i x_i^* : (\gamma_i)_i \in Ba(\ell_q), \\ \gamma_i \in \mathbb{Q}, (x_i^*)_i \text{ is } S_{n_{2k}} \text{ admissible and } (x_i^*)_i \subset \mathcal{N}\Big\},$$

$$\mathcal{N}_\infty^q = \bigcup_{k=0}^{\infty} \left\{ \frac{1}{m_{2k+1}} \sum_i \gamma_i E x_i^* : (\gamma_i)_i \in 2^{1/p} Ba(\ell_q), \gamma_i \in \mathbb{Q}, \right.$$
$$\left. E \text{ is an interval} \subset \mathbb{N},\ (x_i^*)_i \text{ is an } S_{n_{2k+1}} \text{ admissible } \subset \bigcup_{j=1}^{\infty} \mathcal{N}_j \right\}$$

we have $\mathcal{N} \subset \bigcup_{j=1}^{\infty} \mathcal{N}_j \cup \mathcal{N}_\infty^q \cup \{\pm e_n : n \in \mathbb{N}\}$ and $\mathcal{N}_j \subset \mathcal{N}$ for all j.

In the case of the Banach space constructed in [3], there are fixed sequences $M = (m_i)_{i=1}^{\infty}$ and $N = (n_i)_{i=1}^{\infty}$. We suppose further that M and N satisfy conditions (i), (ii). It follows directly from the definition that the norming set for this space is $(M, N, 2)$-Schreier.

For the asymptotic ℓ_p HI space, $X_{(p)}$, found in [13] the reasoning is similar. Assume that the sequences M and N prescribed in [13] satisfy conditions (i), (ii). For a fixed p and $1/q + 1/p = 1$, we must show that the norming set $\mathcal{N}$ (denoted K in [13]) is (M, N, q)-Schreier. The reader should refer to [13] for the precise definitions of K, K^n, and K_j^n. $K = \bigcup_{n=1}^{\infty} K^n$ where $K^n = \bigcup_{j=1}^{\infty} K_j^n$. By definition, for $j, n \in \mathbb{N}$, $K_{2j}^n \subset \mathcal{N}_j$, and $K_{2j+1}^n \subset \mathcal{N}_\infty^q$ (for the latter inclusion the factor $2^{1/p}$ in the definition of $\mathcal{N}_\infty^q$ is required). Thus, $K \subset \bigcup_j \mathcal{N}_j \cup \mathcal{N}_\infty^q \cup \{\pm e_n : n \in \mathbb{N}\}$. If $x^* \in \mathcal{N}_j$, then $x^* = 1/m_{2j} \sum_i \gamma_i x_i^*$ where $(x_i^*)_i \subset \mathcal{N} = K$, so again by the definition of K, we see that $\mathcal{N}_j \subset \mathcal{N}$.

It is convenient to view an element of $\mathcal{N}$ as successive blocks of the basis $(e_i^*)_i$. This decomposition into blocks is not unique, and thus our goal is to find a decomposition that is the most suitable. To this end, we associate each element of $\mathcal{N}$ with a rooted tree. A finite set with a partial ordering $(\mathcal{T}, \preceq)$ is called a tree if for every $\alpha \in \mathcal{T}$ the set $\{\beta \in \mathcal{T} : \beta \preceq \alpha\}$ is linearly ordered. Each element of the tree $\mathcal{T}$ is called a node. A node $\alpha \in \mathcal{T}$, such that there is no β with $\alpha \prec \beta$ is called terminal ($\alpha \prec \beta$ means $\alpha \preceq \beta$ and $\alpha \neq \beta$). If $\beta \prec \alpha$, we say α is a successor of β. For $\alpha \in \mathcal{T}$, let $D_\alpha(\mathcal{T})$ denote the set of immediate successors of α in $\mathcal{T}$. A branch of $\mathcal{T}$ is a maximal linearly ordered subset.

For each $\alpha \in \mathcal{T}$, we define corresponding $\gamma_\alpha \in \mathbb{Q}$, $m_\alpha \in (m_i)_{i=1}^{\infty}$ and $n_\alpha \in (n_i)_{i=1}^{\infty}$. We associate to each $x^* \in \mathcal{N}$ a rooted tree $\mathcal{T}$ (i.e., a tree with a unique first node) in the following way: Let α_0 be the root of $\mathcal{T}$. There is an $x_{\alpha_0}^* \in \mathcal{N}$, such that $x^* = \gamma_{\alpha_0} x_{\alpha_0}^*$ and

$$x_{\alpha_0}^* = \frac{1}{m_{\alpha_0}} \sum_{\beta \in D_{\alpha_0}(\mathcal{T})} \gamma_\beta x_\beta^*.$$

In this definition $(x_\beta^*)_{\beta \in D_{\alpha_0}(\mathcal{T})}$ is $S_{n_{\alpha_0}}$ admissible, $m_{\alpha_0} = m_j$ for some $j \in \mathbb{N}$ and $(\gamma_\beta)_{\beta \in D_{\alpha_0}(\mathcal{T})} \in Ba(\ell_q)$ if j is even and $(\gamma_\beta)_{\beta \in D_{\alpha_0}(\mathcal{T})} \in 2^{1/p} Ba(\ell_q)$ if j is odd. Thus, for any pairwise incomparable collection A of $\mathcal{T}$ which intersects every branch of $\mathcal{T}$ we have

$$x^* = \sum_{\alpha \in A} \frac{\prod_{\beta \preceq \alpha} \gamma_\beta}{\prod_{\beta \prec \alpha} m_\beta} x^*_\alpha. \tag{1}$$

Call $(x^*_\alpha)_{\alpha \in \mathcal{T}}$ the functional tree of x^*. For $\beta \in \mathcal{T}$, the functional x^*_β has a corresponding tree $\mathcal{T}_\beta$, which is a subset of $\mathcal{T}$.

3. Main results

We start this section by stating the main theorem of the paper. The proof of this theorem can be found at the end of this section. The majority of this section is devoted to proving auxiliary lemmas and remarks.

THEOREM 3.1. *Let $\mathcal{N}$ be a norming set which is (M, N, q)-Schreier and $X_\mathcal{N}$ be the corresponding Banach space. There is an operator on $X_\mathcal{N}$ which is not a compact perturbation of a multiple of the identity. If $X_\mathcal{N}$ is HI, this operator is strictly singular. Moreover, ℓ_∞ embeds isomorphically into $\mathcal{L}(X_\mathcal{N})$.*

The existence of the following sequence in the space $X^*_\mathcal{N}$ is the main ingredient in the construction of the desired operator. The definition below is tailored to fit our construction.

Definition 3.2. Let $(x_k)_k$ be a block basic sequence of $(e_k)_k$. If there is a $C > 0$ such that for all $l \in \mathbb{N}$, $F \subset \mathbb{N}$ with $F \geq l$ and $(x_k)_{k \in F}$ being S_{f_l} admissible we have $\| \sum_{k \in F} \beta_k x_k \| \leq C \| (\beta_k)_{k \in F} \|_q$ for every scalar sequence $(\beta_k)_{k \in F}$, we say $(x_k)_k$ satisfies an upper ℓ^ω_q estimate with constant C.

For the rest of the section, we fix p, q, and $\mathcal{N}$, such that $1/p + 1/q = 1$ and $\mathcal{N}$ is a (M, N, q)-Schreier norming set. It follows easily that any normalized block sequence $(x_i)_{i=1}^m$ with $m \leq \operatorname{supp} x_1$ in $X_\mathcal{N}$, satisfies a lower ℓ_p estimate with constant $1/m_2$. The next remark demonstrates that $X_\mathcal{N}$ is an asymptotic ℓ_p space by verifying that is satisfies an upper ℓ_p estimate on normalized blocks.

REMARK 3.3. *Let $(x_i)_{i=1}^m \in X_\mathcal{N}$ be a normalized block basic sequence of $(e_i)_i$. For any sequence of scalars $(a_i)_i$, the following holds:*

$$\left\| \sum_{i=1}^m a_i x_i \right\|_\mathcal{N} \leq 12 \left(\sum_{i=1}^m |a_i|^p \right)^{\frac{1}{p}}.$$

Proof. For $x^* \in \mathcal{N}$, let $o(x^*)$ denote the height (i.e., the length of the longest branch) of the tree $\mathcal{T}$ associated with x^*. We proceed by induction on $o(x^*)$.

We will show that for all $x^* \in \mathcal{N}$, such that $o(x^*) = n$, $x^*(\sum_{i=1}^m a_i x_i) \leq 12(\sum_{i=1}^m |a_i|^p)^{\frac{1}{p}}$ holds for any normalized block basic sequence $(x_i)_{i=1}^m$ of $(e_i)_i$ and any sequence of scalars $(a_i)_i$. For $x^* \in \mathcal{N}$, such that $o(\mathcal{T}) = 1$, the assertion follows easily. Assume the claim for all $y^* \in \mathcal{N}$, such that $o(y^*) < n$ and let $x^* = 1/m_k \sum_j \gamma_j x_j^* \in \mathcal{N}$ with $o(x^*) = n$. By definition of $\mathcal{N}$, $(x_j^*)_j$ is S_{n_k} admissible and $(\gamma_j)_j \in 2^{1/p} Ba(\ell_q)$. Define the following two sets:

$$Q(1) = \{1 \leq i \leq m : \text{there is exactly one } j \text{ such that } r(x_j^*) \cap r(x_i) \neq \emptyset\},$$

and $Q(2) = \{1, \dots, m\} \setminus Q(1)$. Apply the functional x^* to $\sum_{i=1}^m a_i x_i$ to obtain

$$\begin{aligned}
&\left| \frac{1}{m_k} \sum_j \gamma_j x_j^* \left(\sum_{i=1}^m a_i x_i \right) \right| \\
&\quad \leq \frac{1}{m_k} \sum_j |\gamma_j| \left| x_j^* \sum_{\substack{i \in Q(1) \\ r(x_j^*) \cap r(x_i) \neq \emptyset}} a_i x_i \right| + \frac{1}{m_k} \left| \sum_j \gamma_j x_j^* \sum_{i \in Q(2)} a_i x_i \right| \\
&\quad \leq \frac{12}{m_k} \sum_j |\gamma_j| \Bigg(\sum_{\substack{i \in Q(1) \\ r(x_j^*) \cap r(x_i) \neq \emptyset}} |a_i|^p \Bigg)^{\frac{1}{p}} + \sum_{i \in Q(2)} |a_i| \left| \frac{1}{m_k} \sum_j \gamma_j x_j^*(x_i) \right|.
\end{aligned}$$

The first inequality follows from the triangle inequality. The second follows from applying the induction hypothesis for $x_j^*(\sum_{\{i \in Q(1): r(x_j^*) \cap r(x_i) \neq \emptyset\}} a_i x_i)$ and using the definition of $Q(2)$. We may apply the induction hypothesis since the height of the trees associated with the functionals x_j^* are each less than n. Before continuing, notice that for each $i \in Q(2)$ the set $J_i = \{j : r(x_j^*) \cap (x_i) \neq \emptyset\}$ is an interval, and therefore

$$\frac{1}{m_k} \sum_{j \in J_i} \frac{\gamma_j}{(\sum_{j \in J_i} |\gamma_j|^q)^{1/q}} x_j^* \in \mathcal{N}. \tag{2}$$

The above estimate continues as follows

$$\begin{aligned}
&\leq 4 \sum_j |\gamma_j| \Bigg(\sum_{\substack{i \in Q(1) \\ r(x_j^*) \cap r(x_i) \neq \emptyset}} |a_i|^p \Bigg)^{\frac{1}{p}} + \sum_{i \in Q(2)} |a_i| \left(\sum_{j \in J_i} |\gamma_j|^q \right)^{\frac{1}{q}} \\
&\leq 4 \left(\sum_j |\gamma_j|^q \right)^{\frac{1}{q}} \Bigg(\sum_j \sum_{\substack{i \in Q(1) \\ r(x_j^*) \cap r(x_i) \neq \emptyset}} |a_i|^p \Bigg)^{\frac{1}{p}} \\
&\quad + \left(\sum_{i \in Q(2)} |a_i|^p \right)^{\frac{1}{p}} \left(\sum_{i \in Q(2)} \sum_{j \in J_i} |\gamma_j|^q \right)^{\frac{1}{q}}
\end{aligned}$$

$$\leq 4\Big(\sum_j |\gamma_j|^q\Big)^{\frac{1}{q}}\Big(\sum_j \sum_{\substack{i\in Q(1)\\ r(x_j^*)\cap r(x_i)\neq\emptyset}} |a_i|^p\Big)^{\frac{1}{p}} + \Big(\sum_{i\in Q(2)} |a_i|^p\Big)^{\frac{1}{p}}\Big(2\sum_j |\gamma_j|^q\Big)^{\frac{1}{q}}$$

$$\leq 12\Big(\sum_i^m |a_i|^p\Big)^{\frac{1}{p}}.$$

In the first inequality, we used the fact that $3 < m_1$ in the first term and (2) in the second term. For the second inequality, we applied Hölders inequality. For the third inequality, we used the fact that for each j there are at most two values of $i \in Q(2)$ such that $r(x_j^*) \cap r(x_i) \neq \emptyset$. For the final inequality, we used $(\sum_\ell |\gamma_\ell|^q)^{1/q} \leq 2$. This finishes the proof. □

The following is a compilation of remarks (variants of which can be found in [16]) regarding the sequences $(m_i)_{i=1}^\infty, (n_i)_{i=1}^\infty$ and $(f_i)_{i=2}^\infty$. In the interest of completeness, we have included the proofs.

(1.1) If $p_k = 5n_1 + \sum_{i<k} s_i n_{2i}$ for $k \geq 2$, then $p_k \leq 2f_k$.

(1.2) If $(a_i)_{i=1}^{k-1}$ is a sequence of nonnegative integers and $a \in \mathbb{N} \cup \{0\}$, such that $m_1^a \prod_{i<k} m_{2i}^{a_i} < m_{2k}$ then $an_1 + \sum_{i<k} a_i n_{2i} < p_k$.

(1.3) Let $(a_\ell)_{\ell=1}^{k-1}$ be a sequence of nonnegative integers, $(x_i^*)_{i=1}^t \in \mathcal{N}$ be $S_{\sum_{l<k} a_l n_{2l}}$ admissible and $(\beta_i)_{i=1}^t \in Ba(\ell_q)$, then we have

$$\frac{1}{\prod_{\ell<k} m_{2\ell}^{a_\ell}} \sum_{i=1}^t \beta_i x_i^* \in \mathcal{N}.$$

The proof of (1.1) follows by induction. For $k = 2$, we have $f_2 = 4n_1 + (s_1 - 1)n_2$. Since $s_1 \geq 2$, the claim follows. Suppose the statement is true for some $k \geq 2$. Let $f_k = \gamma n_1 + \sum_{i<k} \gamma_i n_{2i}$, and observe that

$$\begin{aligned} p_{k+1} &= 5n_1 + \sum_{i<k} s_i n_{2i} + s_k n_{2k} \\ &\leq 2\Big(\gamma n_1 + \sum_{i<k} \gamma_i n_{2i}\Big) + s_k n_{2k} \quad \text{(by the induction hypothesis)} \\ &\leq 2\Big(\gamma n_1 + \sum_{i<k} \gamma_i n_{2i} + s_k n_{2k}\Big) \leq 2 f_{k+1}. \end{aligned}$$

We obtained the third inequality by noting that $m_1^\gamma \prod_{i<k} m_{2i}^{\gamma_i} m_{2k}^{s_k} < m_{2k} m_{2k}^{s_k} = m_{2k+2}$ and using the maximality of f_{k+1}.

To prove (1.2), again proceed by induction. For $k = 2$, deduce from the hypothesis that $a + 5a_1 < 5s_1$. Clearly, $a_1 < s_1$. If $a < 5$, we are done. Suppose $5n \leq a < 5(n+1)$ for some $n \in \mathbb{N}$. This implies that $a_1 < s_1 - n$. The following inequality finishes the proof of the base step

$$an_1 + a_1 n_2 < an_1 + (s_1 - n)n_2 \le s_1 n_2 + 5(n+1)n_1 - nn_2 < 5n_1 + s_1 n_2.$$

The final inequality follows from $5n_1 < n_2$.

Assume the statement is true for some $k \ge 2$. By assumption, $m_1^a \prod_{i<k+1} m_{2i}^{a_i} < m_{2k+2}$ and by definition $m_{2k+2} = m_{2k}^{s_k+1}$. Clearly, $m_1^a \prod_{i<k} m_{2i}^{a_i} < m_{2k}^{s_k - a_k + 1}$. Thus, $s_k + 1 \ge a_k$. This leaves two possibilities, either $s_k = a_k$ or $s_k > a_k$. In the former case, $m_1^a \prod_{i<k} m_{2i}^{a_i} < m_{2k}$. By the induction hypothesis, $an_1 + \sum_{i<k} a_i n_{2i} < p_k$ and thus, $an_1 + \sum_{i<k+1} a_i n_{2i} < p_{k+1}$. If $s_k > a_k$, we claim that $an_1 + \sum_{i<k+1} a_i n_{2i} < s_k n_{2k}$, which clearly finishes the proof. To see this, we start by showing that $an_1 + \sum_{i<k} a_i n_{2i} \le 2(s_k - a_k + 1) f_k$. By assumption $m_1^a \prod_{i<k} m_{2i}^{a_i} < m_{2k}^{s_k - a_k + 1}$, which implies that

$$m_1^{\lfloor \frac{a}{s_k - a_k + 1} \rfloor} \prod_{i<k} m_{2i}^{\lfloor \frac{a_i}{s_k - a_k + 1} \rfloor} < m_{2k},$$

where $\lfloor x \rfloor$ is the greatest integer of x. By the maximality of f_k, we have

$$\left\lfloor \frac{a}{s_k - a_k + 1} \right\rfloor n_1 + \sum_{i<k} \left\lfloor \frac{a_i}{s_k - a_k + 1} \right\rfloor n_{2i} \le f_k.$$

Since $x \le 2\lfloor x \rfloor$ for $x \ge 0$, we see that

$$\begin{aligned} &\frac{a}{s_k - a_k + 1} n_1 + \sum_{i<k} \frac{a_i}{s_k - a_k + 1} n_{2i} \\ &\quad \le 2\left(\left\lfloor \frac{a}{s_k - a_k + 1} \right\rfloor n_1 + \sum_{i<k} \left\lfloor \frac{a_i}{s_k - a_k + 1} \right\rfloor n_{2i} \right) \le 2 f_k. \end{aligned}$$

Finally, using $4f_k < n_{2k}$ to observe that

$$\begin{aligned} an_1 + \sum_{i<k+1} a_i n_{2i} &\le 2(s_k - a_k + 1) f_k + a_k n_{2k} < n_{2k}\big((s_k+1)/2 - a_k\big) + a_k n_{2k} \\ &< s_k n_{2k}. \end{aligned}$$

The proof of (1.3) requires a complicated induction. For simplicity, we prove the case where $a_j = a_l = 1$ for some $j, l \le k$. Suppose, $(x_i^*)_{i=1}^t \in \mathcal{N}$ is $S_{n_{2l}+n_{2j}}$ admissible. Let $(\beta_i)_{i=1}^t \in Ba(\ell_q)$. We wish to show that

$$\frac{1}{m_{2l} m_{2j}} \sum_{i=1}^t \beta_i x_i^* \in \mathcal{N}.$$

Do this by carefully grouping the functionals. Let $(J_k)_{k=1}^m$ be successive intervals of integers such that $\bigcup_{k=1}^m J_k = \{1, \dots, t\}$, $(x_i^*)_{i \in J_k}$ is $S_{n_{2l}}$ admissible for each $k \le m$ and $(x_{\min J_k}^*)_{k=1}^m$ is $S_{n_{2j}}$ admissible. Now, define a sequence

$(z_k^*)_{k=1}^m$ by

$$\frac{1}{m_{2l}m_{2j}}\sum_{i=1}^t \beta_i x_i^* = \frac{1}{m_{2l}m_{2j}}\sum_{k=1}^m\sum_{i\in J_k}\beta_i x_i^*$$
$$= \frac{1}{m_{2j}}\sum_{k=1}^m\Big(\sum_{i\in J_k}|\beta_i|^q\Big)^{\frac{1}{q}}\frac{1}{m_{2l}}\sum_{i\in J_k}\frac{\beta_i}{(\sum_{i\in J_k}|\beta_i|^q)^{1/q}}x_i^*$$
$$= \frac{1}{m_{2j}}\sum_{k=1}^m\Big(\sum_{i\in J_k}|\beta_i|^q\Big)^{\frac{1}{q}} z_k^*.$$

It is straightforward to check that $z_k^*\in\mathcal{N}_l$ for all $k\leq m$. The claim follows by observing that $(z_k^*)_{k=1}^m$ is $S_{n_{2j}}$ admissible since $(x^*_{\min J_k})_{k=1}^p$ is $S_{n_{2j}}$ admissible and $((\sum_{i\in J_k}|\beta_i|^q)^{1/q})_{k=1}^m\in Ba(\ell_q)$.

Before proceeding further, we pause briefly to discuss the structure of the proof of Theorem 3.1. The proof begins by introducing some auxiliary remarks and lemmas. Remark 3.4 and Lemma 3.5 follow from the technical definitions of the sequences $(n_i)_i$ and $(m_i)_i$ and the tree structure of the functionals in $\mathcal{N}$. Lemma 3.5 is quite specific to spaces which are (M,N,p) Schreier and will be used throughout the proof of Theorem 3.1. The main task at hand is to construct a sequence of functionals in $\mathcal{N}$ which are seminormalized and satisfy an upper ℓ_p^ω estimate with constant 1. We do this in Lemma 3.8. The construction of these functionals is rather straightforward; it is in proving that they possess the desired properties that we must make use of Lemma 3.5 and Corollary 3.6. Once we have constructed these norming functionals (and after making a few easy remarks), we are ready to define the operator. This is done in a very natural way. The fact that the operator is bounded and noncompact follows from the properties of the norming functionals from which it is built.

For any functional tree $\mathcal{T}$, we define a function $\varphi:\mathcal{T}\to\mathbb{N}\cup\{0\}$ in the following way.

$$\varphi(\beta)=\begin{cases} n_{2i} & \text{if } n_\beta=n_{2i} \text{ for some } i,\\ n_1 & \text{if } n_\beta=n_{2i+1} \text{ for some } i,\\ 0 & \text{if } \beta \text{ is terminal.}\end{cases}$$

Remark 3.4. *Let $(x_\alpha^*)_{\alpha\in\mathcal{T}}$ be a functional tree for some $x^*\in\mathcal{N}$, such that for $\alpha\in T$, $(x_\beta^*)_{\beta\in D_\alpha(\mathcal{T})}$ is $S_{\varphi(\mathcal{T})}$ admissible. For every subset A of $\mathcal{T}$ consisting of pairwise incomparable nodes, the collection $(x_\alpha^*)_{\alpha\in A}$ is S_d admissible where $d=\max\{\sum_{\beta\prec\alpha}\varphi(\beta):\alpha\in A\}$.*

Proof. We proceed by induction on $o(\mathcal{T})$. The base step is trivial. Let $k\geq 1$ assume the statement for $\mathcal{T}$ such that $o(\mathcal{T})<k+1$ and suppose $o(\mathcal{T})=k+1$. Let α_0 be the root of $\mathcal{T}$ and for $\alpha\in D_{\alpha_0}(\mathcal{T})$ let $\mathcal{T}_\alpha$ be the tree corresponding to x_α^*. For the given collection A and $\alpha\in D_{\alpha_0}(\mathcal{T})$ we can define $A_\alpha=\{\beta:\beta\in$

$T_\alpha \cap A\}$. Notice that $A = \bigcup_{\alpha\in D_{\alpha_0}(\mathcal{T})} A_\alpha$ or $A = \{\alpha_0\}$. Apply the induction hypothesis for each collection A_α to conclude that $(x^*_\beta)_{\beta\in A_\alpha}$ is S_{d_α} admissible for $d_\alpha = \max\{\sum_{\alpha_0\prec\gamma\prec\beta}\varphi(\gamma) : \beta\in A_\alpha\}$. Let $d_{\alpha_0} = \max_{\alpha\in D_{\alpha_0}(\mathcal{T})} d_\alpha$. The block sequence $((x^*_\beta)_{\beta\in A_\alpha})_{\alpha\in D_{\alpha_0}(\mathcal{T})}$ is $S_{d_{\alpha_0}+\varphi(\alpha_0)}$ admissible by the convolution property of Schreier families. Finish by observing that $d_{\alpha_0}+\varphi(\alpha_0) = \max\{\sum_{\beta\prec\alpha}\varphi(\beta) : \alpha\in A\}$ and $((x^*_\beta)_{\beta\in A_\alpha})_{\alpha\in D_{\alpha_0}(\mathcal{T})} = (x^*_\alpha)_{\alpha\in A}$. □

Our next lemma allows us to decompose norming functionals. Decompositions are extremely useful when attempting to find tight upper estimates on the norm of vectors in the space.

LEMMA 3.5 (Decomposition lemma). *Let $k\in\mathbb{N}$ and $x^*\in\mathcal{N}$ such that $\operatorname{supp} x^* \geq 2k$. There is an $m\in\mathbb{N}$, $x^*_1 < \cdots < x^*_m \in\mathcal{N}$, a partition I_1, I_2 of $\{1,\dots,m\}$ and scalars $(\lambda_i)_{i=1}^m$, such that:*

(a) $x^* = \sum_{i=1}^m \lambda_i x^*_i$.
(b) $x^*_i = \pm e^*_{j_i}$ *for* $i\in I_1$ *and* $\{j_i : i\in I_1\}\in S_{p_k-1}$.
(c) $(\sum_{i\in I_2}|\lambda_i|^q)^{1/q} \leq 2/m_{2k}$ *and* $(\sum_{i\in I_1\cup I_2}|\lambda_i|^q)^{1/q}\leq 2$.

Proof. Let $x^*\in\mathcal{N}$ and $k\in\mathbb{N}$. Let $\mathcal{T}$ be the tree corresponding to x^*. For each node β, there are corresponding m_β, n_β, and γ_β. Let $\mathcal{B}$ denote the set of branches of $\mathcal{T}$. For each branch $b\in\mathcal{B}$, let $\alpha(b)$ denote the node of b, such that either $\alpha(b)$ is the first node β for which $\prod_{\alpha\prec\beta} m_\alpha \geq m_{2k}$ holds, or the terminal node of b if no such β exists. Set $A = \{\alpha(b) : b\in\mathcal{B}\}$. Notice that A is a collection of pairwise incomparable nodes intersecting every branch of $\mathcal{B}$. Let A_1 denote the set of terminal nodes of A and $A_2 = A\setminus A_1$. Enumerate A with the set $\{1,\dots,m\}$ for some $m\in\mathbb{N}$ and define $I_t = \{i : x^*_i \in (x^*_\alpha)_{\alpha\in A_t}\}$ for $t\in\{1,2\}$. By (1) we have

$$x^* = \sum_{\alpha\in A}\frac{\prod_{\beta\preceq\alpha}\gamma_\beta}{\prod_{\beta\prec\alpha} m_\beta}x^*_\alpha, \quad \text{so,} \quad \text{set } \lambda_i = \frac{\prod_{\beta\preceq\alpha}\gamma_\beta}{\prod_{\beta\prec\alpha} m_\beta} \quad \text{if } x^*_i = x^*_\alpha.$$

It is left to verify that conditions (b) and (c) hold. Condition (c) follows from the fact that for each α, $(\gamma_\beta)_{\beta\in D_\alpha(\mathcal{T})}\in 2^{1/p}Ba(\ell_q)$, and observing that

$$\Big(\sum_{i\in I_2}|\lambda_i|^q\Big)^{1/q} = \Big(\sum_{\alpha\in A_2}\Big|\frac{\prod_{\beta\preceq\alpha}\gamma_\beta}{\prod_{\beta\prec\alpha} m_\beta}\Big|^q\Big)^{\frac{1}{q}} \leq \frac{1}{m_{2k}}\Big(\sum_{\alpha\in A_2}\Big|\prod_{\beta\preceq\alpha}\gamma_\beta\Big|^q\Big)^{\frac{1}{q}}$$
$$\leq \frac{2^{1/p}}{m_{2k}} < \frac{2}{m_{2k}}.$$

The second part of (c) follows similarly. The first part of (b) follows from the definition. For the second part of (b), we employ Remark 3.4. Let $\mathcal{R} = \bigcup_{\alpha\in A_1}\{\beta : \beta\prec\alpha\}$. For $\alpha\in\mathcal{R}$, such that $m_\alpha = m_{2j+1}$ for some $j\in\mathbb{N}$, $(x^*_\beta)_{\beta\in D_\alpha(\mathcal{R})}$ is S_1 and hence, S_{n_1} admissible. To see this, first note that for all $\beta\in\mathcal{R}$, $m_\beta < m_{2k}$. By the injectivity of the function σ (defined in $\mathcal{N}^q_\infty$) for

$\beta, \gamma \in D_\alpha(\mathcal{R})$, $m_\beta \neq m_\gamma < m_{2k}$. Since $\operatorname{supp} x^* \geq 2k$ we have that $(x^*_\beta)_{\beta \in D_\alpha(\mathcal{R})}$ is S_1 admissible. Thus, for $\alpha \in A_1$, $(x^*_\beta)_{\beta \in D_\alpha(\mathcal{R})}$ is $S_{\varphi(\alpha)}$ admissible. By Remark 3.4, $(x^*_\alpha)_{\alpha \in A_1}$ is S_d admissible where $d = \max\{\sum_{\beta \prec \alpha} \varphi(\beta) : \alpha \in A_1\}$.

Let $\alpha \in A_1$. We have $\prod_{\beta \prec \alpha} m_\beta = m_1^{b_1} \prod_{i<k} m_{2i}^{b_{2i}+5b_{2i+1}} < m_{2k}$, where $b_j = |\{\beta : \beta \prec \alpha, m_\beta = m_j\}|$. Apply (1.2) for $b_1 =$"a" and $b_{2i} + 5b_{2i+1} =$"a_i", to conclude that

$$b_1 n_1 + \sum_{i<k} (b_{2i} + 5b_{2i+1}) n_{2i} < \sum_{i<k} s_i n_{2i} = p_k.$$

We also have

$$\sum_{\beta \prec \alpha} \varphi(\beta) = \left(\sum_{0 \leq i < k} b_{2i+1} \right) n_1 + \sum_{1 \leq i < k} b_{2i} n_{2i} < b_1 n_1 + \sum_{1 \leq i < k} (b_{2i} + 5b_{2i+1}) n_{2i}.$$

This holds for all $\alpha \in A_1$ and thus, $\max\{\sum_{\beta \prec \alpha} \varphi(\beta) : \alpha \in A_1\} \leq p_k - 1$. □

COROLLARY 3.6. *Let $x^* \in \mathcal{N}$ and $k \in \mathbb{N}$. Decompose x^* as*

$$x^* = \sum_{\beta \in \max \mathcal{T}} \frac{\prod_{\alpha \preceq \beta} \gamma_\alpha}{\prod_{\alpha \prec \beta} m_\alpha} e^*_{j_\beta}.$$

Then the set

$$\left\{ j_\beta : |x^*(e_{j_\beta})| \geq \frac{2 \prod_{\alpha \preceq \beta} \gamma_\alpha}{m_{2k}}, j_\beta \geq 2k \right\}$$

is $S_{p_k - 1}$ admissible.

Proof. For $k \in \mathbb{N}$, we can assume without loss of generality that $\operatorname{supp} x^* \geq 2k$. Apply the decomposition lemma to x^* to obtain I_1 and I_2, such that

$$x^* = \sum_{i \in I_1} \lambda_i e^*_{j_i} + \sum_{i \in I_2} \lambda_i x^*_i,$$

where $\{j_i : i \in I_1\} \in S_{p_k - 1}$. We claim that,

$$\left\{ j_\beta : |x^*(e_{j_\beta})| \geq \frac{2 \prod_{\alpha \preceq \beta} \gamma_\alpha}{m_{2k}}, j_\beta \geq 2k \right\} \subset \{j_i : i \in I_1\}.$$

If this were not the case, then for some $i_0 \in I_2$

$$\frac{2 \prod_{\alpha \preceq \beta} \gamma_\alpha}{m_{2k}} \leq |x^*(e_{j_\beta})| = |\lambda_{i_0} x^*_{i_0}(e_{j_\beta})| \leq |\lambda_{i_0}|.$$

From the proof of the decomposition lemma,

$$\lambda_{i_0} = \frac{\prod_{\alpha \preceq \beta} \gamma_\alpha}{\prod_{\alpha \prec \beta} m_\alpha} \quad \text{for some } \beta \in A_2.$$

For $\beta \in A_2$, we have that $\prod_{\alpha \prec \beta} m_\alpha \geq m_{2k}$ serving as our contradiction. □

Before passing to the main lemma of the paper, we state the following fact concerning the existence of a particular sequence of scalars. These scalars are called repeated hierarchy averages and were first studied in [9] and later in [8, 18]. These averages are defined in [3] for $q=2$. In [16], a similar fact is established for $q=1$.

Fact 3.7. For any $1\le q<\infty$ and $\varepsilon>0$, there exist successive subsets of $\mathbb{N}$, $(F_k)_{k=1}^\infty$, and scalars $(a_{k,i})_{i\in F_k}$, such that for each $k\in\mathbb{N}$, $F_k\ge 2k$, $F_k\in S_{p_k}$, $\|(a_{k,i})_{i\in F_k}\|_q=1$ and $(\sum_{i\in G}|a_{k,i}|^q)^{1/q}<\varepsilon$ for $G\in S_{p_k-1}$.

The next lemma establishes the existence of a seminormalized block sequence satisfying an upper ℓ_q^ω estimate with constant 1 in $X_{\mathcal{N}}$. These blocks are constructed using Fact 3.7 and used to construct the desired operator on $X_{\mathcal{N}}$.

LEMMA 3.8. *Let $(F_k)_{k=1}^\infty$ be successive subsets of $\mathbb{N}$ and scalars $(a_{k,i})_{i\in F_k}$ be such that $F_k\ge 2k$, $F_k\in S_{p_k}$, $\|(a_{k,i})_{i\in F_k}\|_q=1$ and $(\sum_{i\in G}|a_{k,i}|^q)^{1/p}<1/m_{2k}$ for all $G\in S_{p_k-1}$ and each $k\in\mathbb{N}$. The sequence of functionals $(x_k^*)_{k=1}^\infty\in\mathcal{N}$ defined by, $x_k^*=1/m_{2k}\sum_{i\in F_k}a_{k,i}e_i^*$, are seminormalized and satisfy an upper ℓ_q^ω-estimate with constant* 1.

Proof. We start by making an observation concerning the decomposition of each x_k^*. For fixed k and $k_0\le k$, write $F_k=\bigcup_{r=1}^{d_k}J_{k,r}$, such that $J_{k,1}<\cdots<J_{k,d_k}$, each $J_{k,r}$ is $S_{p_k-p_{k_0}}$ admissible and $(J_{k,r})_{r=1}^{d_k}$ is $S_{p_{k_0}}$ admissible (we can do this because F_k is S_{p_k} admissible). Then

$$x_k^*=\frac{1}{m_{2k_0}}\sum_{r=1}^{d_k}\biggl(\sum_{i\in J_{k,r}}|a_{k,i}|^q\biggr)^{\frac{1}{q}}z_{k,r}^*\quad\text{for}$$
$$z_{k,r}^*=\frac{m_{2k_0}}{m_{2k}}\sum_{i\in J_{k,r}}\frac{a_{k,i}}{(\sum_{i\in J_{k,r}}|a_{k,i}|^q)^{1/q}}e_i^*.$$

Since $m_{2k_0}/m_{2k}=1/\prod_{2k_0\le\ell<2k}m_\ell^{s_\ell}$ and $(e_i^*)_{i\in J_{k,r}}$ is $S_{p_k-p_{k_0}}$ admissible, we conclude by (1.3) that $z_{k,r}^*\in\mathcal{N}$ for all $r\le d_k$. Since, $\min J_{k,r}=\min\operatorname{supp}z_{k,r}^*$, we have that $(z_{k,r}^*)_{r=1}^{d_k}$ is $S_{p_{k_0}}$ admissible.

We now show that $(x_k^*)_k$ satisfies and upper ℓ_q^ω estimate with constant 1. For starters, let $k_0\in\mathbb{N}$ and $F\subset\mathbb{N}$ with $F\ge k_0$, such that $(x_k^*)_{k\in F}$ is $S_{f_{k_0}}$ admissible. For every $k\in F$ we apply the above (since $F\ge k_0$) to define $(z_{k,r}^*)_{r=1}^{d_k}$. The block sequence $((z_{k,r}^*)_{r=1}^{d_k})_{k\in F}$ is $S_{p_{k_0}+f_{k_0}}$ admissible, by the convolution property of Schreier families. Hence, it is $S_{n_{2k_0}}$ admissible by (1.1) and the hereditary property of Schreier families. To conclude, it suffices to let $(\beta_k)_{k\in F}\in Ba(\ell_q)$ and show that $\sum_{i\in F}\beta_i x_i^*\in\mathcal{N}$. We do this by observing the following equality:

$$\sum_{k\in F}\beta_k x_k^* = \sum_{k\in F}\beta_k\frac{1}{m_{2k_0}}\sum_{r=1}^{d_k}\bigg(\sum_{i\in J_{k,r}}|a_{k,i}|^q\bigg)^{\frac{1}{q}} z_{k,r}^*$$
$$=\frac{1}{m_{2k_0}}\sum_{k\in F}\sum_{r=1}^{d_k}\beta_k\bigg(\sum_{i\in J_{k,r}}|a_{k,i}|^q\bigg)^{\frac{1}{q}} z_{k,r}^*.$$

Since $(\beta_k(\sum_{i\in J_{k,r}}|a_{k,i}|^q)^{\frac{1}{q}})_k\in Ba(\ell_q)$ and $((z_{k,r}^*)_{r=1}^{d_k})_{k\in F}$ is $S_{n_{2k_0}}$ admissible, it follows that $\sum_{k\in F}\beta_k x_k^*\in\mathcal{N}$. Thus, $(x_k^*)_k$ satisfies a upper ℓ_q^ω-estimate with constant 1.

To show that $(x_k^*)_k$ is seminormalized, it suffices to find a uniform lower bound. For each k, define $x_k=\sum_{j\in F_k}a_{k,j}^{q/p}e_j$. It suffices to show that $\|x_k\|\leq 26/m_{2k}$. From this, it follows easily that $\|x_k^*\|\geq 1/26$ for all $k\in\mathbb{N}$. Let $x^*\in\mathcal{N}$ be an arbitrary norming functional which we may assume without loss of generality satisfies $\operatorname{supp}x^*\geq 2k$ (since $F_k\geq 2k$). By applying the decomposition lemma for $k\in\mathbb{N}$ and x^*, we can estimate $\|x_k\|$ from above as follows:

$$|x^*(x_k)|\leq\bigg|\sum_{i\in I_1}\lambda_i e_{j_i}^*(x_k)\bigg|+\bigg|\sum_{i\in I_2}\lambda_i y_i^*(x_k)\bigg|$$
$$\leq\sum_{i\in I_1}|\lambda_i||a_{k,j_i}|^{\frac{q}{p}}+\sum_{i\in I_2}|\lambda_i|12\bigg(\sum_{\{j:j\in\operatorname{supp}y_i\cap F_k\}}|a_{k,j}|^q\bigg)^{\frac{1}{p}}$$
$$\leq\bigg(\sum_{i\in I_1}|\lambda_i|^q\bigg)^{\frac{1}{q}}\bigg(\sum_{i\in I_1}|a_{k,j_i}|^q\bigg)^{\frac{1}{p}}$$
$$+12\bigg(\sum_{i\in I_2}|\lambda_i|^q\bigg)^{\frac{1}{q}}\bigg(\sum_{i\in I_2}\sum_{\{j:j\in\operatorname{supp}y_i\cap F_k\}}|a_{k,j}|^q\bigg)^{\frac{1}{p}}$$
$$\leq 2\frac{1}{m_{2k}}+12\frac{2}{m_{2k}}=\frac{26}{m_{2k}}.$$

The first inequality follows from the decomposition lemma and the triangle inequality. The second inequality follows from the triangle inequality, the definition of x_k, and Remark 3.3. The third follows from two applications of Hölders inequality. For the last inequality, we used condition (c) of the decomposition lemma, the fact that $(j_i)_{i\in I_1}$ is S_{p_k-1} admissible (by condition (b) of the decomposition lemma) and the definition of $(a_{k,i})_{i\in F_k}$. This concludes the proof. □

We make two final remarks before proceeding with the proof of the main theorem.

REMARK 3.9. *Let $(y_i^*)_i$ be the even subsequence of the seminormalized block sequence $(x_i^*)_i$ satisfying an upper ℓ_q^ω estimate with constant 1 defined in Lemma 3.8. Let $k\in\mathbb{N}$, $F\subset\mathbb{N}$, with $F\geq k$ such that $(y_i^*)_{i\in F}$ is $S_{n_{2k}}$ admissible. Then $\|\sum_{i\in F}\beta_i y_i^*\|\leq 1$ for all $(\beta_i)_i\in Ba(\ell_q)$.*

Proof. Let $k\in\mathbb{N}$, F be a subset of $\mathbb{N}$, with $F\geq k$ and $(y_i^*)_{i\in F}$ being $S_{n_{2k}}$ admissible. Set $G=\{i: i=2j, j\in F\}$ and note that $(y_i)_{i\in F}=(x_i^*)_{i\in G}$. Since $F\geq k$, we have $i\geq k+1$ for all $i\in G$. Since $(x_i^*)_i$ satisfies an upper ℓ_q^ω, estimate $G\geq k+1$ and $(x_i^*)_{i\in G}$ is $S_{n_{2k}}$ admissible and thus, $S_{f_{k+1}}$ admissible we have:

$$\Big\|\sum_{i\in F}\beta_i y_i^*\Big\|=\Big\|\sum_{i\in G}\beta_i x_i^*\Big\|\leq 1.$$

This concludes the proof. ☐

REMARK 3.10. *Let $(y_i^*)_i$ be the subsequence from Remark 3.9. For every $x\in S(X)$, $k\in\mathbb{N}$, $F\subset\mathbb{N}$ with $F\geq k$ and $(y_i^*)_{i\in F}$ being $S_{n_{2k}}$ admissible we have $(y_i^*(x))_{i\in F}\in Ba(\ell_p)$.*

Proof. Let $x\in S(X)$, $k\in\mathbb{N}$ and $F\subset\mathbb{N}$ with $F\geq k$ such that $(y_i^*)_{i\in F}$ is $S_{n_{2k}}$ admissible. By Remark 3.9 for all $(\beta_i)_{i\in F}\in Ba(\ell_q)$, we have $\|\sum_{i\in F}\beta_i y_i^*\|\leq 1$. Apply this for

$$\beta_i=\frac{|y_i^*(x)|^{p/q}\,\mathrm{sign}(y_i^*(x))}{(\sum_{j\in F}|y_j^*(x)|^p)^{1/q}}$$

and estimate $\|\sum_{i\in F}\beta_i y_i^*\|$ from below with x. ☐

Proof of Theorem 3.1. We are now ready to define the desired operator on $X_{\mathcal{N}}$. Let $(y_i^*)_i$ be the seminormalized block sequence from Remark 3.9. For $x\in c_{00}$, define the operator $T: c_{00}\to c_{00}$ by $Tx=\sum_{i=1}^\infty y_i^*(x)e_i$. Once we show that T is a bounded operator, it can be extended as on operator defined on $X_{\mathcal{N}}$.

Since $(y_i^*)_i$ is a seminormalized block sequence, it follows that T is noncompact. In the case that $X_{\mathcal{N}}$ is an HI space, T must be strictly singular. Since $dim(\operatorname{Ker}T)=\infty$, if there was an infinite dimensional subspace Y of $X_{\mathcal{N}}$, such that $T|_Y$ was an isomorphism. $Y+\operatorname{Ker}(T)$ would be a direct sum. Contradicting the fact that $X_{\mathcal{N}}$ is HI. (It is known that the spaces constructed in [13] have few operators. Using similar techniques, it can be further shown that the space constructed in [3] has few operators.)

Our final task is to demonstrate that T is bounded. Let $x\in S(X)$ and $x^*\in\mathcal{N}$. If $x^*=\pm e_j^*$ for some j, then $|x^*(Tx)|\leq 1$. Thus, assume $x^*\in\mathcal{N}$, such that $|\operatorname{supp}x^*|>1$. Suppose x^* has the following decomposition:

$$x^*=\sum_{\beta\in\max\mathcal{T}}\frac{\prod_{\alpha\preceq\beta}\gamma_\alpha}{\prod_{\alpha\prec\beta}m_\alpha}e_{j_\beta}^*.$$

Define

$$H_2 = \left\{ j_\beta : \frac{2\prod_{\alpha\preceq\beta}\gamma_\alpha}{m_4} \leq |x^*(e_{j_\beta})| < \frac{\prod_{\alpha\preceq\beta}\gamma_\alpha}{m_1} \right\}.$$

For $k>2$, define

$$H_k = \left\{ j_\beta : \frac{2\prod_{\alpha\preceq\beta}\gamma_\alpha}{m_{2k}} \leq |x^*(e_{j_\beta})| < \frac{2\prod_{\alpha\preceq\beta}\gamma_\alpha}{m_{2(k-1)}} \right\}.$$

For $k>2$, define $G_k = \{j_\beta \in H_k : j_\beta \geq 2k\}$. Clearly, $\operatorname{supp} x^* = \bigcup_{k=2}^\infty H_k$. Apply Corollary 3.6 to deduce that $G_k \in S_{p_k-1}$. By (1.1) and (ii), $G_k \in S_{n_{2k}}$. By the spreading property of Schreier families, $(y_i)_{i\in G_k}$ is $S_{n_{2k}}$ admissible for all k. For each k, apply Remark 3.10 to deduce that

$$\left(\sum_{i\in G_k}|y_i^*(x)|^p\right)^{\frac{1}{p}} \leq 1, \qquad \left(\sum_{i\in H_2}|y_i^*(x)|^p\right)^{\frac{1}{p}} \leq 1. \tag{3}$$

Estimate $|x^*(Tx)|$ from above in the following way:

$$\begin{aligned}
x^*\left(\sum_{i=1}^\infty y_i^*(x)e_i\right) &\leq \sum_{i\in H_2}|y_i^*(x)||x^*(e_i)| \\
&\quad + \sum_{k=3}^\infty\Bigg[\sum_{i\in G_k}|y_i^*(x)||x^*(e_i)| + \sum_{i\in H_k\setminus G_k}|y_i^*(x)||x^*(e_i)|\Bigg] \\
&\leq \left(\sum_{i\in H_2}|y_i^*(x)|^p\right)^{\frac{1}{p}}\left(\sum_{i\in H_2}|x^*(e_i)|^q\right)^{\frac{1}{q}} \\
&\quad + \sum_{k=3}^\infty\Bigg[\left(\sum_{i\in G_k}|y_i^*(x)|^p\right)^{\frac{1}{p}}\left(\sum_{i\in G_k}|x^*(e_i)|^q\right)^{\frac{1}{q}} \\
&\quad + \left(\sum_{i\in H_k\setminus G_k}|y_i^*(x)|^p\right)^{\frac{1}{p}}\left(\sum_{i\in H_k\setminus G_k}|x^*(e_i)|^q\right)^{\frac{1}{q}}\Bigg] \\
&< \left(\sum_{j_\beta\in H_2}\left|\prod_{\alpha\preceq\beta}\gamma_\alpha\right|^q\right)^{\frac{1}{q}}\frac{1}{m_1} \\
&\quad + \sum_{k=3}^\infty\Bigg[\left(\sum_{j_\beta\in G_k}\left|\prod_{\alpha\preceq\beta}\gamma_\alpha\right|^q\right)^{\frac{1}{q}}\frac{2}{m_{2(k-1)}} \\
&\quad + \left(\sum_{j_\beta\in H_k\setminus G_k}\left|\prod_{\alpha\preceq\beta}\gamma_\alpha\right|^q\right)^{\frac{1}{q}}\frac{2(k-1)}{m_{2(k-1)}}\Bigg] \\
&\leq \frac{2}{m_1} + \sum_{k=3}^\infty\frac{4}{m_{2(k-1)}} + \frac{4(k-1)}{m_{2(k-1)}} = M.
\end{aligned}$$

The first inequality follows from the triangle inequality and the definitions of H_k, G_k. For the second, we apply Hölders inequality to each of the terms. For the first and second terms of the third inequality, we used (3) and the definition of H_k. For the third term of the third inequality, we used the fact that $|x^*(e_i)| \leq 1$ for all i, $|H_k \setminus G_k| \leq k-1$ and the definition of H_k. For the final inequality, we used the fact that $(|\prod_{\alpha \preceq \beta} \gamma_\alpha|)_{\beta \in A} \in 2Ba(\ell_q)$ for $A = H_k, G_k$ or $H_k \setminus G_k$. Thus, $\|T\| \leq \max\{M, 1\}$.

We conclude by noting that ℓ_∞ embeds isomorphically into $\mathcal{L}(X_{\mathcal{N}})$ via the mapping

$$(a_i)_{i=1}^\infty \longmapsto SOT\text{-}\lim_{n\to\infty} \sum_{i=1}^n a_i y_i^* \otimes e_i.$$

Here, "SOT-lim" denotes the strong operator topology limit. To see that this is a bounded isomorphism one merely follows, almost identically, the previous calculation. □

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