ON THE ZETA FUNCTION OF A PROJECTIVE COMPLETE INTERSECTION

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ABSTRACT. We compute a basis for the p-adic Dwork cohomology of a smooth complete intersection in projective space over a finite field and use it to give p-adic estimates for the action of Frobenius on this cohomology. In particular, we prove that the Newton polygon of the characteristic polynomial of Frobenius lies on or above the associated Hodge polygon. This result was first proved by B. Mazur using crystalline cohomology.

1. Introduction

Let $X \subseteq \mathbf{P}^n$ be a smooth complete intersection of codimension r over a finite field $\mathbf{F}_q$ of characteristic p. The zeta function of X may be written in the form

$$Z(X/\mathbf{F}_q, t) = \frac{P(t)^{(-1)^{n-r-1}}}{(1-t)(1-qt)\cdots(1-q^{n-r}t)},$$

where $P(t) \in 1 + t\mathbf{Z}[t]$. The reciprocal roots of $P(t)$ are units at all primes except the archimedian primes and those primes lying over p. At any archimedian prime, they have absolute value $q^{(n-r)/2}$ by Deligne [7]; the p-adic Newton polygon of $P(t)$ lies over its Hodge polygon by Mazur [10, 11]. (The Hodge polygon is determined by the multidegree of the complete intersection.)

The hypersurface case of Mazur's result was originally proved by Dwork [9]. Adolphson–Sperber [2], using a toric approach, extended Dwork's method to (among other cases) the case where X is the intersection of smooth hypersurfaces of degrees prime to p that meet transversally and which also meet all coordinate varieties transversally (this latter condition can be ensured by a coordinate change if q is sufficiently large), but technical difficulties prevented the extension of Dwork's method to general complete intersections until now.

Recently, we gave an algebraic description of the Jacobian ring of a complete intersection (see [4]). This provides the algebraic basis for a proof of Mazur's theorem by a generalization of Dwork's work.

We describe our results. Suppose that X is defined by nonconstant homogeneous polynomials $f_1, \ldots, f_r \in \mathbf{F}_q[x_0, x_1, \ldots, x_n]$ of degrees $d_1, \ldots, d_r$, respectively. Put

$$F = \sum_{i=1}^{r} y_i f_i \in \mathbf{F}_q[x_0, \ldots, x_n, y_1, \ldots, y_r].$$

We consider the complex $(\Omega^\bullet_{\mathbf{F}_q[x,y]/\mathbf{F}_q}, dF \wedge)$, where $dF \in \Omega^1_{\mathbf{F}_q[x,y]/\mathbf{F}_q}$ denotes the exterior derivative of F and the boundary map sends $\omega \in \Omega^k_{\mathbf{F}_q[x,y]/\mathbf{F}_q}$ to $dF \wedge \omega \in \Omega^{k+1}_{\mathbf{F}_q[x,y]/\mathbf{F}_q}$. This complex has a bigrading defined by taking $(\Omega^k_{\mathbf{F}_q[x,y]/\mathbf{F}_q})^{(e_1,e_2)}$ to be the $\mathbf{F}_q$-span of those differential forms

$$x_0^{a_0} \cdots x_n^{a_n} y_1^{b_1} \cdots y_r^{b_r} \, dx_{i_1} \cdots dx_{i_l} \, dy_{j_1} \cdots dy_{j_m}$$

with $l + m = k$,

$$e_1 = \sum_{i=0}^{n} a_i + l - \sum_{j=1}^{r} b_j d_j - d_{j_1} - \cdots - d_{j_m},$$

and

$$e_2 = b_1 + \cdots + b_r + m$$

i.e., we are defining the bidegrees of x_i and dx_i to be $(1,0)$ and the bidegrees of y_j and dy_j to be $(-d_j, 1)$. Since the boundary map $\omega \mapsto dF \wedge \omega$ is bihomogeneous of bidegree $(0,1)$, there is an induced bigrading on cohomology that we denote by

$$H^k\big(\Omega^\bullet_{\mathbf{F}_q[x,y]/\mathbf{F}_q}, dF \wedge\big)^{(e_1,e_2)}.$$

Note that as $\mathbf{F}_q[x,y]$-module, $H^{n+r+1}(\Omega^\bullet_{\mathbf{F}_q[x,y]/\mathbf{F}_q}, dF \wedge) = \mathbf{F}_q[x,y]/I$, where I is the ideal

$$I = \left(f_1, \ldots, f_r, \sum_{j=1}^{r} y_j \frac{\partial f_j}{\partial x_0}, \ldots, \sum_{j=1}^{r} y_j \frac{\partial f_j}{\partial x_n} \right).$$

When $f_1 = \cdots = f_r = 0$ defines a smooth complete intersection X in $\mathbf{P}^n$, we proved in [4, Theorem 1.6] that

$$H^{n+r+1}(\Omega^\bullet_{\mathbf{F}_q[x,y]/\mathbf{F}_q}, dF \wedge)^{(0,e)} = 0 \quad \text{for } e < r \text{ or } e > n.$$

For $r \le e \le n$, put

$$h_e = \dim_{\mathbf{F}_q} H^{n+r+1}\big(\Omega^\bullet_{\mathbf{F}_q[x,y]/\mathbf{F}_q}, dF \wedge\big)^{(0,e)}$$

$$- \begin{cases} 1 & \text{if } p \mid d_1 \cdots d_r, \ n+r \text{ is odd, and } e = (n+r+1)/2, \\ 0 & \text{otherwise.} \end{cases}$$

We gave an explicit formula for the polynomial $\sum_{e=r}^{n} h_e t^e$ in [4, Section 5]. In particular, it depends only on n, r, and the multidegree $(d_1, \ldots, d_r)$ of X (see [4, Corollary 1.14] for additional properties of the h_e).

By a *monomial form* we mean an $(n+r+1)$-form of the type

$$x_0^{a_0} \cdots x_n^{a_n} y_1^{b_1} \cdots y_r^{b_r} \, dx_0 \cdots dx_n \, dy_1 \cdots dy_r.$$

In the exceptional case where $p \mid d_1 \cdots d_r$, $n+r$ is odd, and $e = (n+r+1)/2$, let $\xi_l^{(e)}$, $l = 1, \ldots, h_e$ be monomial forms such that

$$\left\{ \xi_l^{(e)} \mid l = 1, \ldots, h_e \right\} \cup \left\{ \tau_{(n+r+1)/2} \right\}$$

is a basis for $H^{n+r+1}(\Omega^\bullet_{\mathbf{F}_q[x,y]/\mathbf{F}_q}, dF\wedge)^{(0,e)}$, where $\tau_{(n+r+1)/2}$ is defined in the proof of Lemma 4.2 below. In the nonexceptional cases, for $e = r, \ldots, n$, let $\xi_l^{(e)}$, $l = 1, \ldots, h_e$ be a basis of monomial forms for

$$H^{n+r+1}\left(\Omega^\bullet_{\mathbf{F}_q[x,y]/\mathbf{F}_q}, dF\wedge\right)^{(0,e)}.$$

Our main result is that the monomial forms $\{\xi_l^{(e)} \mid e = r, \ldots, n, \ l = 1, \ldots, h_e\}$ are a basis for the Dwork cohomology group $H^{n+r+1}(\Omega^\bullet_{b,0})$ of X. (The definition of Dwork cohomology is recalled in Section 2; a formal statement of this result is given in Theorem 5.7 below.) For (generic) smooth affine and toric complete intersections, Bourgeois [5, 6] has given an explicit isomorphism between Dwork cohomology and rigid cohomology. It seems likely that his results extend to the projective case, so that the image of our basis under Bourgeois's isomorphism is a basis for the rigid cohomology of smooth projective complete intersections.

Let ord_q denote the p-adic valuation normalized by the condition $\text{ord}_q q = 1$. As an application of our work we prove the following, which is the main result of Mazur ([10, 11]).

THEOREM 1.1. *Suppose that $f_1 = \cdots = f_r = 0$ defines a smooth complete intersection X in $\mathbf{P}^n$. Then the Newton polygon of $P(t)$ with respect to ord_q lies on or above the Newton polygon with respect to ord_q of the polynomial*

$$\prod_{e=0}^{n-r} (1 - q^e t)^{h_{e+r}}.$$

REMARK 1.2. The Newton polygon of $\prod_{e=0}^{n-r}(1 - q^e t)^{h_{e+r}}$ is equal to the middle-dimensional primitive Hodge polygon of X (see [4, Section 1]).

The outline of the paper is as follows. In Section 2, we review Dwork's p-adic cohomology theory. In Section 3, we define a p-adic filtration on the Dwork complex and use it to compute some of the cohomology. We introduce some auxiliary complexes in Section 4 and compute their cohomology. The results of Section 4 are applied in Section 5 to finish the computation of the cohomology of the Dwork complex (Theorem 5.7). In Section 6, we study the

action of Frobenius on cohomology and prove Theorem 1.1. In the Appendix, we have collected some general results relating the cohomology of a p-adic complex to the cohomology of its reduction mod p. These results are used at various points in the paper.

2. Dwork cohomology

We retain the notation of the previous section. Let $X' \subseteq \mathbf{A}^{n+1}$ be the affine variety defined by the equations $f_1 = \cdots = f_r = 0$. Let N_m (resp. N'_m) be the number of $\mathbf{F}_{q^m}$-rational points of X (resp. X'). Then

$$Z(X/\mathbf{F}_q; t) = \exp\left(\sum_{m=1}^{\infty} N_m \frac{t^m}{m}\right),$$

$$Z(X'/\mathbf{F}_q; t) = \exp\left(\sum_{m=1}^{\infty} N'_m \frac{t^m}{m}\right).$$

The obvious relation

$$N_m = \frac{N'_m - 1}{q^m - 1}$$

is equivalent to

(2.1) $$Z(X'/\mathbf{F}_q; t) = \frac{Z(X/\mathbf{F}_q; qt)}{(1-t)Z(X/\mathbf{F}_q; t)}.$$

If we write

$$Z(X/\mathbf{F}_q; t) = \frac{P(t)^{(-1)^{n-r-1}}}{(1-t)(1-qt)\cdots(1-q^{n-r}t)},$$

then (2.1) implies

(2.2) $$Z(X'/\mathbf{F}_q; t) = \frac{1}{1-q^{n-r+1}t}\left(\frac{P(t)}{P(qt)}\right)^{(-1)^{n-r}}.$$

This zeta function is closely related to the L-function of a certain exponential sum. Let $\Psi : \mathbf{F}_q \to \mathbf{C}^{\times}$ be a nontrivial additive character and consider the exponential sums

$$S_m = \sum_{(x,y)\in\mathbf{A}^{n+1+r}(\mathbf{F}_{q^m})} \Psi(\mathrm{Tr}_{\mathbf{F}_{q^m}/\mathbf{F}_q}(F(x,y))).$$

The associated L-function is

$$L(\mathbf{A}^{n+1+r}, \Psi, F; t) = \exp\left(\sum_{m=1}^{\infty} S_m \frac{t^m}{m}\right).$$

It is easily seen that

$$\sum_{y\in\mathbf{A}^r(\mathbf{F}_q)} \Psi(\mathrm{Tr}_{\mathbf{F}_{q^m}/\mathbf{F}_q}(F(x,y))) = \begin{cases} q^{mr} & \text{if } f_1(x) = \cdots = f_r(x) = 0, \\ 0 & \text{otherwise,} \end{cases}$$

hence,

$$L(\mathbf{A}^{n+1+r}, \Psi, F; t) = Z(X'/\mathbf{F}_q; q^r t).$$

By (2.2), we thus have

(2.3) $$L(\mathbf{A}^{n+1+r}, \Psi, F; t)^{(-1)^{n+r}} = (1 - q^{n+1}t)^{(-1)^{n+r-1}} \frac{P(q^r t)}{P(q^{r+1}t)}.$$

We recall the construction of the Dwork complex whose cohomology describes this L-function. Let $\mathbf{Q}_p$ be the field of p-adic numbers, ζ_p a primitive pth root of unity, and $\Lambda_1 = \mathbf{Q}_p(\zeta_p)$. The field Λ_1 is a totally ramified extension of $\mathbf{Q}_p$ of degree $p - 1$. Write $q = p^a$ and let K be the unramified extension of $\mathbf{Q}_p$ of degree a. Set $\Lambda_0 = K(\zeta_p)$. The Frobenius automorphism $x \mapsto x^p$ of $\mathrm{Gal}(\mathbf{F}_q/\mathbf{F}_p)$ lifts to a generator τ of $\mathrm{Gal}(\Lambda_0/\Lambda_1)(\simeq \mathrm{Gal}(K/\mathbf{Q}_p))$ by requiring $\tau(\zeta_p) = \zeta_p$. Let Λ be the completion of an algebraic closure of Λ_0. Denote by "ord" the additive valuation on Λ normalized by $\mathrm{ord}\, p = 1$ and by "ord_q" the additive valuation normalized by $\mathrm{ord}_q q = 1$.

Let $E(t)$ be the Artin–Hasse exponential series:

$$E(t) = \exp\left(\sum_{i=0}^{\infty} \frac{t^{p^i}}{p^i}\right).$$

Let $\gamma \in \Lambda_1$ be a solution of $\sum_{i=0}^{\infty} t^{p^i}/p^i = 0$ satisfying $\mathrm{ord}\, \gamma = 1/(p-1)$ and consider

(2.4) $$\theta(t) = E(\gamma t) = \sum_{i=0}^{\infty} \lambda_i t^i \in \Lambda_1[[t]].$$

The series $\theta(t)$ is a splitting function in Dwork's terminology [8, Section 4a]. Its coefficients satisfy

(2.5) $$\mathrm{ord}\, \lambda_i \geq i/(p-1).$$

We consider the following spaces of p-adic functions. Let b be a positive rational number and choose a positive integer M such that both $Mb/(p(p-1))$ and $M/(p-1)$ are integers. Let π be such that

(2.6) $$\pi^M = p$$

and put $\tilde{\Lambda}_1 = \Lambda_1(\pi)$, $\tilde{\Lambda}_0 = \Lambda_0(\pi)$. The element π is a uniformizing parameter for the rings of integers of $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_0$. We extend $\tau \in \mathrm{Gal}(\Lambda_0/\Lambda_1)$ to a generator of $\mathrm{Gal}(\tilde{\Lambda}_0/\tilde{\Lambda}_1)$ by requiring $\tau(\pi) = \pi$. For $v = (v_1, \ldots, v_r) \in \mathbf{R}^r$, we put $|v| = v_1 + \cdots + v_r$. Define

(2.7) $$C(b) = \left\{ \sum_{(u,v) \in \mathbf{N}^{n+1+r}} A_{u,v} \pi^{Mb|v|} x^u y^v \mid A_{u,v} \in \tilde{\Lambda}_0 \right.$$

$$\left. \text{and } A_{u,v} \to 0 \text{ as } (u,v) \to \infty \right\}.$$

We construct a de Rham-type complex using $C(b)$. Let

$$\Omega_b^k = \bigoplus_{l+m=k} \bigoplus_{\substack{0 \leq i_1 < \cdots < i_l \leq n \\ 1 \leq j_1 < \cdots < j_m \leq r}} C(b) \, dx_{i_1} \cdots dx_{i_l} \, dy_{j_1} \cdots dy_{j_m}.$$

Define an exterior derivative $d \colon \Omega_b^k \to \Omega_b^{k+1}$ by linearity and the formula

$$d(\xi \, dx_{i_1} \cdots dx_{i_l} \, dy_{j_1} \cdots dy_{j_m})$$
$$= \left(\sum_{i=0}^n \frac{\partial \xi}{\partial x_i} \, dx_i + \sum_{j=1}^r \frac{\partial \xi}{\partial y_j} \, dy_j \right) \wedge dx_{i_1} \cdots dx_{i_l} \, dy_{j_1} \cdots dy_{j_m}$$

for $\xi \in C(b)$. Let $\hat{f}_i = \sum_u \hat{a}_{u,i} x^u \in K[x]$ be the Teichmüller lifting of f_i, let

$$(2.8) \qquad \hat{F} = \sum_{i=1}^r y_i \hat{f}_i(x) = \sum_{u,i} \hat{a}_{u,i} x^u y_i \in K[x,y]$$

be the Teichmüller lifting of F, and put $\gamma_l = \sum_{i=0}^l \gamma^{p^i}/p^i$. From the definition of γ we have

$$(2.9) \qquad \operatorname{ord} \gamma_l \geq \frac{p^{l+1}}{p-1} - l - 1.$$

Set

$$(2.10) \qquad H = \sum_{l=0}^\infty \gamma_l \hat{F}^{\tau^l}(x^{p^l}, y^{p^l}) = \sum_{l=0}^\infty \gamma_l \sum_{i=1}^r y_i^{p^l} \hat{f}_i^{\tau^l}(x^{p^l}).$$

It follows from (2.9) that $\partial H/\partial x_i, \partial H/\partial y_j \in C(b)$, so $dH \in \Omega_b^1$. Define $D \colon \Omega_b^k \to \Omega_b^{k+1}$ by

$$D(\omega) = \pi^{Mb} \gamma^{-1}(d\omega + dH \wedge \omega).$$

(The reason for the normalizing factor $\pi^{Mb} \gamma^{-1}$ will become apparent in Section 3.) We thus obtain a complex $(\Omega_b^\bullet, D)$.

We define the Frobenius operator on this complex. Set (see (2.8))

$$(2.11) \qquad G(x) = \prod_{u,i} \theta(\hat{a}_{u,i} x^u y_i),$$

$$(2.12) \qquad G_0(x) = \prod_{j=0}^{a-1} \prod_{u,i} \theta((\hat{a}_{u,i} x^u y_i)^{p^j}).$$

The estimate (2.5) implies that $G \in C(b)$ for all $b < 1/(p-1)$ and $G_0 \in C(b)$ for all $b < p/q(p-1)$. Define an operator ψ on formal power series by

$$(2.13) \qquad \psi\left(\sum_{(u,v) \in \mathbf{N}^{n+1+r}} A_{u,v} x^u y^v \right) = \sum_{(u,v) \in \mathbf{N}^{n+1+r}} A_{pu,pv} x^u y^v.$$

It is clear that $\psi(C(b)) \subseteq C(pb)$. For $0 < b < p/(p-1)$, let $\alpha = \psi^a \circ G_0$ be the composition

$$C(b) \hookrightarrow C(b/q) \xrightarrow{G_0} C(b/q) \xrightarrow{\psi^a} C(b),$$

where the middle arrow is multiplication by G_0. Then α is a completely continuous $\tilde{\Lambda}_0$-linear endomorphism of $C(b)$. We shall also consider $\beta = \tau^{-1} \circ \psi \circ G$, which is a completely continuous $\tilde{\Lambda}_1$-linear (or $\tilde{\Lambda}_0$-semilinear) endomorphism of $C(b)$. Note that $\alpha = \beta^a$.

Define a map $\alpha_\bullet : \Omega_b^\bullet \to \Omega_b^\bullet$ by additivity and the formula $(k = l + m)$

(2.14)
$$\alpha_k(\xi\, dx_{i_1} \cdots dx_{i_l}\, dy_{j_1} \cdots dy_{j_m})$$
$$= \frac{q^{n+1+r-k}}{x_{i_1} \cdots x_{i_l} y_{j_1} \cdots y_{j_m}}$$
$$\times \alpha(x_{i_1} \cdots x_{i_l} y_{j_1} \cdots y_{j_m} \xi)\, dx_{i_1} \cdots dx_{i_l}\, dy_{j_1} \cdots dy_{j_m}.$$

Then $\alpha_\bullet$ is a map of complexes (see, e.g., [3, equation (2.11)]). Similarly, we define a map of complexes (see [3, equation (2.12)]) $\beta_\bullet : \Omega_b^\bullet \to \Omega_b^\bullet$ by additivity and the formula

(2.15)
$$\beta_k(\xi\, dx_{i_1} \cdots dx_{i_l}\, dy_{j_1} \cdots dy_{j_m})$$
$$= \frac{p^{n+1+r-k}}{x_{i_1} \cdots x_{i_l} y_{j_1} \cdots y_{j_m}}$$
$$\times \beta(x_{i_1} \cdots x_{i_l} y_{j_1} \cdots y_{j_m} \xi)\, dx_{i_1} \cdots dx_{i_l}\, dy_{j_1} \cdots dy_{j_m}.$$

The Dwork trace formula, as formulated by Robba [12], then gives

(2.16)
$$L(\mathbf{A}^{n+1+r}, \Psi, F; t) = \prod_{k=0}^{n+r+1} \det(I - t\alpha_k \,|\, \Omega_b^k)^{(-1)^{k+1}}.$$

So far, we have followed closely the description of Dwork's theory as given, for example, in [3]. However, since the f_i are homogeneous polynomials, we can replace $\Omega_b^\bullet$ by a smaller complex that is more easily analyzed. For $s \in \mathbf{Z}$, let $\Omega_{b,s}^k$ be the subspace of Ω_b^k spanned by those k-forms $(l + m = k)$

$$\sum_{u,v} A_{u,v} \pi^{Mb|v|} x^u y^v\, dx_{i_1} \cdots dx_{i_l}\, dy_{j_1} \cdots dy_{j_m}$$

with

(2.17)
$$\sum_{i=0}^{n} u_i + l - \sum_{j=1}^{r} v_j d_j - d_{j_1} - \cdots - d_{j_m} = s$$

for all (u, v) with $A_{u,v} \neq 0$. There is an obvious decomposition as $\tilde{\Lambda}_0$-vector space

$$\Omega_b^k = \bigoplus_{s \in \mathbf{Z}} \Omega_{b,s}^k.$$

Note that every monomial $x^u y^v$ that appears in any of $F, \hat{F}, G, G_0$ with nonzero coefficient satisfies $\sum_{i=0}^{n} u_i = \sum_{j=1}^{r} v_j d_j$. It then follows easily that

$$(2.18) \qquad\qquad D(\Omega_{b,s}^k) \subseteq \Omega_{b,s}^{k+1}$$

and that

$$(2.19) \qquad\qquad \alpha_k(\Omega_{b,s}^k) \subseteq \Omega_{b,s/q}^k.$$

In particular, if $q \nmid s$, α_k is zero on $\Omega_{b,s}^k$; so for fixed $s \neq 0$, α_k^N is zero on $\Omega_{b,s}^k$ for all $N \gg 0$. We thus see that $\alpha_\bullet$ is stable on the subcomplex $(\Omega_{b,0}^\bullet, D)$ of $(\Omega_b^\bullet, D)$ while

$$\det(I - t\alpha_k \mid \Omega_b^k / \Omega_{b,0}^k) = 1.$$

Equation (2.16) and Serre [13, Proposition 9] now imply

$$L(\mathbf{A}^{n+1+r}, \Psi, F; t) = \prod_{k=0}^{n+r+1} \det(I - t\alpha_k \mid \Omega_{b,0}^k)^{(-1)^{k+1}},$$

and passing to cohomology gives

$$(2.20) \qquad L(\mathbf{A}^{n+1+r}, \Psi, F; t) = \prod_{k=0}^{n+r+1} \det\big(I - t\alpha_k \mid H^k(\Omega_{b,0}^\bullet, D)\big)^{(-1)^{k+1}}.$$

3. The π-adic filtration on $(\Omega_b^\bullet, D)$

The p-adic Banach space $C(b)$ has a decreasing filtration $\{F^s C(b)\}_{s=-\infty}^{\infty}$ defined by setting

$$F^s C(b) = \left\{ \sum_{(u,v) \in \mathbf{N}^{n+1+r}} A_{u,v} \pi^{Mb|v|} x^u y^v \in C(b) \mid A_{u,v} \in \pi^s \mathcal{O}_{\tilde{\Lambda}_0} \text{ for all } u, v \right\},$$

where $\mathcal{O}_{\tilde{\Lambda}_0}$ denotes the ring of integers of $\tilde{\Lambda}_0$. We extend this to a filtration on $\Omega_b^\bullet$ by defining

$$F^s \Omega_b^k = \bigoplus_{l+m=k} \bigoplus_{\substack{0 \leq i_1 < \cdots < i_l \leq n \\ 1 \leq j_1 < \cdots < j_m \leq r}} \pi^{Mbm} F^s C(b) \, dx_{i_1} \cdots dx_{i_l} \, dy_{j_1} \cdots dy_{j_m}.$$

We assume from now on that $1/(p-1) < b < p/(p-1)$. A calculation shows that under this condition, $D(F^s \Omega_b^k) \subset F^s \Omega_b^{k+1}$ (this is the reason for introducing the normalizing factor $\pi^{Mb} \gamma^{-1}$ in Section 2) hence, $(\Omega_b^\bullet, D)$ is a filtered complex. We shall use this filtration to compute the cohomology of $(\Omega_{b,0}^\bullet, D)$ and then use (2.20) and (2.3) to estimate the Newton polygon of $P(t)$.

Consider the map $F^0\Omega_b^k \to \Omega_{\mathbf{F}_q[x,y]/\mathbf{F}_q}^k$ defined by additivity and the formula

$$\sum_{(u,v)\in\mathbf{N}^{n+1+r}} A_{u,v}\pi^{Mb(|v|+m)}x^u y^v\, dx_{i_1}\cdots dx_{i_l}\, dy_{j_1}\cdots dy_{j_m}$$

$$\mapsto \sum_{(u,v)\in\mathbf{N}^{n+1+r}} \bar{A}_{u,v}x^u y^v\, dx_{i_1}\cdots dx_{i_l}\, dy_{j_1}\cdots dy_{j_m},$$

where $\bar{A}_{u,v}$ denotes the reduction of $A_{u,v}$ modulo the maximal ideal of $\tilde{\Lambda}_0$. (Since $A_{u,v} \to 0$ as $(u,v) \to \infty$, the sum on the right-hand side is finite.) It is clear that this map is surjective with kernel $F^1\Omega_b^k$, hence we get an isomorphism

$$(3.1) \qquad F^0\Omega_b^k/F^1\Omega_b^k \cong \Omega_{\mathbf{F}_q[x,y]/\mathbf{F}_q}^k.$$

For $\omega \in F^0\Omega_b^k$, we have

$$D(\omega) \equiv \pi^{Mb}\gamma^{-1}dH \wedge \omega \,(\mathrm{mod}\, F^1\Omega_b^{k+1})$$

$$\equiv \pi^{Mb}\left(\sum_{i=0}^n \frac{\partial \hat{F}}{\partial x_i}\, dx_i + \sum_{j=1}^r \frac{\partial \hat{F}}{\partial y_j}\, dy_j\right) \wedge \omega \,(\mathrm{mod}\, F^1\Omega_b^{k+1})$$

$$\equiv \pi^{Mb}\, d\hat{F} \wedge \omega \,(\mathrm{mod}\, F^1\Omega_b^{k+1}).$$

It follows that under the isomorphism (3.1), D is identified with $dF\wedge$, i.e., there is an isomorphism of complexes

$$(F^0\Omega_b^\bullet/F^1\Omega_b^\bullet, D) \cong \left(\Omega_{\mathbf{F}_q[x,y]/\mathbf{F}_q}^\bullet, dF\wedge\right).$$

Since multiplication by π^s defines an isomorphism of complexes,

$$(3.2) \qquad (F^0\Omega_b^\bullet, D) \cong (F^s\Omega_b^\bullet, D),$$

we have in fact isomorphisms for all $s \in \mathbf{Z}$

$$(3.3) \qquad (F^s\Omega_b^\bullet/F^{s+1}\Omega_b^\bullet, D) \cong \left(\Omega_{\mathbf{F}_q[x,y]/\mathbf{F}_q}^\bullet, dF\wedge\right).$$

Referring to the bigrading defined in the Introduction, we set

$$\left(\Omega_{\mathbf{F}_q[x,y]/\mathbf{F}_q}^k\right)^{(0)} = \bigcup_{e=0}^\infty \left(\Omega_{\mathbf{F}_q[x,y]/\mathbf{F}_q}^k\right)^{(0,e)} \subseteq \Omega_{\mathbf{F}_q[x,y]/\mathbf{F}_q}^k.$$

Since $dF \in \left(\Omega_{\mathbf{F}_q[x,y]/\mathbf{F}_q}^1\right)^{(0)}$, it follows that $\left(\left(\Omega_{\mathbf{F}_q[x,y]/\mathbf{F}_q}^\bullet\right)^{(0)}, dF\wedge\right)$ is a subcomplex of $\left(\Omega_{\mathbf{F}_q[x,y]/\mathbf{F}_q}^\bullet, dF\wedge\right)$. The isomorphism (3.1) induces

$$F^0\Omega_{b,0}^k/F^1\Omega_{b,0}^k \cong \left(\Omega_{\mathbf{F}_q[x,y]/\mathbf{F}_q}^k\right)^{(0)},$$

so as above we get isomorphisms for all s

$$(3.4) \qquad (F^s\Omega_{b,0}^\bullet/F^{s+1}\Omega_{b,0}^\bullet, D) \cong \left(\left(\Omega_{\mathbf{F}_q[x,y]/\mathbf{F}_q}^\bullet\right)^{(0)}, dF\wedge\right).$$

Our general approach will be as follows. The results of [4] describe the cohomology of $((\Omega^\bullet_{\mathbf{F}_q[x,y]/\mathbf{F}_q})^{(0)}, dF\wedge)$. We shall use the isomorphism (3.4) to infer a description of the cohomology of $(F^s\Omega^\bullet_{b,0}, D)$, and from that we shall calculate the cohomology of $(\Omega^\bullet_{b,0}, D)$. This explicit description will lead to the desired p-adic estimates for the action of Frobenius on the cohomology of $\Omega^\bullet_{b,0}$.

For example, by [4, Theorem 1.6], we have

(3.5) $$H^k\big((\Omega^\bullet_{\mathbf{F}_q[x,y]/\mathbf{F}_q})^{(0)}, dF\wedge\big) = 0 \quad \text{for } k \neq 2r, n+r, n+r+1.$$

An application of Proposition A.1 then gives the following result.

PROPOSITION 3.1. *If the equations $f_1 = \cdots = f_r = 0$ define a smooth complete intersection X in $\mathbf{P}^n$, then for all $s \in \mathbf{Z}$*

$$H^k(F^s\Omega^\bullet_{b,0}, D) = 0 \quad \text{for } k \neq 2r, n+r, n+r+1.$$

We next compute $H^{2r}(F^s\Omega^\bullet_{b,0}, D)$. Define

(3.6) $$\xi_1 = \pi^{Mb}\gamma^{-1}\sum_{j=1}^{r} d_1\cdots\hat{d}_j\cdots d_r\left(\sum_{l=0}^{\infty}\frac{\gamma_l}{p^l}d(\hat{f}_j^{\tau^l}(x^{p^l}))\wedge d(y_j^{p^l})\right)$$

$$= \pi^{Mb}\gamma^{-1}\sum_{j=1}^{r} d_1\cdots\hat{d}_j\cdots d_r\left(\sum_{l=0}^{\infty}\gamma_l y_j^{p^l-1}d(\hat{f}_j^{\tau^l}(x^{p^l}))\,dy_j\right) \in \Omega^2_{b,0}.$$

For $1 < k \leq r$, we define inductively

(3.7)
$$\xi_k = \frac{(-1)^{k-1}}{kd_1\cdots d_r}\xi_1 \wedge \xi_{k-1}$$
$$= \frac{(-1)^{k(k-1)/2}}{k!\,(d_1\cdots d_r)^{k-1}}\xi_1 \wedge \cdots \wedge \xi_1 \quad (k \text{ factors}).$$

Put

$$\sigma_j = \sum_{l=0}^{\infty}\gamma_l y_j^{p^l-1}d(\hat{f}_j^{\tau^l}(x^{p^l})).$$

Then we have explicitly

(3.8) $$\xi_k = (\pi^{Mb}\gamma^{-1})^k \sum_{1\leq j_1<\cdots<j_k\leq r}\left(\prod_{j\notin\{j_1,\ldots,j_k\}} d_j\right)$$
$$\times \sigma_{j_1}\wedge\cdots\wedge\sigma_{j_k}\wedge dy_{j_1}\wedge\cdots\wedge dy_{j_k}.$$

If $\xi \in \Omega^i_{b,0}$, then $d(\xi\wedge\eta) = d\xi\wedge\eta + (-1)^i\xi\wedge d\eta$. Since $d\circ d = 0$, it follows from (3.6) and (3.7) that $d\xi_k = 0$ for all k. From (2.10), we have

$$dH = \sum_{j=1}^{r}\left(y_j\sigma_j + \left(\sum_{l=0}^{\infty}\gamma_l p^l y_j^{p^l-1}\hat{f}_j^{\tau^l}(x^{p^l})\right)dy_j\right).$$

It now follows immediately from (3.8) that $dH \wedge \xi_r = 0$, hence

$$(3.9) \qquad\qquad\qquad D(\xi_r) = 0.$$

It is straightforward to check that $\xi_k \in F^0 \Omega_{b,0}^{2k}$ and that under the isomorphism (3.4) (with $s = 0$), ξ_k is mapped to

$$\bar{\xi}_k = \sum_{1 \leq j_1 < \cdots < j_k \leq r} \left(\prod_{j \notin \{j_1, \ldots, j_k\}} d_j \right) df_{j_1} \wedge \cdots \wedge df_{j_k} \wedge dy_{j_1} \wedge \cdots \wedge dy_{j_k}.$$

By [4, Theorem 1.6], if $r < n$, then

$$(3.10) \qquad\qquad H^{2r}\left(\left(\Omega_{\mathbf{F}_q[x,y]/\mathbf{F}_q}^{\bullet} \right)^{(0)}, dF\wedge \right) = \mathbf{F}_q \cdot [\bar{\xi}_r].$$

PROPOSITION 3.2. *If* $r < n$ *and the equations* $f_1 = \cdots = f_r = 0$ *define a smooth complete intersection* X *in* $\mathbf{P}^n$, *then* $H^{2r}(F^s \Omega_{b,0}^{\bullet}, D)$ *is a free* $\mathcal{O}_{\tilde{\Lambda}_0}$-*module with basis* $[\pi^s \xi_r]$.

Proof. The isomorphism (3.4) is equivalent to the exactness of

$$(3.11) \qquad 0 \to (F^s \Omega_{b,0}^{\bullet}, D) \xrightarrow{\pi} (F^s \Omega_{b,0}^{\bullet}, D) \to \left(\left(\Omega_{\mathbf{F}_q[x,y]/\mathbf{F}_q}^{\bullet} \right)^{(0)}, dF\wedge \right) \to 0,$$

where the second arrow is multiplication by π. By (3.5), the associated long exact sequence gives an exact sequence

$$0 \to H^{2r}(F^s \Omega_{b,0}^{\bullet}, D) \xrightarrow{\pi} H^{2r}(F^s \Omega_{b,0}^{\bullet}, D) \to H^{2r}\left(\left(\Omega_{\mathbf{F}_q[x,y]/\mathbf{F}_q}^{\bullet} \right)^{(0)}, dF\wedge \right)$$

$$\xrightarrow{\delta} H^{2r+1}(F^s \Omega_{b,0}^{\bullet}, D) \xrightarrow{\pi} H^{2r+1}(F^s \Omega_{b,0}^{\bullet}, D).$$

Since $\bar{\xi}_r \in (\Omega_{\mathbf{F}_q[x,y]/\mathbf{F}_q}^{2r})^{(0)}$ is the image of $\pi^s \xi_r \in F^s \Omega_{b,0}^{2r}$ under the isomorphism (3.4), we have $\delta([\bar{\xi}_r]) = [D(\pi^s \xi_r)] = 0$ by (3.9). By (3.10), the connecting homomorphism δ is the zero map. It follows that multiplication by π is injective on $H^{2r}(F^s \Omega_{b,0}^{\bullet}, D)$ and $H^{2r+1}(F^s \Omega_{b,0}^{\bullet}, D)$. The assertion of the proposition now follows immediately from Proposition A.2. $\square$

Propositions 3.1 and 3.2 compute $H^k(F^s \Omega_{b,0}^{\bullet}, D)$ for $0 \leq k < n + r$. To get information about this cohomology when $k = n + r, n + r + 1$, we need to introduce some related complexes.

4. The θ-map

Define $\theta : \Omega_b^k \to \Omega_b^{k-1}$ by $C(b)$-linearity and the formula

$$(4.1) \qquad \theta(dx_{i_1} \cdots dx_{i_l} \, dy_{j_1} \cdots dy_{j_m})$$

$$= \sum_{s=1}^{l} (-1)^{s-1} x_{i_s} \, dx_{i_1} \cdots \widehat{dx_{i_s}} \cdots dx_{i_l} \, dy_{j_1} \cdots dy_{j_m}$$

$$+ \sum_{t=1}^{m} (-1)^{l+t-1} (-d_{j_t} y_{j_t}) \, dx_{i_1} \cdots dx_{i_l} \, dy_{j_1} \cdots \widehat{dy}_{j_t} \cdots dy_{j_m}.$$

One has $\theta^2 = 0$, $\theta(\Omega_{b,s}^k) \subseteq \Omega_{b,s}^{k-1}$, and $\theta(F^s\Omega_b^k) \subseteq F^s\Omega_b^{k-1}$.

PROPOSITION 4.1. *The sequence*

$$0 \to F^s\Omega_{b,0}^{n+r+1} \xrightarrow{\theta} F^s\Omega_{b,0}^{n+r} \xrightarrow{\theta} \cdots \xrightarrow{\theta} F^s\Omega_{b,0}^0 \to \pi^s\mathcal{O}_{\tilde{\Lambda}_0} \to 0$$

is exact for all $s \in \mathbf{Z}$ (where the map $F^s\Omega_{b,0}^0 \to \pi^s\mathcal{O}_{\tilde{\Lambda}_0}$ sends a power series to its constant term).

Proof. Define $\theta: (\Omega_{\mathbf{F}_q[x,y]/\mathbf{F}_q}^k)^{(0)} \to (\Omega_{\mathbf{F}_q[x,y]/\mathbf{F}_q}^{k-1})^{(0)}$ by $\mathbf{F}_q[x,y]$-linearity and the formula (4.1). As in (3.4), one has isomorphisms of complexes

$$(4.2) \qquad (F^s\Omega_{b,0}^\bullet/F^{s+1}\Omega_{b,0}^\bullet, \theta) \cong \left((\Omega_{\mathbf{F}_q[x,y]/\mathbf{F}_q}^\bullet)^{(0)}, \theta\right)$$

for all s. By [4, Proposition 4.6], we have

$$H_k\left((\Omega_{\mathbf{F}_q[x,y]/\mathbf{F}_q}^\bullet)^{(0)}, \theta\right) = \begin{cases} 0 & \text{if } k > 0, \\ \mathbf{F}_q & \text{if } k = 0. \end{cases}$$

It now follows from Propositions A.1 and A.2 that

$$H_k(F^s\Omega_{b,0}^\bullet, \theta) = \begin{cases} 0 & \text{if } k > 0, \\ \pi^s\mathcal{O}_{\tilde{\Lambda}_0} & \text{if } k = 0, \end{cases}$$

which is the assertion of the proposition.   □

For $k \geq 0$, put

$$(4.3) \qquad F^s\widetilde{\Omega}_{b,0}^k = \theta(F^s\Omega_{b,0}^{k+1}) \quad \left(= \ker(\theta: F^s\Omega_{b,0}^k \to F^s\Omega_{b,0}^{k-1}) \text{ if } k > 0\right).$$

It is straightforward to check that

$$(4.4) \qquad \theta \circ D + D \circ \theta = 0$$

on $\Omega_{b,0}^\bullet$, hence $D(F^s\widetilde{\Omega}_{b,0}^k) \subset F^s\widetilde{\Omega}_{b,0}^{k+1}$. We thus obtain a complex $(F^s\widetilde{\Omega}_{b,0}^\bullet, D)$.

We define a related complex $F^s\widehat{\Omega}_{b,0}^\bullet$ as follows. Let

$$F^s\widehat{\Omega}_{b,0}^0 = \pi^s\mathcal{O}_{\tilde{\Lambda}_0}$$

and let $F^s\widehat{\Omega}_{b,0}^k = F^s\widetilde{\Omega}_{b,0}^{k-1}$ for $k \geq 1$. We define the boundary map $F^s\widehat{\Omega}_{b,0}^k \to F^s\widehat{\Omega}_{b,0}^{k+1}$ to be 0 if $k = 0$ and $-D$ if $k \geq 1$. Thus,

$$(4.5) \qquad H^0(F^s\widehat{\Omega}_{b,0}^\bullet) = \pi^s\mathcal{O}_{\tilde{\Lambda}_0}$$

and

$$(4.6) \qquad H^k(F^s\widehat{\Omega}_{b,0}^\bullet) = H^{k-1}(F^s\widetilde{\Omega}_{b,0}^\bullet) \quad \text{for } k \geq 1.$$

Define maps $F^s\Omega_{b,0}^k \to F^s\widehat{\Omega}_{b,0}^k$ as follows. For $k = 0$, take the map $F^s\Omega_{b,0}^0 \to \pi^s\mathcal{O}_{\tilde{\Lambda}_0}$ that sends a power series to its constant term. By Proposition 4.1, this map defines an isomorphism $F^s\Omega_{b,0}^0/\theta(F^s\Omega_{b,0}^1) \cong \pi^s\mathcal{O}_{\tilde{\Lambda}_0}$. For $k \geq 1$, take the

map $\theta : F^s\Omega^k_{b,0} \to F^s\widehat{\Omega}^k_{b,0}$. It follows from Proposition 4.1 that we have a short exact sequence of complexes

$$(4.7) \qquad 0 \to F^s\widetilde{\Omega}^\bullet_{b,0} \to F^s\Omega^\bullet_{b,0} \to F^s\widehat{\Omega}^\bullet_{b,0} \to 0,$$

and this gives rise to an exact cohomology sequence

$$(4.8) \qquad \begin{aligned} \cdots &\to H^k(F^s\widetilde{\Omega}^\bullet_{b,0}) \to H^k(F^s\Omega^\bullet_{b,0}) \\ &\to H^k(F^s\widehat{\Omega}^\bullet_{b,0}) \xrightarrow{\delta} H^{k+1}(F^s\widetilde{\Omega}^\bullet_{b,0}) \to \cdots . \end{aligned}$$

Our goal in this section is to describe the cohomology groups $H^k(F^s\widetilde{\Omega}^\bullet_{b,0})$ $(= H^{k+1}(F^s\widehat{\Omega}^\bullet_{b,0})$ by (4.6)) for $k < n + r$. We begin by describing some differential forms $\eta_k \in F^0\widehat{\Omega}^{2k}_{b,0}$ $(= F^0\widetilde{\Omega}^{2k-1}_{b,0}$ for $k > 0)$ that will play a key role in what follows.

LEMMA 4.2. *Set* $\eta_0 = 1 \in F^0\widehat{\Omega}^0_{b,0}$. *For* $k \geq 1$, *there exist differential forms* $\eta_k \in F^0\widetilde{\Omega}^{2k-1}_{b,0}$ *such that*

(a) $[\eta_k] = \delta([\eta_{k-1}])$, *where* δ *is the connecting homomorphism in* (4.8), *and*
(b) $\bar{\eta}_k \in (\Omega^{2k-1}_{\mathbf{F}_q[x,y]/\mathbf{F}_q})^{(0,k)}$, *where* $\bar{\eta}_k$ *is the image of* η_k *under the isomorphism* (3.4).

Proof. The proof is by induction on k. Define $\eta_1 = D(1)$. Then $\bar{\eta}_1 = dF \in (\Omega^1_{\mathbf{F}_q[x,y]/\mathbf{F}_q})^{(0,1)}$, so the result is true for $k = 1$. Suppose the result true for some $k \geq 1$. Now $\theta(\eta_k) = 0$, so $\theta(\bar{\eta}_k) = 0$. It follows from [4, Proposition 4.6] that there exists $\zeta_k \in (\Omega^{2k}_{\mathbf{F}_q[x,y]/\mathbf{F}_q})^{(0,k)}$ such that $\theta(\zeta_k) = \bar{\eta}_k$. By Proposition A.1, we can choose $\tau_k \in F^0\Omega^{2k}_{b,0}$ such that $\theta(\tau_k) = \eta_k$ and $\bar{\tau}_k = \zeta_k$. Define

$$(4.9) \qquad \qquad \eta_{k+1} = D(\tau_k).$$

Then $\delta([\eta_k]) = \eta_{k+1}$ by the definition of the connecting homomorphism. Furthermore, by (4.9),

$$\bar{\eta}_{k+1} = dF \wedge \zeta_k \in \left(\Omega^{2k+1}_{\mathbf{F}_q[x,y]/\mathbf{F}_q}\right)^{(0,k+1)},$$

and by induction the proof is complete. $\qquad\square$

PROPOSITION 4.3. *For* $0 \leq k < 2r$, *if* k *is even, then*

$$H^k(F^s\widetilde{\Omega}^\bullet_{b,0}) = H^{k+1}(F^s\widehat{\Omega}^\bullet_{b,0}) = 0$$

and if k *is odd, then*

$$H^k(F^s\widetilde{\Omega}^\bullet_{b,0}) = H^{k+1}(F^s\widehat{\Omega}^\bullet_{b,0}) \cong \mathcal{O}_{\bar{\Lambda}_0} \cdot \left[\pi^s\eta_{(k+1)/2}\right]$$

(a free, rank-one $\mathcal{O}_{\bar{\Lambda}_0}$-module with basis $[\pi^s\eta_{(k+1)/2}]$).

Proof. By Proposition 3.1, $H^k(F^s\Omega^\bullet_{b,0}) = 0$ for $k < 2r$, so

$$(4.10) \qquad\qquad\qquad H^0(F^s\widetilde{\Omega}^\bullet_{b,0}) = 0$$

and the connecting homomorphism δ in (4.8) gives isomorphisms

$$(4.11) \qquad H^k(F^s\widehat{\Omega}^\bullet_{b,0}) \cong H^{k+1}(F^s\widetilde{\Omega}^\bullet_{b,0}) \quad \text{for } 0 \le k < 2r - 1.$$

Equation (4.10) establishes the proposition for $k = 0$, and (4.5) and (4.11) establish it for $k = 1$. Using (4.6), we may regard (4.11) as isomorphisms

$$(4.12) \qquad H^k(F^s\widetilde{\Omega}^\bullet_{b,0}) \cong H^{k+2}(F^s\widetilde{\Omega}^\bullet_{b,0}) \quad \text{for } 0 \le k < 2r - 2.$$

Using (4.12) and Lemma 4.2, the proposition follows by induction from the cases $k = 0$ and $k = 1$. $\qquad\square$

LEMMA 4.4. *For* $k = 2, \ldots, r$, $\theta(\xi_k) = (-1)^{k-1} D(\xi_{k-1})$.

Proof. A straightforward calculation shows that

$$(4.13) \qquad\qquad \theta(\xi_1) = \pi^{Mb} \gamma^{-1}(d_1 \cdots d_r) \, dH = D(d_1 \cdots d_r).$$

It is convenient to set $\xi_0 = d_1 \cdots d_r$, so that (4.13) establishes the case $k = 1$ of the lemma and we can proceed by induction on k. Let $k \ge 2$ and suppose the assertion of the lemma is true for $1, \ldots, k-1$. If ω_1 is an l-form, then

$$\theta(\omega_1 \wedge \omega_2) = \theta(\omega_1) \wedge \omega_2 + (-1)^l \omega_1 \wedge \theta(\omega_2),$$

so by (3.7) and induction, we have

$$
\begin{aligned}
(4.14) \qquad \theta(\xi_k) &= \frac{(-1)^{k-1}}{kd_1 \cdots d_r} \big(\theta(\xi_1) \wedge \xi_{k-1} + \xi_1 \wedge \theta(\xi_{k-1}) \big) \\
&= \frac{(-1)^{k-1}}{kd_1 \cdots d_r} \big(D(\xi_0) \wedge \xi_{k-1} + \xi_1 \wedge (-1)^{k-2} D(\xi_{k-2}) \big).
\end{aligned}
$$

We observed earlier that $d\xi_l = 0$, hence $D(\xi_l) = (\pi^{Mb}\gamma^{-1}) \, dH \wedge \xi_l$ for all l. Substituting this into (4.14) and using (3.7) gives

$$
\begin{aligned}
\theta(\xi_k) &= \frac{(-1)^{k-1}}{kd_1 \cdots d_r} \big(\pi^{Mb}\gamma^{-1}(kd_1 \cdots d_r) dH \wedge \xi_{k-1} \big) \\
&= (-1)^{k-1} D(\xi_{k-1}),
\end{aligned}
$$

which proves the lemma. $\qquad\square$

The following result is key to describing the cohomology groups $H^k(F^s\widetilde{\Omega}^\bullet_{b,0})$ for $k \ge 2r$.

PROPOSITION 4.5. *Let* $r < n$. *Relative to the bases* $[\pi^s\xi_r]$ *for* $H^{2r}(F^s\Omega^\bullet_{b,0})$ *and* $[\pi^s\eta_r]$ *for* $H^{2r}(F^s\widehat{\Omega}^\bullet_{b,0})$, *the map*

$$\theta : H^{2r}(F^s\Omega^\bullet_{b,0}) \to H^{2r}(F^s\widehat{\Omega}^\bullet_{b,0})$$

is multiplication by $(-1)^{r(r-1)/2} d_1 \cdots d_r$.

Proof. It suffices to prove the assertion when $s = 0$. We prove inductively that for $k = 1, \ldots, r$,

$$\theta(\xi_k) = (-1)^{k(k-1)/2}(d_1 \cdots d_r)\eta_k + D(\theta(\sigma_k)) \tag{4.15}$$

for some $\sigma_k \in F^0\Omega_{b,0}^{2k-1}$. The assertion of the proposition then follows by taking $k = r$ in (4.15). By (4.13),

$$\theta(\xi_1) = (d_1 \cdots d_r)D(1) = (d_1 \cdots d_r)\eta_1,$$

so the assertion is true for $k = 1$. Assume inductively that (4.15) holds for some k, $1 \le k < r$. Choose $\tau_k \in F^0\Omega_{b,0}^{2k}$, as in the proof of Lemma 4.10, so that $\theta(\tau_k) = \eta_k$. Substitution into (4.15) then gives (since $D \circ \theta = -\theta \circ D$)

$$\theta\big(\xi_k - (-1)^{k(k-1)/2}(d_1 \cdots d_r)\tau_k + D(\sigma_k)\big) = 0.$$

By Proposition 4.1, there exists $\sigma_{k+1} \in F^0\Omega_{b,0}^{2k+1}$ such that

$$\xi_k = (-1)^{k(k-1)/2}(d_1 \cdots d_r)\tau_k - D(\sigma_k) + \theta\big((-1)^k\sigma_{k+1}\big). \tag{4.16}$$

From Lemma 4.4, we have

$$\theta(\xi_{k+1}) = (-1)^k D(\xi_k). \tag{4.17}$$

Substituting (4.16) into (4.17) now gives

$$\theta(\xi_{k+1}) = (-1)^{k(k+1)/2}(d_1 \cdots d_r)D(\tau_k) + D(\theta(\sigma_{k+1})).$$

Since $\eta_{k+1} = D(\tau_k)$ (see (4.9)), this is just (4.15) with k replaced by $k+1$, so by induction the proof is complete. $\qquad\square$

PROPOSITION 4.6. *Let $2r \le k < n + r$.*

(a) *If $(p, d_1 \cdots d_r) = 1$, then*

$$H^k(F^s\widetilde{\Omega}_{b,0}^\bullet) = H^{k+1}(F^s\widehat{\Omega}_{b,0}^\bullet) = 0.$$

(b) *If $p \mid d_1 \cdots d_r$, then*

$$H^k(F^s\widetilde{\Omega}_{b,0}^\bullet) = H^{k+1}(F^s\widehat{\Omega}_{b,0}^\bullet) \cong \begin{cases} \mathcal{O}_{\tilde{\Lambda}_0}/(d_1 \cdots d_r)\mathcal{O}_{\tilde{\Lambda}_0} & \text{if } k \text{ is odd,} \\ 0 & \text{if } k \text{ is even.} \end{cases}$$

Proof. If $r = n$ there is nothing to prove, so suppose $r < n$. From (4.8) and Proposition 4.3, we have an exact sequence

$$0 \to H^{2r}(F^s\widetilde{\Omega}_{b,0}^\bullet) \to H^{2r}(F^s\Omega_{b,0}^\bullet) \xrightarrow{\theta} H^{2r}(F^s\widehat{\Omega}_{b,0}^\bullet). \tag{4.18}$$

By Proposition 4.5, θ is injective, so

$$H^{2r}(F^s\widetilde{\Omega}_{b,0}^\bullet) = H^{2r+1}(F^s\widehat{\Omega}_{b,0}^\bullet) = 0. \tag{4.19}$$

If $r = n - 1$, there is nothing more to prove, so suppose $r < n - 1$. By Proposition 3.1, we have $H^{2r+1}(F^s\Omega_{b,0}^\bullet) = 0$, so (4.8) gives an exact sequence

$$H^{2r}(F^s\Omega_{b,0}^\bullet) \xrightarrow{\theta} H^{2r}(F^s\widehat{\Omega}_{b,0}^\bullet) \xrightarrow{\delta} H^{2r+1}(F^s\widetilde{\Omega}_{b,0}^\bullet) \to 0. \tag{4.20}$$

Proposition 4.5 then implies

(4.21) $$H^{2r+1}(F^s\widetilde{\Omega}^\bullet_{b,0}) = H^{2r+2}(F^s\widehat{\Omega}^\bullet_{b,0}) \cong \mathcal{O}_{\tilde{\Lambda}_0}/(d_1\cdots d_r)\mathcal{O}_{\tilde{\Lambda}_0}.$$

Assume first that $(p, d_1\cdots d_r) = 1$, so that by (4.19) and (4.21) we have

$$H^{2r}(F^s\widetilde{\Omega}^\bullet_{b,0}) = H^{2r+1}(F^s\widetilde{\Omega}^\bullet_{b,0}) = 0.$$

Suppose that for some k, $2r \le k < n+r-2$, we have proved

$$H^k(F^s\widetilde{\Omega}^\bullet_{b,0}) = H^{k+1}(F^s\widetilde{\Omega}^\bullet_{b,0}) = 0.$$

Then by (4.6), we also have

$$H^{k+1}(F^s\widehat{\Omega}^\bullet_{b,0}) = H^{k+2}(F^s\widehat{\Omega}^\bullet_{b,0}) = 0.$$

The exact sequence (4.8) then implies

$$H^{k+2}(F^s\widetilde{\Omega}^\bullet_{b,0}) \cong H^{k+2}(F^s\Omega^\bullet_{b,0}).$$

But $H^{k+2}(F^s\Omega^\bullet_{b,0}) = 0$ by Proposition 3.1, so

$$H^{k+2}(F^s\widetilde{\Omega}^\bullet_{b,0}) = H^{k+3}(F^s\widehat{\Omega}^\bullet_{b,0}) = 0.$$

Part (a) of the proposition now follows by induction on k.

Now assume that $p \mid d_1\cdots d_r$. Consider first the case of even k. The assertion

(4.22) $$H^k(F^s\widetilde{\Omega}^\bullet_{b,0}) = H^{k+1}(F^s\widehat{\Omega}^\bullet_{b,0}) = 0$$

holds for $k = 2r$ by (4.19). Suppose it holds for some even k, $2r \le k \le n+r-3$. By Proposition 3.1, we have

$$H^{k+1}(F^s\Omega^\bullet_{b,0}) = H^{k+2}(F^s\Omega^\bullet_{b,0}) = 0,$$

so (4.8) gives an isomorphism

(4.23) $$H^{k+1}(F^s\widehat{\Omega}^\bullet_{b,0}) \cong H^{k+2}(F^s\widetilde{\Omega}^\bullet_{b,0}).$$

Equation (4.22) then implies that

(4.24) $$H^{k+2}(F^s\widetilde{\Omega}^\bullet_{b,0}) = H^{k+3}(F^s\widehat{\Omega}^\bullet_{b,0}) = 0,$$

so part (b) of the proposition follows for even k by induction on k.

Now consider the case of odd k. The assertion

(4.25) $$H^k(F^s\widetilde{\Omega}^\bullet_{b,0}) = H^{k+1}(F^s\widehat{\Omega}^\bullet_{b,0}) \cong \mathcal{O}_{\tilde{\Lambda}_0}/(d_1\cdots d_r)\mathcal{O}_{\tilde{\Lambda}_0}$$

holds for $k = 2r+1$ by (4.21). Suppose it holds for some odd k, $2r+1 \le k \le n+r-3$. The isomorphism (4.23) then gives

(4.26) $$H^{k+2}(F^s\widetilde{\Omega}^\bullet_{b,0}) = H^{k+3}(F^s\widehat{\Omega}^\bullet_{b,0}) \cong \mathcal{O}_{\tilde{\Lambda}_0}/(d_1\cdots d_r)\mathcal{O}_{\tilde{\Lambda}_0},$$

so part (b) of the proposition follows for odd k by induction on k also.   □

We describe generators for the torsion modules of Proposition 4.6.

PROPOSITION 4.7. *Suppose $p \mid d_1 \cdots d_r$. For k odd, $2r+1 \leq k < n+r$, the cohomology class $[\pi^s \eta_{(k+1)/2}]$ generates the torsion module*

$$H^k(F^s\widetilde{\Omega}^\bullet_{b,0}) = H^{k+1}(F^s\widehat{\Omega}^\bullet_{b,0}) \cong \mathcal{O}_{\tilde{\Lambda}_0}/(d_1 \cdots d_r)\mathcal{O}_{\tilde{\Lambda}_0}.$$

Proof. It suffices to prove the assertion for $s = 0$. The proof is by induction on k. Consider first $k = 2r+1$. The exact sequence (4.20) shows that $\delta([\eta_r])$ is a generator for $H^{2r+1}(F^0\widetilde{\Omega}^\bullet_{b,0})$. But $\delta([\eta_r]) = [\eta_{r+1}]$ by Lemma 4.2, so the result is true for $k = 2r+1$. Suppose inductively that for some odd k, $2r+1 \leq k < n+r$, the proposition is true. By (4.23), $\delta([\eta_{(k+1)/2}])$ generates $H^{k+2}(F^0\widetilde{\Omega}^\bullet_{b,0})$. But by Lemma 4.2, $\delta([\eta_{(k+1)/2}]) = [\eta_{(k+3)/2}]$, so by induction on k the proof is complete. $\qquad\square$

5. Computation of $H^{n+r}(F^s\Omega^\bullet_{b,0})$ and $H^{n+r+1}(F^s\Omega^\bullet_{b,0})$

LEMMA 5.1. $H^{n+r}(F^s\Omega^\bullet_{b,0})$ *is a free, finitely-generated $\mathcal{O}_{\tilde{\Lambda}_0}$-module.*

Proof. The isomorphism (3.4) is equivalent to the exactness of the sequence

$$(5.1) \qquad 0 \to (F^0\Omega^\bullet_{b,0}, D) \xrightarrow{\pi} (F^0\Omega^\bullet_{b,0}, D) \to \left((\Omega^\bullet_{\mathbf{F}_q[x,y]/\mathbf{F}_q})^{(0)}, dF\wedge \right) \to 0,$$

where the second arrow is multiplication by π. From the associated sequence of cohomology groups we get the exact sequence

$$(5.2) \qquad H^{n+r-1}\left((\Omega^\bullet_{\mathbf{F}_q[x,y]/\mathbf{F}_q})^{(0)} \right) \xrightarrow{\delta} H^{n+r}(F^0\Omega^\bullet_{b,0}) \xrightarrow{\pi} H^{n+r}(F^0\Omega^\bullet_{b,0}).$$

If $r \neq n-1$ (so that $n+r-1 \neq 2r$), equation (3.5) implies that multiplication by π is injective on $H^{n+r}(F^0\Omega^\bullet_{b,0})$. If $r = n-1$, then (5.2) becomes

$$(5.3) \qquad H^{2r}\left((\Omega^\bullet_{\mathbf{F}_q[x,y]/\mathbf{F}_q})^{(0)} \right) \xrightarrow{\delta} H^{n+r}(F^0\Omega^\bullet_{b,0}) \xrightarrow{\pi} H^{n+r}(F^0\Omega^\bullet_{b,0}).$$

But by (3.10), this connecting homomorphism δ is the zero map:

$$\delta([\bar{\xi}_r]) = [D(\xi_r)] = 0$$

by (3.9). Thus, in all cases multiplication by π is injective on $H^{n+r}(F^0\Omega^\bullet_{b,0})$ hence, $H^{n+r}(F^0\Omega^\bullet_{b,0})$ is torsion-free. To show it is free, we are thus reduced to showing that it is finitely generated.

Since $F^0\widetilde{\Omega}^k_{b,0} = 0$ for $k > n+r$, we get from (4.8) the exact sequence

$$H^{n+r}(F^0\widetilde{\Omega}^\bullet_{b,0}) \to H^{n+r}(F^0\Omega^\bullet_{b,0}) \to H^{n+r}(F^0\widehat{\Omega}^\bullet_{b,0}) \to 0.$$

By Proposition 4.6, $H^{n+r}(F^0\widehat{\Omega}^\bullet_{b,0})$ is finitely generated, so it suffices to show that $H^{n+r}(F^0\widetilde{\Omega}^\bullet_{b,0})$ is finitely generated. By (4.6), we have

$$H^{n+r}(F^0\widetilde{\Omega}^\bullet_{b,0}) = H^{n+r+1}(F^0\widehat{\Omega}^\bullet_{b,0}),$$

and from (4.8) we get $H^{n+r+1}(F^0\Omega^\bullet_{b,0}) \cong H^{n+r+1}(F^0\widehat{\Omega}^\bullet_{b,0})$, so we are finally reduced to showing that $H^{n+r+1}(F^0\Omega^\bullet_{b,0})$ is finitely generated. But by [4,

Theorem 1.6], $H^{n+r+1}((\Omega^\bullet_{\mathbf{F}_q[x,y]/\mathbf{F}_q})^{(0)})$ is finitely generated. We can choose a "monomial" basis for it consisting of $(n+r+1)$-forms of the type

$$x_0^{a_0} \cdots x_n^{a_n} y_1^{b_1} \cdots y_r^{b_r} \, dx_0 \cdots dx_n \, dy_1 \cdots dy_r.$$

After multiplying by an appropriate power of π, such a form may be regarded as an element of $F^0\Omega^{n+r+1}_{b,0}$. These normalized forms span $H^{n+r+1}(F^0\Omega^\bullet_{b,0})$ by Proposition A.3. This completes the proof of the lemma. $\qquad\square$

PROPOSITION 5.2. *If* $(p, d_1 \cdots d_r) = 1$ *or if* $n + r$ *is even, then* $H^{n+r+1}(F^s\Omega^\bullet_{b,0})$ *is a free* $\mathcal{O}_{\tilde{\Lambda}_0}$-*module.*

Proof. It suffices to prove the assertion for $s = 0$. From (4.8), we get the exact sequence

$$(5.4) \qquad H^{n+r-1}(F^0\Omega^\bullet_{b,0}) \xrightarrow{\theta} H^{n+r-1}(F^0\widehat{\Omega}^\bullet_{b,0}) \to H^{n+r}(F^0\widetilde{\Omega}^\bullet_{b,0})$$
$$\to H^{n+r}(F^0\Omega^\bullet_{b,0}).$$

If $r = n$, then $H^{n+r-1}(F^0\widehat{\Omega}^\bullet_{b,0}) = 0$ by Proposition 4.3. If $r = n - 1$ and $(p, d_1 \cdots d_r) = 1$, then the first arrow in (5.4) is surjective by Proposition 4.5. The case $r = n - 1$ cannot occur when $n + r$ is even. Finally, if $r < n - 1$, then $H^{n+r-1}(F^0\widehat{\Omega}^\bullet_{b,0}) = 0$ by Proposition 4.6. It follows that in all cases, the last arrow in (5.4) is injective, so $H^{n+r}(F^0\widetilde{\Omega}^\bullet_{b,0})$ is free by Lemma 5.1. But $H^{n+r}(F^0\widetilde{\Omega}^\bullet_{b,0}) = H^{n+r+1}(F^0\widehat{\Omega}^\bullet_{b,0})$ by (4.6), and from (4.8), $H^{n+r+1}(F^s\Omega^\bullet_{b,0}) \cong H^{n+r+1}(F^0\widehat{\Omega}^\bullet_{b,0})$. Thus $H^{n+r+1}(F^s\Omega^\bullet_{b,0})$ is also free. $\qquad\square$

PROPOSITION 5.3. *If* $p \mid d_1 \cdots d_r$ *and* $n+r$ *is odd, then the torsion submodule of* $H^{n+r+1}(F^s\Omega^\bullet_{b,0})$ *is isomorphic to* $\mathcal{O}_{\tilde{\Lambda}_0}/(d_1 \cdots d_r)\mathcal{O}_{\tilde{\Lambda}_0}$ *and is generated by the cohomology class* $[\pi^s \tau_{(n+r+1)/2}]$ *(where* $\tau_{(n+r+1)/2}$ *is defined in the proof of Lemma 4.2). Furthermore, there is an exact sequence*

$$0 \to \left\langle [\pi^s \tau_{(n+r+1)/2}] \right\rangle \to H^{n+r+1}(F^s\Omega^\bullet_{b,0}) \xrightarrow{\theta} H^{n+r}(F^s\Omega^\bullet_{b,0}) \to 0,$$

where $\left\langle [\pi^s \tau_{(n+r+1)/2}] \right\rangle$ *denotes the submodule generated by* $[\pi^s \tau_{(n+r+1)/2}]$.

Proof. Since $n + r$ is odd, the case $r = n$ is impossible so we have $r < n$. From (4.8) and Proposition 4.6(b), we get the exact sequence

$$(5.5) \qquad H^{n+r-1}(F^0\Omega^\bullet_{b,0}) \xrightarrow{\theta} H^{n+r-1}(F^0\widehat{\Omega}^\bullet_{b,0}) \xrightarrow{\delta} H^{n+r}(F^0\widetilde{\Omega}^\bullet_{b,0})$$
$$\to H^{n+r}(F^0\Omega^\bullet_{b,0}) \to 0.$$

By Proposition 4.3, if $r = n - 1$ and by Proposition 4.7 if $r < n - 1$, it follows that $H^{n+r-1}(F^0\widehat{\Omega}^\bullet_{b,0})$ is generated by $[\eta_{(n+r-1)/2}]$. Hence, by Lemma 4.2, the image of δ is generated by $[\eta_{(n+r+1)/2}]$. By Proposition 4.5, if $r = n - 1$ and by Propositions 3.1 and 4.6(b), if $r < n - 1$, it follows that the cokernel of θ in (5.5) is isomorphic to $\mathcal{O}_{\tilde{\Lambda}_0}/(d_1 \cdots d_r)\mathcal{O}_{\tilde{\Lambda}_0}$. Thus, (5.5) gives an exact sequence

$$(5.6) \qquad 0 \to \langle [\eta_{(n+r+1)/2}] \rangle \to H^{n+r}(F^0\widetilde{\Omega}^\bullet_{b,0}) \to H^{n+r}(F^0\Omega^\bullet_{b,0}) \to 0,$$

where $\langle[\eta_{(n+r+1)/2}]\rangle \cong \mathcal{O}_{\tilde{\Lambda}_0}/(d_1\cdots d_r)\mathcal{O}_{\tilde{\Lambda}_0}$. By (4.6), we have $H^{n+r}(F^0\widetilde{\Omega}_{b,0}^\bullet) = H^{n+r+1}(F^0\widehat{\Omega}_{b,0}^\bullet)$, and (4.8) shows that the map θ gives an isomorphism

$$H^{n+r+1}(F^0\Omega_{b,0}^\bullet) \cong H^{n+r+1}(F^0\widehat{\Omega}_{b,0}^\bullet).$$

Under this isomorphism, the cohomology class $[\tau_{(n+r+1)/2}] \in H^{n+r+1}(F^0\Omega_{b,0}^\bullet)$ corresponds to $[\eta_{(n+r+1)/2}] \in H^{n+r+1}(F^0\widehat{\Omega}_{b,0}^\bullet)$. With these identifications, the sequence (5.6) becomes

$$0 \to \langle[\tau_{(n+r+1)/2}]\rangle \to H^{n+r+1}(F^0\Omega_{b,0}^\bullet) \xrightarrow{\theta} H^{n+r}(F^0\Omega_{b,0}^\bullet) \to 0,$$

with $\langle[\tau_{(n+r+1)/2}]\rangle \cong \mathcal{O}_{\tilde{\Lambda}_0}/(d_1\cdots d_r)\mathcal{O}_{\tilde{\Lambda}_0}$. This proves the proposition. $\qquad\square$

REMARK 5.4. The exact sequence (5.1) implies that there is an isomorphism

$$H^{n+r+1}(F^0\Omega_{b,0}^\bullet)/\pi H^{n+r+1}(F^0\Omega_{b,0}^\bullet) \cong H^{n+r+1}\big((\Omega_{\mathbf{F}_q[x,y]/\mathbf{F}_q}^\bullet)^{(0)}\big).$$

From Proposition 5.3, it then follows that the cohomology class $[\bar{\tau}_{(n+r+1)/2}]$ is not zero in $H^{n+r+1}((\Omega_{\mathbf{F}_q[x,y]/\mathbf{F}_q}^\bullet)^{(0)})$.

In Section 1, we defined h_e to be the dimension of $H^{n+r+1}(\Omega_{\mathbf{F}_q[x,y]/\mathbf{F}_q}^\bullet)^{(0,e)}$ for $r \le e \le n$ except in the exceptional case $p \mid d_1\cdots d_r$, $n+r$ odd, $e = (n+r+1)/2$, where this dimension is $h_e + 1$. Choose elements of $(\Omega_{\mathbf{F}_q[x,y]/\mathbf{F}_q}^{n+r+1})^{(0,e)}$ of the form

$$(5.7) \qquad \bar{\xi}_l^{(e)} = \prod_{i=0}^{n} x_i^{a_i(e;l)} \prod_{j=1}^{r} y_j^{b_j(e;l)}\, dx_0\cdots dx_n\, dy_1\cdots dy_r$$

for $l = 1,\ldots,h_e$ so that $\{[\bar{\xi}_l^{(e)}]\}_{l=1}^{h_e}$ is a basis for $H^{n+r+1}(\Omega_{\mathbf{F}_q[x,y]/\mathbf{F}_q}^\bullet)^{(0,e)}$ in the nonexceptional cases and $\{[\bar{\xi}_l^{((n+r+1)/2)}]\}_{l=1}^{h_{(n+r+1)/2}} \cup \{[\bar{\tau}_{(n+r+1)/2}]\}$ is a basis in the exceptional case. As noted in Section 1,

$$H^{n+r+1}\big((\Omega_{\mathbf{F}_q[x,y]/\mathbf{F}_q}^\bullet)^{(0)}\big) = \bigcup_{e=r}^{n} H^{n+r+1}\big(\Omega_{\mathbf{F}_q[x,y]/\mathbf{F}_q}^\bullet\big)^{(0,e)},$$

so $\{[\bar{\xi}_l^{(e)}] \mid e = r,\ldots,n,\ l = 1,\ldots,h_e\}$ is a basis for $H^{n+r+1}((\Omega_{\mathbf{F}_q[x,y]/\mathbf{F}_q}^\bullet)^{(0)})$ if either $(p, d_1\cdots d_r) = 1$ or $n+r$ is even, and $\{[\bar{\xi}_l^{(e)}] \mid e = r,\ldots,n,\ l = 1,\ldots,h_e\} \cup \{[\bar{\tau}_{(n+r+1)/2}]\}$ is a basis if $p \mid d_1\cdots d_r$ and $n+r$ is odd. Put

$$(5.8) \quad \xi_l^{(e)} = \pi^{Mbe} \prod_{i=0}^{n} x_i^{a_i(e;l)} \prod_{j=1}^{r} y_j^{b_j(e;l)}\, dx_0\cdots dx_n\, dy_1\cdots dy_r \in F^0\Omega_{b,0}^{n+r+1}.$$

The image of $\xi_l^{(e)}$ under the isomorphism (3.4) is $\bar{\xi}_l^{(e)}$, so by Propositions 5.2 and A.2 we get the following result.

COROLLARY 5.5. *If* $(p, d_1 \cdots d_r) = 1$ *or if* $n + r$ *is even, then the cohomology classes* $\{[\pi^s \xi_l^{(e)}] \mid e = r, \ldots, n, \, l = 1, \ldots, h_e\}$ *form a basis for* $H^{n+r+1}(F^s \Omega_{b,0}^\bullet)$.

The following result describes the situation in the exceptional case.

COROLLARY 5.6. *If* $p \mid d_1 \cdots d_r$ *and* $n + r$ *is odd, then*

$$H^{n+r+1}(F^s \Omega_{b,0}^\bullet) = H_s \oplus \left\langle \left[\pi^s \tau_{(n+r+1)/2} \right] \right\rangle,$$

where H_s *is a free submodule of* $H^{n+r+1}(F^s \Omega_{b,0}^\bullet)$ *with basis the cohomology classes* $\{[\pi^s \xi_l^{(e)}] \mid e = r, \ldots, n, \, l = 1, \ldots, h_e\}$.

Proof. The fact that $H^{n+r+1}(F^s \Omega_{b,0}^\bullet)$ is the sum of these two submodules follows from Proposition A.3. Suppose there are $c_{e,l} \in \mathcal{O}_{\tilde{\Lambda}_0}$ such that

$$(5.9) \qquad \sum_{e,l} c_{e,l} \left[\xi_l^{(e)} \right] = 0$$

in $H^{n+r+1}(F^0 \Omega_{b,0}^\bullet)$. Then in $H^{n+r+1}((\Omega_{\mathbf{F}_q[x,y]/\mathbf{F}_q}^\bullet)^{(0)})$, we have

$$(5.10) \qquad \sum_{e,l} \bar{c}_{e,l} \left[\bar{\xi}_l^{(e)} \right] = 0.$$

If some $c_{e,l}$ is a unit in $\mathcal{O}_{\tilde{\Lambda}_0}$, then the relation (5.10) is nontrivial, contradicting the definition of the $\bar{\xi}_l^{(e)}$. If all $c_{e,l}$ are divisible by π but some $c_{e,l}$ is nonzero, we can choose w so that $c_{e,l} = \pi^w c_{e,l}'$ for all e, l, where $c_{e,l}' \in \mathcal{O}_{\tilde{\Lambda}_0}$ and some $c_{e,l}'$ is a unit. Then (5.9) says that $\sum_{e,l} c_{e,l}' [\xi_l^{(e)}]$ is a torsion element of $H^{n+r+1}(F^0 \Omega_{b,0}^\bullet)$, so by Proposition 5.3 we have

$$(5.11) \qquad \sum_{e,l} c_{e,l}' \left[\xi_l^{(e)} \right] = c \left[\tau_{(n+r+1)/2} \right]$$

for some $c \in \mathcal{O}_{\tilde{\Lambda}_0}$. But this implies the nontrivial relation

$$(5.12) \qquad \sum_{e,l} \bar{c}_{e,l}' \left[\bar{\xi}_l^{(e)} \right] = \bar{c} \left[\bar{\tau}_{(n+r+1)/2} \right]$$

in $H^{n+r+1}((\Omega_{\mathbf{F}_q[x,y]/\mathbf{F}_q}^\bullet)^{(0)})$, again contradicting the definition of the $\bar{\xi}_l^{(e)}$. $\square$

The following result is the key to estimating the Newton polygon. It is an immediate consequence of Corollaries 5.5 and 5.6 and Proposition A.4.

THEOREM 5.7. *In all cases, the cohomology classes* $\{[\xi_l^{(e)}] \mid e = r, \ldots, n, \, l = 1, \ldots, h_e\}$ *form a basis for* $H^{n+r+1}(\Omega_{b,0}^\bullet)$.

6. Frobenius action and Newton polygon

It follows from Proposition 4.1 that the sequence

$$(6.1) \qquad 0 \to \Omega_{b,0}^{n+r+1} \xrightarrow{\theta} \Omega_{b,0}^{n+r} \xrightarrow{\theta} \cdots \xrightarrow{\theta} \Omega_{b,0}^{0} \to \tilde{\Lambda}_0 \to 0$$

is exact. Put

$$\widetilde{\Omega}_{b,0}^{k} = \theta(\Omega_{b,0}^{k+1}) = \bigcup_{s \in \mathbf{Z}} F^s \widetilde{\Omega}_{b,0}^{k}$$

and define

$$\widehat{\Omega}_{b,0}^{0} = \tilde{\Lambda}_0,$$
$$\widehat{\Omega}_{b,0}^{k} = \widetilde{\Omega}_{b,0}^{k-1} = \bigcup_{s \in \mathbf{Z}} F^s \widehat{\Omega}_{b,0}^{k} \quad \text{for } k \geq 1.$$

The boundary maps $F^s \widetilde{\Omega}_{b,0}^{k} \to F^s \widetilde{\Omega}_{b,0}^{k+1}$ and $F^s \widehat{\Omega}_{b,0}^{k} \to F^s \widehat{\Omega}_{b,0}^{k+1}$ defined in Section 4 give complexes $\widetilde{\Omega}_{b,0}^{\bullet}$ and $\widehat{\Omega}_{b,0}^{\bullet}$. As in (4.5) and (4.6), we have

$$(6.2) \qquad H^0(\widehat{\Omega}_{b,0}^{\bullet}) = \tilde{\Lambda}_0$$

and

$$(6.3) \qquad H^k(\widehat{\Omega}_{b,0}^{\bullet}) = H^{k-1}(\widetilde{\Omega}_{b,0}^{\bullet}) \quad \text{for } k \geq 1,$$

and as in (4.7) we have a short exact sequence of complexes

$$(6.4) \qquad 0 \to \widetilde{\Omega}_{b,0}^{\bullet} \to \Omega_{b,0}^{\bullet} \to \widehat{\Omega}_{b,0}^{\bullet} \to 0.$$

It is straightforward to check from (2.14) and (4.1) that

$$(6.5) \qquad \theta \circ q\alpha_k = \alpha_{k-1} \circ \theta$$

for $k = 1, \ldots, n+r+1$. It follows that α_k is stable on $\widetilde{\Omega}_{b,0}^{k}$ hence, the Frobenius structure on $\Omega_{b,0}^{\bullet}$ induces a Frobenius structure on $\widetilde{\Omega}_{b,0}^{\bullet}$ and the inclusion $\widetilde{\Omega}_{b,0}^{\bullet} \hookrightarrow \Omega_{b,0}^{\bullet}$ is a morphism of complexes with Frobenius structure. For the Frobenius structure on $\widehat{\Omega}_{b,0}^{\bullet}$, we define $\hat{\alpha}_k : \widehat{\Omega}_{b,0}^{k} \to \widehat{\Omega}_{b,0}^{k}$ as follows. For $k = 0$, we have $\widehat{\Omega}_{b,0}^{0} = \tilde{\Lambda}_0$, and we define $\hat{\alpha}_0$ to be multiplication by q^{n+r+1}. For $k > 0$ we have $\widehat{\Omega}_{b,0}^{k} = \widetilde{\Omega}_{b,0}^{k-1}$ and we define $\hat{\alpha}_k$ to be $q^{-1}\alpha_{k-1}$. Then (6.5) implies that the map $\Omega_{b,0}^{\bullet} \to \widehat{\Omega}_{b,0}^{\bullet}$ is a morphism of complexes with Frobenius structure. From (6.4), we then get an exact sequence of cohomology spaces with Frobenius structure

$$(6.6) \qquad \cdots \to H^k(\widetilde{\Omega}_{b,0}^{\bullet}) \to H^k(\Omega_{b,0}^{\bullet}) \to H^k(\widehat{\Omega}_{b,0}^{\bullet}) \xrightarrow{\delta} H^{k+1}(\widetilde{\Omega}_{b,0}^{\bullet}) \to \cdots,$$

where δ denotes the connecting homomorphism. Note in particular that the identification (6.3) is *not* an isomorphism of Frobenius modules. One has instead

$$(6.7) \qquad \det(I - qt\hat{\alpha}_k \mid H^k(\widehat{\Omega}_{b,0}^{\bullet})) = \det(I - t\alpha_{k-1} \mid H^{k-1}(\widetilde{\Omega}_{b,0}^{\bullet})).$$

From Propositions 4.3 and A.4, we get the following result.

PROPOSITION 6.1. *Let $0 \le k < 2r$. For k even, $H^k(\widetilde{\Omega}_{b,0}^\bullet) = H^{k+1}(\widehat{\Omega}_{b,0}^\bullet) = 0$, and for k odd, $H^k(\widetilde{\Omega}_{b,0}^\bullet) = H^{k+1}(\widehat{\Omega}_{b,0}^\bullet)$ is a one-dimensional vector space with basis $[\eta_{(k+1)/2}]$.*

By Lemma 4.2, we have $\delta([\eta_i]) = [\eta_{i+1}]$, so for k even, $0 \le k \le 2r - 2$, the connecting homomorphism δ of (6.6) is an isomorphism of one-dimensional vector spaces with Frobenius and we get

$$(6.8) \qquad \det\big(I - t\hat{\alpha}_k \mid H^k(\widehat{\Omega}_{b,0}^\bullet)\big) = \det\big(I - t\alpha_{k+1} \mid H^{k+1}(\widetilde{\Omega}_{b,0}^\bullet)\big).$$

Combining this with the observation (6.7) gives for k even, $0 \le k \le 2r - 4$,

$$(6.9) \qquad \det\big(I - t\hat{\alpha}_k \mid H^k(\widehat{\Omega}_{b,0}^\bullet)\big) = \det\big(I - qt\hat{\alpha}_{k+2} \mid H^{k+2}(\widehat{\Omega}_{b,0}^\bullet)\big).$$

From the definition of $\hat{\alpha}_0$ we have $\det(I - t\hat{\alpha}_0 \mid H^0(\widehat{\Omega}_{b,0}^\bullet)) = (1 - q^{n+r+1}t)$, hence for k even, $0 \le k \le 2r - 2$, we have by (6.9) that

$$(6.10) \qquad \det\big(I - t\hat{\alpha}_k \mid H^k(\widehat{\Omega}_{b,0}^\bullet)\big) = \big(1 - q^{n+r+1-(k/2)}t\big).$$

From (6.8), we then get for these same k that

$$(6.11) \qquad \det\big(I - t\alpha_{k+1} \mid H^{k+1}(\widetilde{\Omega}_{b,0}^\bullet)\big) = \big(1 - q^{n+r+1-(k/2)}t\big).$$

Finally, taking $k = 2r - 2$ in (6.11) and using (6.7) gives

$$(6.12) \qquad \det\big(I - t\hat{\alpha}_{2r} \mid H^{2r}(\widehat{\Omega}_{b,0}^\bullet)\big) = (1 - q^{n+1}t).$$

By Propositions 3.2 and A.4, the cohomology class $[\xi_r]$ is a basis for $H^{2r}(\Omega_{b,0}^\bullet)$, and by Propositions 4.3 and A.4, the cohomology class $[\eta_r]$ is a basis for $H^{2r}(\widehat{\Omega}_{b,0}^\bullet)$.

PROPOSITION 6.2. *Let $r < n$. Relative to the bases $[\xi_r]$ for $H^{2r}(\Omega_{b,0}^\bullet)$ and $[\eta_r]$ for $H^{2r}(\widehat{\Omega}_{b,0}^\bullet)$, the map*

$$\theta : H^{2r}(\Omega_{b,0}^\bullet) \to H^{2r}(\widehat{\Omega}_{b,0}^\bullet)$$

is multiplication by $(-1)^{r(r-1)/2}d_1 \cdots d_r$. In particular, this map is an isomorphism of Frobenius modules hence,

$$\det\big(I - t\alpha_{2r} \mid H^{2r}(\Omega_{b,0}^\bullet)\big) = (1 - q^{n+1}t).$$

Proof. The first assertion is an immediate consequence of Proposition 4.5. The second is an immediate consequence of the first and equation (6.12). □

From Propositions 4.6 and A.4, we get the following result.

PROPOSITION 6.3. *For $2r \le k < n + r$, $H^k(\widetilde{\Omega}_{b,0}^\bullet) = H^{k+1}(\widehat{\Omega}_{b,0}^\bullet) = 0$.*

We can now state the main consequence of our cohomological computations.

THEOREM 6.4.

$$L(\mathbf{A}^{n+r+1}, \Psi, F; t)$$

$$= (1 - q^{n+1}t)^{-1} \left(\frac{\det(I - t\alpha_{n+r+1} \mid H^{n+r+1}(\Omega^{\bullet}_{b,0}))}{\det(I - qt\alpha_{n+r+1} \mid H^{n+r+1}(\Omega^{\bullet}_{b,0}))} \right)^{(-1)^{n+r}}.$$

Proof. From (2.20) and Proposition 3.1, if $r < n$ then

$$L(\mathbf{A}^{n+1+r}, \Psi, F; t)$$

$$= \det(I - t\alpha_{2r} \mid H^{2r}(\Omega^{\bullet}_{b,0}))^{-1} \det(I - t\alpha_{n+r} \mid H^{n+r}(\Omega^{\bullet}_{b,0}))^{(-1)^{n+r+1}}$$

$$\times \det(I - t\alpha_{n+r+1} \mid H^{n+r+1}(\Omega^{\bullet}_{b,0}))^{(-1)^{n+r}},$$

while if $r = n$ (so that $2r = n + r$), then

$$L(\mathbf{A}^{n+1+r}, \Psi, F; t)$$

$$= \det(I - t\alpha_{n+r} \mid H^{n+r}(\Omega^{\bullet}_{b,0}))^{-1} \det(I - t\alpha_{n+r+1} \mid H^{n+r+1}(\Omega^{\bullet}_{b,0})).$$

If $r < n$, then by Proposition 6.2, Theorem 6.4 reduces to proving that

$$(6.13) \quad \det(I - t\alpha_{n+r} \mid H^{n+r}(\Omega^{\bullet}_{b,0})) = \det(I - qt\alpha_{n+r+1} \mid H^{n+r+1}(\Omega^{\bullet}_{b,0})).$$

If $r = n$, then Theorem 6.4 reduces to proving that

$$(6.14) \qquad \det(I - t\alpha_{n+r} \mid H^{n+r}(\Omega^{\bullet}_{b,0}))$$

$$= (1 - q^{n+1}t) \det(I - qt\alpha_{n+r+1} \mid H^{n+r+1}(\Omega^{\bullet}_{b,0})).$$

Note that since $\widetilde{\Omega}^k_{b,0} = 0$ for $k > n + r$, the exact sequence (6.6) gives

$$(6.15) \qquad\qquad H^{n+r+1}(\Omega^{\bullet}_{b,0}) \cong H^{n+r+1}(\widehat{\Omega}^{\bullet}_{b,0}).$$

Suppose $r < n - 1$. Then

$$H^{n+r-1}(\widehat{\Omega}^{\bullet}_{b,0}) = H^{n+r}(\widehat{\Omega}^{\bullet}_{b,0}) = 0$$

by Proposition 6.3. Using this in (6.6) gives

$$(6.16) \qquad\qquad H^{n+r}(\Omega^{\bullet}_{b,0}) \cong H^{n+r}(\widetilde{\Omega}^{\bullet}_{b,0}).$$

Equation (6.13) now follows from (6.15), (6.16), and (6.7). If $r = n - 1$, then $H^{n+r}(\widehat{\Omega}^{\bullet}_{b,0}) = 0$ by Proposition 6.3, so (6.6) gives an exact sequence

$$H^{2r}(\widehat{\Omega}^{\bullet}_{b,0}) \xrightarrow{\delta} H^{n+r}(\widetilde{\Omega}^{\bullet}_{b,0}) \to H^{n+r}(\Omega^{\bullet}_{b,0}) \to 0.$$

By Proposition 6.2, the image of δ is spanned by

$$\delta([\theta(\xi_r)]) = [D(\xi_r)] = 0,$$

i.e., δ is the zero map, so (6.16) holds in this case also. Equation (6.13) now follows as in the case $r < n - 1$. Finally, suppose $r = n$. Then $H^{n+r-1}(\widehat{\Omega}^{\bullet}_{b,0}) = 0$ by Proposition 6.1, so (6.6) gives an exact sequence

$$(6.17) \qquad 0 \to H^{n+r}(\widetilde{\Omega}^{\bullet}_{b,0}) \to H^{n+r}(\Omega^{\bullet}_{b,0}) \xrightarrow{\theta} H^{n+r}(\widehat{\Omega}^{\bullet}_{b,0}) \to 0.$$

By (6.7) and (6.15), we have

$$(6.18) \quad \det\big(I - t\alpha_{n+r} \mid H^{n+r}(\widetilde{\Omega}^\bullet_{b,0})\big) = \det\big(I - qt\alpha_{n+r+1} \mid H^{n+r+1}(\Omega^\bullet_{b,0})\big).$$

Equation (6.14) then follows from (6.17), (6.18), and (6.12). This completes the proof of the theorem. $\qquad\square$

COROLLARY 6.5. *Let $P(t)$ be as defined in Section 1. Then*

$$P(q^r t) = \det\big(I - t\alpha_{n+r+1} \mid H^{n+r+1}(\Omega^\bullet_{b,0})\big).$$

Proof. Let $g(t) \in 1 + t\tilde\Lambda_0[t]$ and define $r(t) = g(t)/g(qt) \in 1 + t\tilde\Lambda_0[[t]]$. Then $\prod_{i=0}^{m-1} r(q^i t) = g(t)/g(q^m t)$. It follows that

$$g(t) = \lim_{m \to \infty} \prod_{i=0}^{m-1} r(q^i t),$$

in the sense that the coefficients of the power series on the right-hand side converge term-by-term to the coefficients of $g(t)$. In particular, $g(t)$ is uniquely determined by $r(t)$. The assertion of the corollary now follows from (2.3) and Theorem 6.4. $\qquad\square$

By Corollary 6.5, Theorem 1.1 is equivalent to the following result.

THEOREM 6.6. *Suppose that $f_1 = \cdots = f_r = 0$ defines a smooth complete intersection X in $\mathbf{P}^n$. Then the Newton polygon of $\det(I - t\alpha_{n+r+1} \mid H^{n+r+1}(\Omega^\bullet_{b,0}))$ with respect to ord_q lies on or above the Newton polygon with respect to ord_q of the polynomial $\prod_{e=r}^{n}(1 - q^e t)^{h_e}$.*

We begin with a reduction step. In (2.15), we defined a $\tilde\Lambda_1$-linear endomorphism β_{n+r+1} of $H^{n+r+1}(\Omega^\bullet_{b,0})$ such that $\alpha_{n+r+1} = (\beta_{n+r+1})^a$. Let ord denote the p-adic valuation normalized by $\mathrm{ord}\, p = 1$. By [9, Lemma 7.1], we have the following.

LEMMA 6.7. *The Newton polygon of $\det(I - t\alpha_{n+r+1} \mid H^{n+r+1}(\Omega^\bullet_{b,0}))$ with respect to the valuation ord_q is obtained from the Newton polygon of $\det(I - t\beta_{n+r+1} \mid H^{n+r+1}(\Omega^\bullet_{b,0}))$ with respect to the valuation ord by shrinking ordinates and abscissas by a factor of $1/a$.*

Theorem 6.6 is thus equivalent to the following result.

THEOREM 6.8. *Suppose that $f_1 = \cdots = f_r = 0$ defines a smooth complete intersection X in $\mathbf{P}^n$. Then the Newton polygon of $\det(I - t\beta_{n+r+1} \mid H^{n+r+1}(\Omega^\bullet_{b,0}))$ with respect to ord lies on or above the Newton polygon with respect to ord of the polynomial $\prod_{e=r}^{n}(1 - p^e t)^{a h_e}$.*

Proof. Let $\{\gamma_m\}_{m=1}^a$ be an integral basis for $\tilde\Lambda_0$ over $\tilde\Lambda_1$. By Theorem 5.7 and the definition of an integral basis, the cohomology classes

$$(6.19) \qquad \big[\gamma_m \xi_l^{(e)}\big], \quad e = r, \ldots, n,\, l = 1, \ldots, h_e,\, m = 1, \ldots, a,$$

form a basis for $H^{n+r+1}(\Omega_{b,0}^\bullet)$ as $\tilde{\Lambda}_1$-vector space. It is straightforward to check from the definitions that

$$\beta\big(F^s C(b/p)\big) \subseteq F^s C(b)$$

for all $s \in \mathbf{Z}$. Using this, one checks that

(6.20)
$$\beta_{n+r+1}\big(\gamma_m \xi_l^{(e)}\big) \in F^{Mbe(p-1)/p}\Omega_{b,0}^{n+r+1}.$$

By Corollary A.5, $[\beta_{n+r+1}(\gamma_m \xi_l^{(e)})]$ is a $\tilde{\Lambda}_1$-linear combination of the $[\gamma_{m'}\xi_{l'}^{(e')}]$ with coefficients in $\pi^{Mbe(p-1)/p}\mathcal{O}_{\tilde{\Lambda}_1}$. This says that in the matrix of β_{n+r+1} relative to the basis (6.19), the column corresponding to $[\gamma_m \xi_l^{(e)}]$ has all entries divisible by $\pi^{Mbe(p-1)/p}$. This implies that the Newton polygon of $\det(I - t\beta_{n+r+1} \mid H^{n+r+1}(\Omega_{b,0}^\bullet))$ with respect to the valuation ord lies on or above the Newton polygon with respect to the valuation ord of the polynomial

$$\prod_{e=r}^n \big(1 - \pi^{Mbe(p-1)/p}t\big)^{ah_e}.$$

But $\det(I - t\beta_{n+r+1} \mid H^{n+r+1}(\Omega_{b,0}^\bullet))$ is independent of b by Corollary 6.5, so we may take the limit as $b \to p/(p-1)$ to conclude (recall that $\pi^M = p$) that its Newton polygon lies on or above the Newton polygon of $\prod_{e=r}^n (1 - p^e t)^{ah_e}$. $\square$

Appendix

In this section, we collect (with references) some basic results on "lifting" cohomology from characteristic p to characteristic zero. Let $\mathcal{O}$ be a complete discrete valuation ring with uniformizer π. Call an $\mathcal{O}$-module M *flat* if multiplication by π is injective and call M *separated* if $\bigcap_{j=1}^\infty \pi^j M = 0$. A separated $\mathcal{O}$-module M has an obvious metric space structure with the $\{\pi^j M\}_{j=1}^\infty$ forming a fundamental system of neighborhoods of 0. Call M $\mathcal{O}$-*complete* if it is complete in this metric. Let

$$C^\bullet = \{0 \to C^0 \xrightarrow{\partial} C^1 \to \cdots\}$$

be a complex of flat, separated, $\mathcal{O}$-complete $\mathcal{O}$-modules with $\mathcal{O}$-linear boundary maps. Let

$$\bar{C}^\bullet = \{0 \to \bar{C}^0 \xrightarrow{\bar{\partial}} \bar{C}^1 \to \cdots\}$$

be the complex obtained by reducing $C^\bullet$ modulo π, i.e., $\bar{C}^i = C^i/\pi C^i$ and the boundary maps of $\bar{C}^\bullet$ are those induced by the boundary maps of $C^\bullet$.

The first assertion of the following result is [1, Theorem A.1(a)]. The second assertion follows from the proof of [1, Theorem A.1(a)].

PROPOSITION A.1. *If $H^i(\bar{C}^\bullet) = 0$ for some i, then $H^i(C^\bullet) = 0$. More precisely, if $\omega \in C^i$ satisfies $\partial(\omega) = 0$ and if $\bar{\omega} = \bar{\partial}(\eta)$ for some $\eta \in \bar{C}^{i-1}$ (where $\bar{\omega}$ denotes the image of ω in $\bar{C}^i$), then there exists $\xi \in C^{i-1}$ such that $\partial(\xi) = \omega$ and $\bar{\xi} = \eta$.*

The next result is [2, Lemma 4.1].

PROPOSITION A.2. *If $H^i(\bar{C}^\bullet)$ is of finite dimension d over $\mathcal{O}/(\pi)$ and multiplication by π is injective on $H^i(C^\bullet)$ and $H^{i+1}(C^\bullet)$, then $H^i(C^\bullet)$ is a free $\mathcal{O}$-module of rank d. Furthermore, if $\xi_1, \ldots, \xi_d \in C^i$ satisfy* (a) $\partial(\xi_j) = 0$ *for $j = 1, \ldots, d$ and* (b) *the cohomology classes $[\bar{\xi}_1], \ldots, [\bar{\xi}_d]$ form a basis for $H^i(\bar{C}^\bullet)$, then the cohomology classes $[\xi_1], \ldots, [\xi_d]$ form a basis for $H^i(C^\bullet)$.*

We shall also need some additional results.

PROPOSITION A.3. *Let $\{\xi_j\}_{j=1}^N \subseteq C^i$ satisfy* (a) $\partial(\xi_j) = 0$ *for all j and* (b) *the cohomology classes $\{[\bar{\xi}_j]\}_{j=1}^N$ span $H^i(\bar{C}^\bullet)$. Then the cohomology classes $\{[\xi_j]\}_{j=1}^N$ span $H^i(C^\bullet)$.*

Proof. Let $\omega \in C^i$ with $\partial(\omega) = 0$. Then $\bar{\partial}(\bar{\omega}) = 0$, so there exist $\alpha_j^{(0)} \in \mathcal{O}$ and $\eta_0 \in C^{i-1}$ such that

$$(A.1) \qquad \omega = \sum_{j=1}^N \alpha_j^{(0)} \xi_j + \partial(\eta_0) + \pi \omega_0$$

for some $\omega_0 \in C^i$. Suppose that for some $m \geq 0$, we have

$$(A.2) \qquad \omega = \sum_{j=1}^N \alpha_j^{(m)} \xi_j + \partial(\eta_m) + \pi^{m+1} \omega_m$$

for some $\alpha_j^{(m)} \in \mathcal{O}$, $\eta_m \in C^{i-1}$, and $\omega_m \in C^i$ with

$$\alpha_j^{(m)} - \alpha_j^{(m-1)} \in \pi^m \mathcal{O} \quad \text{and} \quad \eta_m - \eta_{m-1} \in \pi^m C^{i-1}.$$

Equation (A.2) implies that $\partial(\omega_m) = 0$, so as in (A.1), we have

$$(A.3) \qquad \omega_m = \sum_{j=1}^N \alpha_j' \xi_j + \partial(\eta') + \pi \omega_{m+1}$$

for some $\alpha_j' \in \mathcal{O}$, $\eta' \in C^{i-1}$, and $\omega_{m+1} \in C^i$. Put

$$\alpha_j^{(m+1)} = \alpha_j^{(m)} + \pi^{m+1} \alpha_j' \quad \text{and} \quad \eta_{m+1} = \eta_m + \pi^{m+1} \eta'.$$

Substituting (A.3) into (A.2) gives

$$\omega = \sum_{j=1}^N \alpha_j^{(m+1)} \xi_j + \partial(\eta_{m+1}) + \pi^{m+2} \omega_{m+1}$$

with

$$\alpha_j^{(m+1)} - \alpha_j^{(m)} \in \pi^{m+1} \mathcal{O} \quad \text{and} \quad \eta_{m+1} - \eta_m \in \pi^{m+1} C^{i-1}.$$

It follows that each sequence $\{\alpha_j^{(m)}\}_{m=0}^\infty$ converges to an element $\alpha_j \in \mathcal{O}$ and $\{\eta_j\}_{j=0}^\infty$ converges to an element $\eta \in C^{i-1}$ satisfying

$$\omega = \sum_{j=1}^N \alpha_j \xi_j + \partial(\eta),$$

which shows that the $[\xi_j]$ span $H^i(C^\bullet)$. $\square$

Let Λ be the quotient field of $\mathcal{O}$ and suppose that $D^\bullet = \{0 \to D^0 \xrightarrow{\partial} D^1 \to \cdots\}$ is a complex of Λ-vector spaces containing $C^\bullet$ as a subcomplex such that

$$D^\bullet = \bigcup_{s \in \mathbf{Z}} \pi^s C^\bullet.$$

PROPOSITION A.4. *Suppose that $H^i(C^\bullet) = H \oplus H'$, where H is a free $\mathcal{O}$-module with basis the cohomology classes $[\xi_1], \ldots, [\xi_d]$ and H' is a torsion $\mathcal{O}$-module spanned by the cohomology classes $[\eta_1], \ldots, [\eta_e]$. Then the cohomology classes $[\xi_1], \ldots, [\xi_d]$ form a basis for $H^i(D^\bullet)$ as Λ-vector space.*

Proof. Since multiplication by π^s is an isomorphism from $C^\bullet$ to $\pi^s C^\bullet$, it follows that $H^i(\pi^s C^\bullet) = H_s \oplus H'_s$, where H_s is a free $\mathcal{O}$-module with basis the cohomology classes $[\pi^s \xi_1], \ldots, [\pi^s \xi_d]$ and H'_s is a torsion $\mathcal{O}$-module spanned by the cohomology classes $[\pi^s \eta_1], \ldots, [\pi^s \eta_e]$. Let $\xi \in D^i$ with $\partial(\xi) = 0$. Since $\xi \in \pi^s C^i$ for some s, we have

$$\xi = \sum_{k=1}^d c_k \pi^s \xi_k + \sum_{l=1}^e c'_l \pi^s \eta_l + \partial(\zeta)$$

for some $c_k, c'_l \in \mathcal{O}$ and some $\zeta \in \pi^s C^{i-1}$. Since the $[\pi^s \eta_l]$ are torsion elements, it follows that there exists a positive integer t such that

$$\pi^t \pi^s \eta_l = \partial(\zeta_l)$$

for $l = 1, \ldots, e$, where $\zeta_l \in \pi^s C^{i-1}$. Substitution then gives

$$\xi = \sum_{k=1}^d c_k \pi^s \xi_k + \partial\left(\zeta + \sum_{l=1}^e c'_l \pi^{-t} \zeta_l\right),$$

which shows that the $[\xi_k]$ span $H^i(D^\bullet)$.

Suppose there are $c_k \in \Lambda$ and $\zeta \in D^{i-1}$ such that

$$\sum_{k=1}^d c_k \xi_k = \partial(\zeta).$$

There exists an integer s such that $\pi^s c_k \in \mathcal{O}$ for all k and $\zeta \in \pi^{-s} C^{i-1}$. Thus, we have

$$\sum_{k=1}^d (\pi^s c_k)(\pi^{-s} \xi_k) = \partial(\zeta).$$

But if some $c_k \neq 0$, this contradicts the fact that the $[\pi^{-s}\xi_k]$ are a basis for the free $\mathcal{O}$-module H_{-s}. This proves that the $[\xi_k]$ are linearly independent in $H^i(D^\bullet)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The proof of Proposition A.4 shows that the following assertion holds.

COROLLARY A.5. *Let* $\xi \in D^i$ *with* $\partial(\xi) = 0$. *If* $\xi \in \pi^s C^i$, *then*

$$[\xi] = \sum_{k=1}^{d} c_k [\xi_k]$$

with $c_k \in \pi^s \mathcal{O}$ *for all* k.

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