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Research Article

Variational and Numerical Analysis for Frictional Contact Problem with Normal Compliance in Thermo-Electro-Viscoelasticity

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In this paper, we consider a mathematical model of a contact problem in thermo-electro-viscoelasticity with the normal compliance conditions and Tresca's friction law. We present a variational formulation of the problem, and we prove the existence and uniqueness of the weak solution. We also study the numerical approach using spatially semidiscrete and fully discrete finite element schemes with Euler's backward scheme. Finally, we derive error estimates on the approximate solutions.

1. Introduction

In the recent years, piezoelectric contact problems have been of great interest to modern engineering. General models of electroelastic characteristics of piezoelectric materials can be found in [1, 2]. The problems of piezoviscoelastic materials have been studied with different contact conditions within linearized elasticity in [3–5] and within nonlinear viscoelasticity in [6–8]. The modeling of these problems does not take into account the thermic effect. Mindlin [9] was the first to propose the thermo-piezoelectric model. The mathematical model which describes the frictional contact between a thermo-piezoelectric body and a conductive foundation is already addressed in the static case in [10, 11].

Sofonea et al. considered in [12] the modeling of quasistatic viscoelastic problem with normal compliance friction and damage; they proved the existence and uniqueness of the weak solution, and they derived error estimates on the approximate solutions.

In the article [13], we find the recent result of a new quasistatic mathematical model which describes the chosen thermo-electro-viscoelastic body behavior and the contact by Signorini condition with nonfrictional and nonconductive foundation; also, the variational formulation of this problem is derived and its unique weak solvability is established.

In this paper, we consider a quasi-static contact problem with Tresca's friction between a thermo-electro viscoelastic body and an electrically and thermally conductive rigid foundation. The novelty in this model, which can be considered as the generalization of the model presented in [13], lies in the use of the penalized normal compliance contact condition:

$$\sigma_\nu(u_\nu - g) = -\frac{1}{\varepsilon}[u_\nu - g]^+, \quad \varepsilon > 0. \tag{1}$$

This means that we allow a weak interpenetration between the body and the foundation. On the contact zone, we consider the following regularized electrical and thermal conditions:

$$\begin{aligned} D\cdot\nu &= \psi(u_\nu - g)\phi_L(\varphi - \varphi_F), \\ \frac{\partial q}{\partial \nu} &= k_c(u_\nu - g)\phi_L(\theta - \theta_F), \end{aligned} \tag{2}$$

which describe both the thermal and electrical conductivities of the foundation. This leads to nonlinear coupling between the mechanical displacement and thermal and electrical fields and hence more complexities on the model.

Since the friction conditions are inequalities, we derive a quasivariational formulation of this problem and we prove the existence and uniqueness of the weak solution based on arguments variational inequalities, Galerkin method, compactness method, and Banach fixed point theorem. We derive error estimates for the numerical approximations based on discrete schemes.

The paper is structured as follows. In Section 2, we present the model of equilibrium process of the thermo-electro-viscoelastic body in frictional contact with a conductive rigid foundation and we introduce the notations and assumptions on the problem data. We also derive the variational formulation of the problem. We state the main results concerning the existence and the uniqueness of a weak solution. We present a spatially semidiscrete scheme and a fully discrete scheme to approximate the contact problem. We then use the finite element method to discretize the domain Ω and Euler's forward scheme to discretize the time derivatives. Finally, the proofs are established in Section 3.

2. Formulation and Main Results

2.1. Problem Setting. We consider a body of a piezoelectric material which occupies the domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ in the reference configuration which will be supposedly bounded with a smooth boundary $\partial\Omega = \Gamma$. This boundary is divided into three open disjoint parts Γ_D, Γ_N, and Γ_C on one hand and a partition of $\Gamma_D \cup \Gamma_N$ into two open parts Γ_a and Γ_b on the other hand, such that $\mathrm{meas}(\Gamma_D) > 0$ and $\mathrm{meas}(\Gamma_a) > 0$. Let $[0; T]$ be the time interval, where $T > 0$.

The body is submitted to the action of body forces of density f_0, a volume electric charge of density q_0, and a heat source of constant strength q_1. It is also submitted to mechanical, electrical, and thermal constants on the boundary. Indeed, the body is assumed to be clamped in Γ_D, and therefore, the displacement field vanishes there. Moreover, we assume that a density of traction forces, denoted by f_2, acts on the boundary part Γ_N. We also assume that the electrical potential vanishes on Γ_a, and a surface electrical charge of density q_2 is prescribed on Γ_b. We consider that the temperature θ_0 is prescribed on the surface $\Gamma_D \cup \Gamma_N$.

In the reference configuration, the body may come in contact over Γ_C with an electrically thermally conductive foundation. Assume that its potential and temperature are maintained at φ_F and θ_F. The contact is frictional, and there may be electrical charges and heat transfer on the contact surface. The normalized gap between Γ_C and the rigid foundation is denoted by g.

In the following sections, we use $\mathbb{S}^d$ to denote the space of second-order symmetric tensors on $\mathbb{R}^d$ while "." and $|.|$ will represent the inner product and the Euclidean norm on $\mathbb{S}^d$ and $\mathbb{R}^d$, that is

$$\begin{gathered} u \cdot v = u_i v_i, \\ |v| = (v \cdot v)^{1/2}, \\ \forall u, v \in \mathbb{R}^d, \\ \sigma \cdot \tau = \sigma_{ij}\tau_{ij}, \\ |\tau| = (\tau \cdot \tau)^{1/2}, \\ \forall \sigma, \tau \in \mathbb{S}^d. \end{gathered} \tag{3}$$

We denote $u : \Omega \times [0; T] \longrightarrow \mathbb{R}^d$ as the displacement field, $\sigma : \Omega \longrightarrow \mathbb{S}^d$ and $\sigma = (\sigma_{ij})$ the stress tensor, $\theta : \Omega \times [0; T] \longrightarrow \mathbb{R}^d$ the temperature, $q : \Omega \longrightarrow \mathbb{R}^d$ and $q = (q_i)$ the heat flux vector, and $D : \Omega \longrightarrow \mathbb{R}^d$ and $D = (D_i)$ the electric displacement field. We also denote $E(\varphi) = (E_i(\varphi))$ as the electric vector field, where $\varphi : \Omega \times [0; T] \longrightarrow \mathbb{R}$ is the electric potential. Moreover, let $\varepsilon(u) = (\varepsilon_{ij}(u))$ denote the linearized strain tensor given by $\varepsilon_{ij}(u) = 1/2(u_{i,j} + u_{j,i})$, and "Div" and "div" denote the divergence operators for tensor and vector valued functions, respectively, i.e., $\mathrm{Div}\,\sigma = (\sigma_{ij,j})$ and $\mathrm{div}\,\xi = (\xi_{j,j})$. We shall adopt the usual notation for normal and tangential components of displacement vector and stress: $v_n = v \cdot n$, $v_\tau = v - v_n n$, $\sigma_n = (\sigma n) \cdot n$, and $\sigma_\tau = \sigma n - \sigma_n n$, where n denotes the outward normal vector on Γ.

Problem (P). Find a displacement field $u : \Omega \times]0, T[\longrightarrow \mathbb{R}^d$, an electric potential $\varphi : \Omega \times]0, T[\longrightarrow \mathbb{R}$, and a temperature field $\theta : \Omega \times]0, T[\longrightarrow \mathbb{R}$ such that

$$\begin{aligned} \sigma = \mathfrak{F}\varepsilon(u) - \varepsilon^* E(\varphi) - \theta\mathscr{M} \\ + C\varepsilon(\dot{u}) \text{ in } \Omega \times (0, T), \end{aligned} \tag{4}$$

$$\begin{aligned} D = \mathscr{E}\varepsilon(u) + \beta E(\varphi) \\ - (\theta - \theta^*)\mathscr{P} \text{ in } \Omega \times (0, T), \end{aligned} \tag{5}$$

$$\mathrm{Div}\,\sigma + f_0 = 0, \text{ in } \Omega \times (0, T), \tag{6}$$

$$\mathrm{div}\,D = q_0, \text{ in } \Omega \times (0, T), \tag{7}$$

$$\dot{\theta} + \mathrm{div}\,q = q_1, \text{ in } \Omega \times (0, T), \tag{8}$$

$$u = 0, \text{ on } \Gamma_D \times (0, T), \tag{9}$$

$$\sigma\nu = f_2, \text{ on } \Gamma_N \times (0, T), \tag{10}$$

$$\varphi = 0, \text{ on } \Gamma_a \times (0, T), \tag{11}$$

$$D \cdot \nu = q_b \text{ on } \Gamma_b \times (0, T), \tag{12}$$

$$\theta = 0, \text{ on } (\Gamma_D \cup \Gamma_N) \times (0, T), \tag{13}$$

$$u(0, x) = u_0, \text{ in } \Omega, \tag{14}$$

$$\theta(0,x) = \theta_0, \text{ in } \Omega, \tag{15}$$

$$\sigma_\nu(u_\nu - g) = -\frac{1}{\varepsilon}[u_\nu - g]^+, \quad \varepsilon > 0, \text{ on } \Gamma_C \times (0,T), \tag{16}$$

$$\|\sigma_\tau\| \le S, \text{ on } \Gamma_C \times (0,T), \tag{17}$$

$$\|\sigma_\tau\| < S \Longrightarrow \dot{u}_\tau = 0, \text{ on } \Gamma_C \times (0,T), \tag{18}$$

$$\begin{gathered} \|\sigma_\tau\| = S \Longrightarrow \exists \lambda \neq 0 \\ \text{such that } \sigma_\tau = -\lambda \dot{u}_\tau \text{ on } \Gamma_C \times (0,T), \end{gathered} \tag{19}$$

$$D \cdot \nu = \psi(u_\nu - g)\phi_L(\varphi - \varphi_F), \text{ on } \Gamma_C \times (0,T), \tag{20}$$

$$\frac{\partial q}{\partial \nu} = k_c(u_\nu - g)\phi_L(\theta - \theta_F), \text{ on } \Gamma_C \times (0,T). \tag{21}$$

Here, the equations (4) and (5) represent the thermo-electro-viscoelastic constitutive law of the material in which $\sigma = (\sigma_{ij})$ denotes the stress tensor, $\varepsilon(u)$ is the linearized strain tensor, $E(\varphi)$ is the electric field. $\mathfrak{F} = (f_{ijkl})$, $\mathscr{E} = (e_{ijk})$, $\mathscr{M} = (m_{ij})$, $\beta = (\beta_{ij})$, $\mathscr{P} = (p_i)$, and $C = (c_{ijkl})$ are, respectively, elastic, piezoelectric, thermal expansion, electric permittivity, pyroelectric tensor, and (fourth-order) viscosity tensor. $\mathscr{E}^*$ is the transpose of $\mathscr{E}$ given by

$$\mathscr{E}^* = (e^*_{ijk}), \quad e^*_{ijk} = e_{kij}, \tag{22}$$

$$\begin{gathered} \mathscr{E}\sigma v = \sigma \mathscr{E}^* v, \\ \forall \sigma \in \mathbb{S}^d, \\ v \in \mathbb{R}^d. \end{gathered} \tag{23}$$

The constant θ^* represents the reference temperature.

Fourier's law of heat conduction is given by

$$q = -\mathscr{K}\nabla\theta, \text{ in } \Omega \times (0,T), \tag{24}$$

where $\mathscr{K} = (k_{ij})$ denotes the thermal conductivity tensor.

Equations (6)–(8) represent the equilibrium equations for the stress. Relations (9) and (10), (11) and (12), and (13) represent the mechanical, the electrical, and the thermal boundary conditions. The unilateral boundary condition (16) represents the normal compliance condition and (17)–(19) represent Tresca's friction law in which S is the coefficient of friction.

Following [14], the contact conditions (20) and (21) on Γ_C are obtained as follows:

When there is no contact at a point on the surface, there is no free electrical charges on the surface and no thermal transfer; that is

$$u_\nu < g \Longrightarrow \begin{cases} D \cdot \nu = 0, \\ q \cdot \nu = 0. \end{cases} \tag{25}$$

If the contact holds, i.e., $u_\nu \ge g$, the normal component of the electric displacement field or the free charge (resp., thermal transfer) is assumed to be proportional to the difference between the potential of foundation and the body's surface potential (resp., to the difference between the temperature of foundation and the body's surface temperature). Thus,

$$u_\nu \ge g \Longrightarrow \begin{cases} D \cdot \nu = k_\varphi(\varphi - \varphi_F), \\ q \cdot \nu = k_\theta(\theta - \theta_F). \end{cases} \tag{26}$$

We combine (25) and (26) to obtain

$$\begin{cases} D \cdot \nu = k_\varphi \chi_{[0,+\infty)}(u_\nu - g)(\varphi - \varphi_F), \\ q \cdot \nu = k_\theta \chi_{[0,+\infty)}(u_\nu - g)(\theta - \theta_F), \end{cases} \tag{27}$$

where $\chi_{[0,+\infty)}$ is the characteristic function of the interval $[0,+\infty)$ defined by

$$\chi_{[0,+\infty)}(s) = \begin{cases} 0, & \text{if } s < 0, \\ 1, & \text{if } s \ge 0. \end{cases} \tag{28}$$

Equation (27) represents the regularization electrical contact condition and the heat flux condition on Γ_C, where

$$\begin{gathered} \phi_L(s) = \begin{cases} -L, & \text{if } s < -L, \\ s, & \text{if } -L \le s \le L, \\ L, & \text{if } s > L, \end{cases} \\ \psi(r) = \begin{cases} 0, & \text{if } r < 0, \\ c\delta r, & \text{if } 0 \le r \le \dfrac{1}{\delta}, \\ c, & \text{if } r > \dfrac{1}{\delta}, \end{cases} \end{gathered} \tag{29}$$

and where $c = k_\varphi, k_\theta$, and L is a large positive constant, $\delta > 0$ is a small parameter, $k_c : r \longrightarrow k_c(r)$ is supposed to be zero for $r < 0$ and positive, otherwise nondecreasing and Lipschitz continuous.

Remark 1. We note that when $\psi \equiv 0$, equality (20) becomes

$$D \cdot \nu = 0 \text{ on } \Gamma_C \times (0,T), \tag{30}$$

which models the case when the foundation is a perfect electric insulator.

Similarly, in equality (21), we have

$$q \cdot \nu = 0 \text{ on } \Gamma_C \times (0,T). \tag{31}$$

2.2. Weak Formulation and Uniqueness Result. To obtain a variational formulation of Problem (P), we need additional notations and need to recall some definitions in the sequel.

We use the following functional Hilbert spaces:

$$L^2(\Omega) = L^2(\Omega)^d,$$
$$H^1(\Omega) = H^1(\Omega)^d, \tag{32}$$
$$\mathscr{H} = \left\{\sigma \in \mathbb{S}^d : \sigma = \sigma_{ij}, \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\right\},$$

$$\mathscr{W} = \left\{D = (D)_i \in H^1(\Omega) : D_i \in L^2(\Omega), \operatorname{div} D \in L^2(\Omega)\right\}, \tag{33}$$

endowed with the canonical inner product given by

$$(u,v)_{L^2(\Omega)} = \int_\Omega u_i v_i dx,$$
$$(\sigma,\tau)_{\mathscr{H}} = \int_\Omega \sigma_i \tau_i dx, \tag{34}$$
$$(u,v)_{H^1(\Omega)} = (u,v)_{L^2(\Omega)} + (\varepsilon(u), \varepsilon(v))_{\mathscr{H}},$$
$$(D \cdot E) = (D \cdot E)_{L^2(\Omega)} + (\operatorname{div} D \cdot \operatorname{div} E)_{L^2(\Omega)},$$

and the associated norms $\|.\|_{L^2(\Omega)}$, $\|.\|_{H^1(\Omega)}$, $\|.\|_{\mathscr{H}}$, and $\|.\|_{\mathscr{W}}$, respectively.

Let V, W, and Q be the closed subspaces of $H^1(\Omega)$ given by

$$V = \left\{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\right\},$$
$$W = \left\{\xi \in H^1(\Omega) : \xi = 0 \text{ on } \Gamma_a\right\}, \tag{35}$$
$$Q = \left\{\eta \in H^1(\Omega) : \eta = 0 \text{ on } \Gamma_D \cup \Gamma_N\right\},$$

and the set of admissible displacements

$$V_{\text{ad}} = \{v \in V : v_\nu - g \le 0 \text{ on } \Gamma_C\}. \tag{36}$$

It is known that V, W, and Q are real Hilbert spaces with the inner products $(u,v)_V = (\varepsilon(u), \varepsilon(v))_{\mathscr{H}}$, $(\varphi,\xi)_W = (\nabla\varphi, \nabla\xi)_{L^2(\Omega)}$, and $(\theta,\eta)_Q = (\nabla\theta, \nabla\eta)_{L^2(\Omega)}$, respectively.

Moreover, the associated norm $\|v\|_V = \|\varepsilon(v)\|_{\mathscr{H}}$ is equivalent on V to the usual norm $\|.\|_{H^1(\Omega)}$ and $\|\xi\|_W = \|\nabla\xi\|_{L^2(\Omega)}$ and $\|\eta\|_Q = \|\nabla\eta\|_{L^2(\Omega)}$ are equivalent on W and Q, respectively, with the usual norms $\|.\|_{H^1(\Omega)}$.

By Sobolev's trace theorem, there exists three positive constants C_{s1}, C_{s2}, and C_{s3} depending on Ω, Γ_C, Γ_N, Γ_D, Γ_a, and Γ_b:

$$\|v\|_{L^2(\Gamma)^d} \le C_{s1}\|v\|_V, \quad \forall v \in V, \tag{37}$$

$$\|\xi\|_{L^2(\Gamma_c)} \le C_{s2}\|\xi\|_W, \quad \forall \xi \in W, \tag{38}$$

$$\|\eta\|_{L^2(\Gamma_c)} \le C_{s3}\|\eta\|_Q, \quad \forall \eta \in Q. \tag{39}$$

Since $\operatorname{meas}(\Gamma_D) > 0$ and Korn's inequality hold

$$\|\varepsilon(v)\|_{\mathscr{H}} \ge C_K\|v\|_{H^1(\Omega)}, \quad \forall v \in V, \tag{40}$$

where C_K in a nonnegative constant depending only on Ω and Γ_D. Notice also that since $\operatorname{meas}(\Gamma_a) > 0$, the following Friedrichs–Poincaré inequalities hold:

$$\|\nabla\xi\|_{\mathscr{W}} \ge C_{F1}\|\xi\|_W, \quad \forall \xi \in W, \tag{41}$$

$$\|\nabla\eta\|_{L^2(\Omega)} \ge C_{F2}\|\eta\|_Q, \quad \forall \eta \in Q, \tag{42}$$

where C_{F1} and C_{F2} are the positive constants which depend only on Ω, Γ_a, Γ_D, and Γ_N.

For a real Banach space X and $1 \le p \le \infty$, we consider the Banach spaces $C(0,T;X)$ and $C^1(0,T;X)$ of continuous and continuously differentiable functions from $[0,T]$ to X with the norms

$$\|u\|_{C(0,T;X)} = \sup_{t\in[0,T]} \|u(t)\|_X,$$
$$\|u\|_{C^1(0,T;X)} = \sup_{t\in[0,T]} \|u(t)\|_X + \sup_{t\in[0,T]} \|\dot{u}(t)\|_X. \tag{43}$$

To simplify the writing, we denote by $\mathfrak{a} : V \times V \longrightarrow \mathbb{R}$, $\mathfrak{b} : W \times W \longrightarrow \mathbb{R}$, $\mathfrak{c} : V \times V \longrightarrow \mathbb{R}$, and $\mathfrak{d} : Q \times Q \longrightarrow \mathbb{R}$ the following bilinear and symmetric applications:

$$\mathfrak{a}(u,v) := (\mathfrak{F}\varepsilon(u), \varepsilon(v))_{\mathscr{H}},$$
$$\mathfrak{b}(\varphi,\xi) := (\beta\nabla\varphi, \nabla\xi)_{L^2(\Omega)},$$
$$\mathfrak{c}(u,v) := (C\varepsilon(u), \varepsilon(v))_{\mathscr{H}}, \tag{44}$$
$$\mathfrak{d}(\theta,\eta) := (\mathscr{K}\nabla\theta, \nabla\eta)_{L^2(\Omega)},$$

and by $\mathfrak{e} : V \times W \longrightarrow \mathbb{R}$, $\mathfrak{m} : Q \times V \longrightarrow \mathbb{R}$, and $\mathfrak{p} : Q \times W \longrightarrow \mathbb{R}$, the following bilinear applications:

$$\mathfrak{e}(v,\xi) := (\mathscr{E}\varepsilon(v), \nabla\xi)_{L^2(\Omega)} = \left(\mathscr{E}^*\nabla\xi, \varepsilon(v)\right)_V,$$
$$\mathfrak{m}(\theta,v) := (\mathscr{M}\theta, \varepsilon(v))_Q, \tag{45}$$
$$\mathfrak{p}(\theta,\xi) := (\mathscr{P}\nabla\theta, \nabla\xi)_{L^2(\Omega)}.$$

In the study of mechanical Problem (P), we make the following assumptions:

HP$_1$. The elasticity operator $\mathfrak{F} : \Omega \times \mathbb{S}^d \longrightarrow \mathbb{S}^d$, the electric permittivity tensor $\beta = (\beta_{ij}) : \longrightarrow \Omega \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$, the viscosity tensor $\mathfrak{c} : \Omega \times \mathbb{S}^d \longrightarrow \mathbb{S}^d$, and the thermal conductivity tensor $\mathscr{K} = (k_{ij}) : \Omega \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ satisfy the usual properties of symmetry, boundedness, and ellipticity:

$$f_{ijkl} = f_{jikl} = f_{lkij} \in L^\infty(\Omega),$$
$$\beta_{ij} = \beta_{ji} \in L^\infty(\Omega),$$
$$c_{ijkl} = c_{jikl} = c_{lkij} \in L^\infty(\Omega), \tag{46}$$
$$k_{ij} = k_{ji} \in L^\infty(\Omega),$$

and there exists that $m_{\mathfrak{F}}, m_\beta, m_{\mathfrak{c}}, m_{\mathscr{K}} > 0$ such that

$$f_{ijkl}(x)\xi_k\xi_l \ge m_{\mathfrak{F}}\|\xi\|^2, \quad \forall \xi \in \mathbb{S}^d, \forall x \in \Omega,$$

$$c_{ijkl}(x)\xi_k\xi_l \ge m_{\mathfrak{c}}\|\xi\|^2, \quad \forall \xi \in \mathbb{S}^d, \forall x \in \Omega,$$

$$\beta_{ij}\zeta_i\zeta_j \ge m_\beta\|\zeta\|^2, k_{ij}\zeta_i\zeta_j \ge m_{\mathscr{K}}\|\zeta\|^2, \quad \forall \zeta \in \mathbb{R}^d. \tag{47}$$

HP_2. The piezoelectric tensor $\mathrm{E} = (e_{ijk}) : \Omega \times \mathbb{S}^d \longrightarrow \mathbb{R}$, the thermal expansion tensor $\mathscr{M} = (m_{ij}) : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$, and the pyroelectric tensor $\mathscr{P} = (p_i) : \Omega \longrightarrow \mathbb{R}^d$ satisfy

$$\begin{aligned} e_{ijk} &= e_{ikj} \in L^{\infty}(\Omega), \\ m_{ij} &= m_{ji} \in L^{\infty}(\Omega), \\ p_i &\in L^{\infty}(\Omega), \end{aligned} \tag{48}$$

and there exist the positive constants $M_{\mathfrak{F}}, M_{\beta}, M_{\mathfrak{c}}, M_{\mathscr{K}}, M_{\mathscr{E}}, M_{\mathscr{M}}$, and $M_{\mathscr{P}}$ such that

$$\begin{aligned} |\mathfrak{a}(u,v)| &\leq M_{\mathfrak{F}} \|u\|_V \|v\|_V, \\ |\mathfrak{b}(\varphi,\xi)| &\leq M_{\beta} \|\varphi\|_W \|\xi\|_W, \\ |\mathfrak{c}(u,v)| &\leq M_{\mathfrak{c}} \|u\|_V \|v\|_V, \\ |\mathfrak{d}(\theta,\eta)| &\leq M_{\mathscr{K}} \|\theta\|_Q \|\eta\|_Q, \\ |\mathfrak{e}(v,\xi)| &\leq M_{\mathrm{E}} \|v\|_V \|\xi\|_W, \\ |\mathfrak{m}(\theta,v)| &\leq M_{\mathscr{M}} \|\theta\|_Q \|v\|_V, \\ |\mathfrak{p}(\theta,\xi)| &\leq M_{\mathscr{P}} \|\theta\|_Q \|\xi\|_W. \end{aligned} \tag{49}$$

HP_3. The surface electrical conductivity $\psi : \Gamma_C \times \mathbb{R} \longrightarrow \mathbb{R}^+$ and the thermal conductance $k_c : \Gamma_C \times \mathbb{R} \longrightarrow \mathbb{R}^+$ satisfy the following hypothesis for $(\pi = \psi, k_c) : \exists M_\pi > 0$ such that $|\pi(x,u)| \leq M_\pi$, $\forall u \in \mathbb{R}$, a.e., $x \in \Gamma_C$ and $x \longrightarrow \pi(x,u)$ is measurable on Γ_C for all $u \in \mathbb{R}$ and is zero for all $u \leq 0$.

The function $u \longrightarrow \pi(x,u)$ is a Lipschitz function on $\mathbb{R}$ for all $x \in \Gamma_3$.

$|\pi(x,u_1) - \pi(x,u_2)| \leq L_\pi |u_1 - u_2|$, $\quad \forall u_1, u_2 \in \mathbb{R}$, where L_π is a positive constant.

HP_4. The forces, the traction, the volume, the surfaces charge densities, and the strength of the heat source are as follows:

$$\begin{aligned} f_0 &\in C\left(0,T; L^2(\Omega)^d\right), \\ f_2 &\in C\left(0,T; L^2(\Gamma_N)^d\right), \\ q_b &\in L^2\left(0,T; L^2(\Gamma_b)\right), \\ q_1 &\in L^2\left(0,T; L^2(\Omega)\right), \\ q_0 &\in L^2\left(0,T; L^2(\Omega)\right). \end{aligned} \tag{50}$$

The potential and the temperature satisfy

$$\begin{aligned} \varphi_F &\in L^2\left(0,T; L^2(\Gamma_C)\right), \\ \theta_F &\in L^2\left(0,T; L^2(\Gamma_C)\right). \end{aligned} \tag{51}$$

The initial conditions, the friction-bounded function, and the gap function satisfy

$$\begin{aligned} &u_0 \in K, \\ &\theta_0 \in Q, \\ &S \in L^{\infty}(\Gamma_C), \\ &S \geq 0, \\ &g \in L^2(\Gamma_C), \\ &g \geq 0. \end{aligned} \tag{52}$$

Using Riesz's representation theorem, we define $\mathfrak{f} : [0,T] \longrightarrow V$, $\mathfrak{q}_{\mathfrak{e}} : [0,T] \longrightarrow W$, and $\mathfrak{q}_{\mathfrak{th}} : [0,T] \longrightarrow Q$ by the following:

$$(\mathfrak{f}(t), v)_V = \int_\Omega f_0(t) \cdot v\, dx + \int_{\Gamma_N} f_2(t) \cdot v\, da, \quad \forall v \in V, \tag{53}$$

$$(\mathfrak{q}_{\mathfrak{e}}(t), \xi)_W = \int_\Omega q_0(t) \cdot \xi\, dx - \int_{\Gamma_b} q_b(t) \cdot \xi\, da, \quad \forall \xi \in V, \tag{54}$$

$$\left(\mathfrak{q}_{\mathfrak{th}}(t), \eta\right)_Q = \int_\Omega q_1(t) \cdot \eta\, dx, \quad \forall \eta \in Q. \tag{55}$$

We define the mappings $\mathfrak{j} : V \longrightarrow \mathbb{R}$, $\varrho : V \times V \longrightarrow \mathbb{R}$, $\ell : V \times W^2 \longrightarrow \mathbb{R}$, and $\chi : V \times Q^2 \longrightarrow \mathbb{R}$, a.e., $t \in]0,T[$ by

$$\mathfrak{j}(v) = \int_{\Gamma_C} S \|v_\tau\| da, \quad \forall v \in V, \tag{56}$$

$$\varrho(u,v) = \int_{\Gamma_C} \left([u_\nu]^+ - g\right) v_\nu da = \langle [u_\nu]^+ - g, v_\nu \rangle_{\Gamma_C}, \quad \forall u, v \in V, \tag{57}$$

$$\ell(u(t), \varphi(t), \xi) = \int_{\Gamma_C} \psi(u_\nu(t) - g) \phi_L(\varphi(t) - \varphi_F) \xi\, da, \quad \forall u \in V, \forall \varphi, \xi \in W, \tag{58}$$

$$\chi(u(t), \theta(t), \eta) = \int_{\Gamma_C} k_c(u_\nu(t) - g) \phi_L(\theta(t) - \theta_F) \eta\, da, \quad \forall u \in V, \forall \theta, \eta \in Q, \tag{59}$$

respectively.

Now, by a standard variational technique, it is straightforward to see that if (u, φ, θ) satisfies the conditions (4)–(21), a.e. $t \in]0;T[$, then

$$\begin{aligned} &(\sigma(t), \varepsilon(v) - \varepsilon(\dot{u}(t)))_{\mathscr{H}} + \frac{1}{\varepsilon} \varrho(u(t), v - \dot{u}(t)) + \mathfrak{j}(v) \\ &\quad - \mathfrak{j}(\dot{u}(t)) \geq (\mathfrak{f}(t), v - \dot{u}(t))_V, \quad \forall v \in V_{ad}, \end{aligned} \tag{60}$$

$$(D(t), \nabla \xi)_{L^2(\Omega)} = \ell(u(t), \varphi(t), \xi) - (\mathfrak{q}_{\mathfrak{e}}(t), \xi)_W, \quad \forall \xi \in W, \tag{61}$$

$$(q(t),\nabla\eta)_{L^2(\Omega)} = (\dot{\theta}(t),\eta)_Q + \chi(u(t),\theta(t),\eta) - \left(\mathfrak{q}_{\mathfrak{th}}(t),\eta\right)_Q, \quad \forall\eta\in Q. \tag{62}$$

We assume that the initial conditions u_0 and θ_0 satisfy the following compatibility condition: there exists $\varphi_0 \in W$ such that

$$b(\varphi_0,\xi) - e(u_0,\xi) - p(\theta_0,\xi) + \ell(u_0,\varphi_0,\xi) = \left(q_{e_0},\xi\right)_W, \quad \forall\xi\in W. \tag{63}$$

This nonlinear problem, has a unique solution φ_0, by using the fixed point theorem.

Using all of these assumptions, notations, and $E = -\nabla\varphi$, we obtain the following variational formulation of the Problem (P).

Problem (PV): find a displacement field $u :]0;T[\longrightarrow V_{\mathrm{ad}}$, an electric potential $\varphi :]0;T[\longrightarrow W$, and a temperature field $\theta :]0;T[\longrightarrow Q$, a.e., $t \in]0,T[$ such that

$$\begin{aligned}
&\mathfrak{c}(\dot{u}(t), v - \dot{u}(t)) + \mathfrak{a}(u(t), v - \dot{u}(t)) + \mathfrak{e}(v - \dot{u}(t),\\
&\quad \varphi(t)) - \mathfrak{m}(\theta(t), v - \dot{u}(t)) + \frac{1}{\epsilon}\varrho(u(t), v - \dot{u}(t))\\
&\quad + \mathfrak{j}(v) - \mathfrak{j}(\dot{u}(t)) \geq (\mathfrak{f}(t), v - \dot{u}(t))_V, \quad \forall v\in V_{ad},
\end{aligned} \tag{64}$$

$$\begin{aligned}
&\mathfrak{b}(\varphi(t),\xi) - \mathfrak{e}(u(t),\xi) - \mathfrak{p}(\theta(t),\xi)\\
&\quad + \ell(u(t),\varphi(t),\xi) = (\mathfrak{q}_{\mathfrak{e}}(t),\xi)_W, \quad \forall\xi\in W,
\end{aligned} \tag{65}$$

$$\mathfrak{d}(\theta(t),\eta) + (\dot{\theta}(t),\eta)_Q + \chi(u(t),\theta(t),\eta) = \left(\mathfrak{q}_{\mathfrak{th}}(t),\eta\right)_Q, \quad \forall\eta\in Q, \tag{66}$$

$$\begin{aligned}
u(0,x) &= u_0(x),\\
\theta(0,x) &= \theta_0(x).
\end{aligned} \tag{67}$$

We present now the existence and the uniqueness of solution to Problem (PV).

Theorem 1. *Assume that the assumptions* $(HP_1)-(HP_4)$, *(37)–(42), for* $\varepsilon > 0$,*and the conditions*

$$\begin{aligned}
m_\beta &> M_\psi c_1^2,\\
m_{\mathscr{K}} &< \frac{c_2\left(M_{k_c}c_2 + L_{k_k}Lc_0\right)}{2},
\end{aligned} \tag{68}$$

hold. Then, Problem (PV) *has a unique solution as follows:*

$$\begin{aligned}
&u\in C^1(0,T;V),\\
&\varphi\in L^2(0,T;W),\\
&\theta\in L^2(0,T;Q).
\end{aligned} \tag{69}$$

2.3. Spatially Semidiscrete Approximation. In this paragraph, we consider a semidiscrete approximation of the Problem (PV) by discretizing the spatial domain, using the finite element method. Let $\mathscr{T}^h$ be a regular finite element partition of the domain Ω compatible with the boundary partition $\Gamma = \overline{\Gamma}_C \cup \overline{\Gamma}_D \cup \overline{\Gamma}_N$. We then define a finite element space $V^h \subset V$ and $V^h_{ad} = V_{ad} \cap V^h$, for the approximates of the displacement field u, $W^h \subset W$ for the electric potential φ, and $Q^h \subset Q$ for the temperature θ defined by

$$\begin{aligned}
V^h &= \left\{v^h \in [C(\overline{\Omega})]^d;\, v^h_{|Tr} \in [\mathbb{P}_1(Tr)]^d \forall Tr\in\mathscr{T}^h;\, v^h = 0 \text{ on } \overline{\Gamma}_C\right\},\\
W^h &= \left\{\xi^h \in C(\overline{\Omega});\, \xi^h_{|Tr} \in \mathbb{P}_1(Tr)\,\forall Tr\in\mathscr{T}^h;\, \xi^h = \text{ on } \overline{\Gamma}_a\right\},\\
Q^h &= \left\{\eta^h \in C(\overline{\Omega});\, \eta^h_{|Tr} \in \mathbb{P}_1(Tr)\,\forall Tr\in\mathscr{T}^h;\, \eta^h = 0 \text{ on } \overline{\Gamma}_D\cup\overline{\Gamma}_N\right\}.
\end{aligned} \tag{70}$$

A spatially semidiscrete scheme can be formed as the following problem:

Problem (PV^h).Find $u^h :]0;T[\longrightarrow V^h$, $\varphi^h :]0;T[\longrightarrow W^h$, and $\theta^h :]0;T[\longrightarrow Q^h$, a.e., $t\in]0,T[$ such that for $v^h \in V^h_{ad}$, $\xi^h \in W^h$, and $\eta^h \in Q^h$

$$\begin{aligned}
&\mathfrak{c}\left(\dot{u}^h(t), v^h - \dot{u}^h(t)\right) + \mathfrak{a}\left(u^h(t), v^h - \dot{u}^h(t)\right)\\
&\quad + \mathfrak{e}\left(v^h - \dot{u}^h(t), \varphi^h(t)\right) - \mathfrak{m}\left(\theta^h(t), v^h - \dot{u}^h\right)\\
&\quad + \frac{1}{\varepsilon}\varrho\left(u^h(t), v^h - \dot{u}^h(t)\right) + j\left(v^h\right) - j\left(\dot{u}^h(t)\right)\\
&\quad \geq \left(\mathfrak{f}(t), v^h - \dot{u}^h(t)\right),
\end{aligned} \tag{71}$$

$$\begin{aligned}
&\mathfrak{b}\left(\varphi^h(t),\xi^h\right) - \mathfrak{e}\left(u^h(t),\xi^h\right) - \mathfrak{p}\left(\theta^h(t),\xi^h\right)\\
&\quad + \ell\left(u^h(t),\varphi^h(t),\xi^h\right) = \left(\mathfrak{q}_{\mathfrak{e}}(t),\xi^h\right),
\end{aligned} \tag{72}$$

$$\begin{aligned}
&\mathfrak{d}\left(\theta^h(t),\eta^h\right) + \left(\dot{\theta}^h(t),\eta^h\right) + \chi\left(u^h(t),\eta^h(t),\eta^h\right)\\
&\quad = \left(\mathfrak{q}_{\mathfrak{th}}(t),\eta^h\right),
\end{aligned} \tag{73}$$

$$\begin{aligned}
u^h(0) &= u^h_0,\\
\varphi^h(0) &= \varphi^h_0,\\
\theta^h(0) &= \theta^h_0.
\end{aligned} \tag{74}$$

Here, $u^h_0 \in V^h$, $\varphi^h_0 \in W^h$, and $\theta^h_0 \in Q^h$ are appropriate approximations of u_0, φ_0, and θ_0, respectively.

Using the same argument presented in Section 2, it can be shown that Problem (PV^h) has a unique solution $u^h \in C^1(0,T;V^h)$, $\varphi^h \in L^2(0,T;W^h)$, and $\theta^h \in L^2(0,T;Q^h)$.

In this paragraph, we are interested in obtaining estimates for the errors $(u - u^h)$, $(\varphi - \varphi^h)$, and $(\theta - \theta^h)$.

Using the initial value condition, we have

$$u(t) = \int_0^t w(s)ds + u_0,$$
$$\theta(t) = \int_0^t \dot{\theta}(s)ds + \theta_0, \quad (75)$$

$$u^h(t) = \int_0^t \dot{u}^h(s)ds + u_0^h,$$
$$\theta^h(t) = \int_0^t \dot{\theta}^h(s)ds + \theta_0^h. \quad (76)$$

Theorem 2. *Assume that the assumptions stated in Theorem 1 are hold, for $\varepsilon > 0$. Then, under the conditions*

$$\left\|u_0 - u_0^h\right\|_V \longrightarrow 0,$$
$$\left\|\theta_0 - \theta_0^h\right\|_V \longrightarrow 0, \quad (77)$$
$$\text{as } h \longrightarrow 0,$$

the semi-discrete solution of (PV^h) converges as follows:

$$\left\|u - u^h\right\|_{L^2(0,T;V)} \longrightarrow 0,$$
$$\left\|\varphi - \varphi^h\right\|_{L^2(0,T;W)} \longrightarrow 0,$$
$$\left\|\theta - \theta^h\right\|_{L^2(0,T;Q)} \longrightarrow 0, \quad (78)$$
$$\text{as } h \longrightarrow 0.$$

2.4. Fully Discrete Approximation. In this paragraph, we consider a fully discrete approximation of Problem (PV). We use the finite element spaces V^h, W^h, and Q^h introduced in Section 2.3. We introduce a partition of the time interval $[0;T] : 0 = t_0 < t_1 < \ldots < t_N = T$. We denote the step size $\Delta t = k_n = t_n - t_{n-1}$ for $n = 1, 2, \ldots, N$ and let $k = \max_n k_n$ be the maximal step size. For a sequence $\{v_n\}_{n=0}^N$, we denote $\delta w = (w_n - w_{n-1})/(k_n)$.

The fully discrete approximation method is based on the backward Euler scheme, and it has the following form.

Problem (PV_n^{hk}). Find a displacement field $\{u_n^{hk}\}_{n=0}^N \subset V_{ad}^h$, an electric potential $\{\varphi_n^{hk}\}_{n=0}^N \subset W^h$, and a temperature field $\{\theta_n^{hk}\}_{n=0}^N \subset Q^h$ for all $v^h \in V^h$, $\xi^h \in W^h$, $\eta^h \in Q^h$, and $n = 1, \ldots, N$ such that

$$\begin{aligned}&\mathfrak{c}\left(w_n^{hk}, v^h - w_n^{hk}\right) + \mathfrak{a}\left(u_n^{hk}, v^h - w_n^{hk}\right) + \mathfrak{e}\left(v^h - w_n^{hk}, \varphi_{n-1}^{hk}\right)\\ &\quad - m\left(\theta_{n-1}^{hk}, v^h - w_n^{hk}\right) + \frac{1}{\varepsilon}\varrho\left(u_n^{hk}, v^h - w_n^{hk}\right) + \mathfrak{j}\left(v^h\right)\\ &\quad - \mathfrak{j}\left(w_n^{hk}\right) \geq \left(\mathfrak{f}_n, v^h - w_n^{hk}\right)_V,\end{aligned} \quad (79)$$

$$\begin{aligned}&\mathfrak{b}\left(\varphi_n^{hk}, \xi^h\right) - \mathfrak{e}\left(u_n^{hk}, \xi^h\right) - \mathfrak{p}\left(\theta_n^{hk}, \xi^h\right) + \ell\left(u_n^{hk}, \varphi_n^{hk}, \xi^h\right)\\ &\quad = \left(\mathfrak{q}_{\mathfrak{e}_n}, \xi^h\right)_W,\end{aligned} \quad (80)$$

$$\mathfrak{d}\left(\theta_n^{hk}, \eta^h\right) + \left(\delta\theta_n^{hk}, \eta^h\right) + \chi\left(u_n^{hk}, \theta_n^{hk}, \eta^h\right) = \left(\mathfrak{q}_{\mathfrak{th}_n}, \eta^h\right)_Q, \quad (81)$$

$$u_0^{hk} = u_0^h,$$
$$\varphi_0^{hk} = \varphi_0^h, \quad (82)$$
$$\theta_0^{hk} = \theta_0^h.$$

Remark 2. The choice of θ_{n-1}^h and φ_{n-1}^h instead of θ_n^h and φ_n^h is motivated by the fixed point method in the proof of the existence and uniqueness. Otherwise, we may get another different condition for the uniqueness of the solution of fixed iteration problem (79)–(82). In addition, this choice will be helpful for the application of discrete Grönwall's lemma in the next.

To simplify again the notation, we introduce the velocity

$$w_n^{hk} = \delta u_n^{hk},$$
$$n = 1, \ldots, N,$$
$$u_n^{hk} = \sum_{j=1}^n k_j w_j^{hk} + u_0^h, \quad (83)$$
$$\theta_n^{hk} = \sum_{j=1}^n k_j \delta\theta_j^{hk} + \theta_0^h,$$
$$n \geq 1.$$

This problem has a unique solution, and the proof is similar to that used in Theorem 1.

We now derive the following convergence result.

Theorem 3. *Assuming that the initial values $u_0^h \in V^h$, $\varphi_0^h \in W^h$, and $\theta_0^h \in Q^h$ are chosen in such a way that*

$$\left\|u_0 - u_0^h\right\|_V \longrightarrow 0,$$
$$\left\|\varphi_0 - \varphi_0^h\right\|_W \longrightarrow 0,$$
$$\left\|\theta_0 - \theta_0^h\right\|_Q \longrightarrow 0, \quad (84)$$
$$\text{as } h \longrightarrow 0,$$

under the condition stated in Theorem 1 and for $\varepsilon > 0$, the fully discrete solution converges, i.e.,

$$\max_{1\leq n\leq N}\left\{\left\|u_n - u_n^{hk}\right\|_V + \left\|w_n - w_n^{hk}\right\|_V + \left\|\varphi_n - \varphi_n^{hk}\right\|_W + \left\|\theta_n - \theta_n^{hk}\right\|_Q\right\} \longrightarrow 0,$$
$$\text{as } h, k \longrightarrow 0. \quad (85)$$

3. Proof of Main Results

In this section, we prove the theorems presented in the previous section.

3.1. Proof of Theorem 1. The proof of Theorem 1 is based on fixed point argument, Galerkin method, and compactness method, similar to those used in [14, 15] but with a different choice of the operators.

We turn now the following existence and uniqueness result.

Let $\mathscr{F} \in C(0,T;V)$ given by

$$(\mathscr{F}(t), v - \dot{u}_{\mathscr{F}}(t))_V = \mathfrak{e}(v - \dot{u}_{\mathscr{F}}(t), \varphi_{\mathscr{F}}(t)) - \mathfrak{m}(\theta_{\mathscr{F}}(t), v - \dot{u}_{\mathscr{F}}(t)). \tag{86}$$

In the first step, we consider the intermediate Problem PV_{df}.

Problem PV_{df}. Find $u_{\mathscr{F}} \in V_{ad}$ for, a.e., $t \in]0,T[$ such that

$$\begin{aligned} &\mathfrak{c}(\dot{u}_{\mathscr{F}}(t), v - \dot{u}_{\mathscr{F}}(t)) + \mathfrak{a}(u_{\mathscr{F}}(t), v - \dot{u}_{\mathscr{F}}(t)) \\ &\quad + (\mathscr{F}(t), v - \dot{u}_{\mathscr{F}}(t))_V + \frac{1}{\varepsilon}\varrho(u_{\mathscr{F}}(t), v - \dot{u}(t)) \\ &\quad + \mathfrak{j}(v) - \mathfrak{j}(\dot{u}_{\mathscr{F}}(t)) \geq (\mathfrak{f}(t), v - \dot{u}_{\mathscr{F}}(t)), \\ &u_{\mathscr{F}}(0) = u_0, \quad \forall v \in V. \end{aligned} \tag{87}$$

We have the following result for PV_{df}.

Lemma 1. *For all $v \in V_{ad}$ and for, a.e., $t \in]0,T[$, the Problem PV_{df} has a unique solution $u_{\mathscr{F}} \in C^1(0,T;V)$.*

Proof. Using Riesz's representation theorem, we define the operator $\mathscr{A} : V \longrightarrow V$ and the element $\mathfrak{f}_{\mathscr{F}}(t) \in V$ by

$$(\mathfrak{f}_{\mathscr{F}}(t), v)_V = (\mathfrak{f}(t), v)_V - (\mathscr{F}(t), v)_V, \tag{88}$$

$$\mathscr{A}(u_{\mathscr{F}}(t), v) = \mathfrak{a}(u_{\mathscr{F}}(t), v) + \frac{1}{\varepsilon}\varrho(u_{\mathscr{F}}(t), v). \tag{89}$$

Then, Problem PV_{df} can be written in the following form:

$$\begin{aligned} &\mathfrak{c}(\dot{u}_{\mathscr{F}}(t), v - \dot{u}_{\mathscr{F}}(t)) + \mathscr{A}(u_{\mathscr{F}}(t), v - \dot{u}_{\mathscr{F}}(t)) \\ &\quad + \mathfrak{j}(v) - \mathfrak{j}(\dot{u}_{\mathscr{F}}(t)) \geq (\mathfrak{f}_{\mathscr{F}}(t), v - \dot{u}_{\mathscr{F}}(t)), \\ &u_{\mathscr{F}}(0) = u_0. \end{aligned} \tag{90}$$

For all $u, v \in V$, there exists a constant $c > 0$ which depends only on $M_{\mathfrak{Z}}$, C_{s1}, and ϵ such that

$$|\mathscr{A}(u, v)| \leq c\|u\|_V \|v\|_V. \tag{91}$$

The assumption (HP_4) and $\mathscr{F} \in C(0,T;V)$ imply that $\mathfrak{f}_{\mathscr{F}} \in C(0,T;V)$ and by $(\mathrm{HP}_1) - (\mathrm{HP}_2)$, the operator $\mathfrak{c}$ is continuous and coercive.

We use now the result presented in pp. 61–65 in [16], and we conclude Problem PV_{df} has a unique solution $u_{\mathscr{F}} \in C^1(0,T;V)$.

Next, we use the displacement field $u_{\mathscr{F}}$ obtained in the first step and we consider the following lemma proved in [15]. □

Lemma 2. (a) *For all $\eta \in Q$ and, a.e., $t \in]0,T[$, the problem*

$$\begin{aligned} &\mathfrak{d}(\theta_{\mathscr{F}}(t), \eta) + (\dot{\theta}_{\mathscr{F}}(t), \eta)_Q + \chi(u_{\mathscr{F}}(t), \theta_{\mathscr{F}}(t), \eta) \\ &\quad = (\mathfrak{q}_{\mathfrak{th}}(t), \eta)_Q, \quad \forall \eta \in Q, \\ &\theta_{\mathscr{F}}(0) = \theta_0, \end{aligned} \tag{92}$$

has a unique solution $\theta_{\mathscr{F}} \in L^2(0,T;Q)$.

(b) *For all $\xi \in W$ and for, a.e., $t \in]0,T[$, the problem*

$$\begin{aligned} &\mathfrak{b}(\varphi_{\mathscr{F}}(t), \xi) - \mathfrak{e}(u_{\mathscr{F}}(t), \xi) - \mathfrak{p}(\theta_{\mathscr{F}}(t), \xi) + \ell(u_{\mathscr{F}}(t), \varphi_{\mathscr{F}}(t), \xi) \\ &\quad = (\mathfrak{q}_{\mathfrak{e}}(t), \xi)_W, \\ &\varphi_{\mathscr{F}}(0) = \varphi_0, \end{aligned} \tag{93}$$

has a unique solution $\varphi_{\mathscr{F}} \in L^2(0,T;W)$.

In the last step, for $\mathscr{F} \in L^2(0,T;V)$, $\varphi_{\mathscr{F}}$ and $\theta_{\mathscr{F}}$ are the functions obtained in Lemma 2 and we consider the operator $\mathscr{L} : C(0,T;V) \longrightarrow C(0,T;V)$ defined by

$$(\mathscr{L}\mathscr{F}(t), v)_V = \mathfrak{e}(v, \varphi_{\mathscr{F}}(t)) - \mathfrak{m}(\theta_{\mathscr{F}}(t), v), \tag{94}$$

for all $v \in V$ and for, a.e., $t \in]0,T[$.

For the operator $\mathscr{L}$, we have the following result obtained in [15].

Lemma 3. *There exists a unique $\widetilde{\mathscr{F}} \in C(0,T;V)$ such that $\mathscr{L}\widetilde{\mathscr{F}} = \widetilde{\mathscr{F}}$.*

We now turn to a proof of Theorem 1.

Existence. Let $\widetilde{\mathscr{F}} \in C(0,T;V)$ be the fixed point of the operator $\mathscr{L}$ and us denote $\tilde{x} = (\tilde{u}_{\mathscr{F}}, \tilde{\varphi}_{\mathscr{F}}, \tilde{\theta}_{\mathscr{F}})$ the solution of variational Problem $(\mathrm{PV}_{\mathscr{F}})$, for $\widetilde{\mathscr{F}} = \mathscr{F}$; the definition of $\mathscr{L}$ and Problem $(\mathrm{PV}_{\mathscr{F}})$ prove that $\tilde{x}$ is a solution of Problem (PV).

Uniqueness. The uniqueness of the solution follows from the uniqueness of the fixed point of the operator $\mathscr{L}$.

3.2. Proof of Theorem 2. To prove Theorem 2, we need the following result

Lemma 4. *Assume that $(HP_1) - (HP_3)$. Then, we have the estimate as follows:*

$$\begin{aligned} &\left\|u(t) - u^h(t)\right\|_{L^2(0,T;V)} + \left\|\varphi(t) - \varphi^h(t)\right\|_{L^2(0,T;W)} \\ &\quad + \left\|\theta(t) - \theta^h(t)\right\|_{L^2(0,T;Q)} \\ &\leq c^* \inf_{\substack{v^h \in L^2(0,T;V^h) \\ \xi^h \in L^2(0,T;W^h) \\ \eta^h \in L^2(0,T;Q^h)}} \left\{ \left\|\dot{u}(t) - v^h\right\|_{L^2(0,T;V)} + \left\|\varphi(t) - \xi^h\right\|_{L^2(0,T;W)} \right. \\ &\quad \left. + \left\|\theta(t) - \eta^h\right\|_{L^2(0,T;Q)} \right\} \\ &\quad + \left|\mathcal{S}(t, v^h, \dot{u}(t))\right|^{1/2}_{L^2(0,T)} \Big\} + \left\|u_0 - u_0^h\right\|_V + \left\|\theta_0 - \theta_0^h\right\|_Q, \end{aligned} \tag{95}$$

for a positive constant c^*.

Proof. Take $v = \dot{u}^h(t)$ in (64) and add the inequality to (71), we have

$$\begin{aligned}
&\mathfrak{c}\left(\dot{u}(t)-\dot{u}^h(t),\dot{u}(t)-\dot{u}^h(t)\right)\\
&\quad\le \mathfrak{c}\left(\dot{u}^h(t),v^h-\dot{u}(t)\right)+\mathfrak{a}\left(u(t),\dot{u}^h(t)-\dot{u}(t)\right)\\
&\qquad+\mathfrak{a}\left(u^h(t),v^h-\dot{u}^h(t)\right)+\mathfrak{e}\left(\dot{u}^h(t)-\dot{u}(t),\varphi(t)\right)\\
&\qquad+\mathfrak{e}\left(v^h-\dot{u}^h(t),\varphi^h(t)\right)-\mathfrak{m}\left(\theta(t),\dot{u}^h(t)-\dot{u}(t)\right)\\
&\qquad-\mathfrak{m}\left(\theta^h(t),v^h-\dot{u}^h(t)\right)+\frac{1}{\varepsilon}\left[\varrho\left(u(t),\dot{u}^h(t)-\dot{u}(t)\right)\right.\\
&\qquad\left.+\varrho\left(u^h(t),v^h-\dot{u}^h(t)\right)\right]+\mathfrak{j}\left(v^h\right)-\mathfrak{j}\left(\dot{u}^h(t)\right)+\mathfrak{j}\left(\dot{u}^h(t)\right)\\
&\qquad-\mathfrak{j}\left(\dot{u}(t)\right)-\left(\mathfrak{f},v^h-\dot{u}(t)\right)-\left(\mathfrak{f},\dot{u}^h-\dot{u}(t)\right).
\end{aligned}\tag{96}$$

Then,

$$\begin{aligned}
&\mathfrak{c}\left(\dot{u}(t)-\dot{u}^h(t),\dot{u}(t)-\dot{u}^h(t)\right)\\
&\quad\le \mathcal{S}\left(t,v^h,\dot{u}(t)\right)+\mathcal{S}_\Phi^h+\mathfrak{c}\left(\dot{u}^h(t)-\dot{u}(t),v^h-\dot{u}(t)\right)\\
&\qquad+\mathfrak{a}\left(u^h(t)-u(t),v^h-\dot{u}^h(t)\right)+\mathfrak{e}\left(v^h-\dot{u}^h(t),\varphi^h(t)-\varphi(t)\right)\\
&\qquad-\mathfrak{m}\left(\theta^h(t)-\theta(t),v^h-\dot{u}^h(t)\right),
\end{aligned}\tag{97}$$

where

$$\begin{aligned}
\mathcal{S}\left(t,v^h,\dot{u}(t)\right)&=\mathfrak{c}\left(\dot{u}(t),v^h-\dot{u}(t)\right)+\mathfrak{a}\left(u(t),v^h-\dot{u}(t)\right)\\
&\quad+\mathfrak{e}\left(v^h-\dot{u}(t),\varphi(t)\right)-\mathfrak{m}\left(\theta(t),v^h-\dot{u}(t)\right)\\
&\quad+\mathfrak{j}\left(v^h\right)-\mathfrak{e}\left(\dot{u}(t)\right)-\left(\mathfrak{f},v^h-\dot{u}(t)\right),
\end{aligned}\tag{98}$$

$$\begin{aligned}
\mathcal{S}_\varrho^h=\frac{1}{\varepsilon}&\left[\varrho\left(u(t),\dot{u}^h(t)-\dot{u}(t)\right)+\varrho\left(u^h(t),v^h-\dot{u}^h(t)\right)\right.\\
&\left.-\varrho\left(u(t),v^h-\dot{u}(t)\right)\right].
\end{aligned}\tag{99}$$

We take $\xi=\xi^h$ in (65) and $\eta=\eta^h$ in (66) and by subtracting to (72) and to (73), respectively, we deduce that

$$\begin{aligned}
\mathfrak{b}\left(\varphi(t)-\varphi^h(t),\varphi(t)-\varphi^h(t)\right)&=\mathcal{S}_\ell^h+\mathfrak{b}\left(\varphi(t)-\varphi^h(t),\varphi(t)-\xi^h\right)\\
&\quad-\mathfrak{e}\left(u(t)-u^h(t),\varphi(t)-\xi^h\right)\\
&\quad-\mathfrak{e}\left(u(t)-u^h(t),\varphi^h(t)-\varphi(t)\right)\\
&\quad-\mathfrak{p}\left(\theta(t)-\theta^h(t),\varphi(t)-\xi^h\right)\\
&\quad-\mathfrak{p}\left(\theta(t)-\theta^h(t),\varphi^h(t)-\varphi(t)\right),
\end{aligned}\tag{100}$$

$$\begin{aligned}
\mathfrak{d}\left(\theta(t)-\theta^h(t),\theta(t)-\theta^h(t)\right)&=\mathcal{S}_\chi^h+\mathfrak{d}\left(\theta(t)-\theta^h(t),\theta(t)-\eta^h\right)\\
&\quad-\left(\dot{\theta}(t)-\dot{\theta}^h(t),\eta^h-\theta(t)\right)\\
&\quad-\left(\dot{\theta}(t)-\dot{\theta}^h(t),\theta(t)-\theta^h(t)\right),
\end{aligned}\tag{101}$$

where

$$\begin{aligned}
\mathcal{S}_\ell^h=&-\ell\left(u(t),\varphi(t),\xi^h-\varphi(t)\right)-\ell\left(u(t),\varphi(t),\varphi(t)-\varphi^h(t)\right)\\
&+\ell\left(u^h(t),\varphi^h(t),\xi^h-\varphi(t)\right)+\ell\left(u^h(t),\varphi^h(t),\varphi(t)-\varphi^h(t)\right),
\end{aligned}\tag{102}$$

$$\begin{aligned}
\mathcal{S}_\chi^h=&\,\chi\left(u^h(t),\theta^h(t),\eta^h-\theta(t)\right)+\chi\left(u^h(t),\theta^h(t),\theta(t)-\theta^h(t)\right)\\
&-\chi\left(u(t),\theta(t),\eta^h-\theta(t)\right)-\chi\left(u(t),\theta(t),\theta(t)-\theta^h(t)\right).
\end{aligned}\tag{103}$$

We now add (87), (100), and (101); we obtain

$$\begin{aligned}
&\mathfrak{c}\left(\dot{u}(t)-\dot{u}^h(t),\dot{u}(t)-\dot{u}^h(t)\right)+\mathfrak{b}\left(\varphi(t)-\varphi^h(t),\varphi(t)-\varphi^h(t)\right)\\
&\quad+\mathfrak{d}\left(\theta(t)-\theta^h(t),\theta(t)-\theta^h(t)\right)\le\mathcal{S}\left(t,v^h,\dot{u}(t)\right)+\mathcal{S}_\varrho^h+\mathcal{S}_\ell+\mathcal{S}_\chi\\
&\quad+\mathfrak{c}\left(\dot{u}^h(t)-\dot{u}(t),v^h-\dot{u}(t)\right)+\mathfrak{a}\left(u^h(t)-u(t),v^h-\dot{u}^h(t)\right)\\
&\quad+\mathfrak{e}\left(v^h-\dot{u}^h(t),\varphi^h(t)-\varphi(t)\right)-\mathfrak{m}\left(\theta^h(t)-\theta(t),v^h-\dot{u}^h(t)\right),\\
&\mathfrak{b}\left(\varphi(t)-\varphi^h(t),\varphi(t)-\xi^h\right)-\mathfrak{e}\left(u(t)-u^h(t),\varphi(t)-\xi^h\right)\\
&\quad-\mathfrak{e}\left(u(t)-u^h(t),\varphi^h(t)-\varphi(t)\right)-\mathfrak{p}\left(\theta(t)-\theta^h(t),\varphi(t)-\xi^h\right)\\
&\quad-\mathfrak{p}\left(\theta(t)-\theta^h(t),\varphi^h(t)-\varphi(t)\right)+\mathfrak{d}\left(\theta(t)-\theta^h(t),\theta(t)-\eta^h\right)\\
&\quad-\left(\dot{\theta}(t)-\dot{\theta}^h(t),\eta^h-\theta(t)\right)-\left(\dot{\theta}(t)-\dot{\theta}^h(t),\theta(t)-\theta^h(t)\right).
\end{aligned}\tag{104}$$

From (HP_1) and the previous inequality, it follows that

$$\begin{aligned}
&m_{\mathfrak{c}}\left\|\dot{u}(t)-\dot{u}^h(t)\right\|_V^2+m_\beta\left\|\varphi(t)-\varphi^h(t)\right\|_W^2+m_{\mathscr{K}}\left\|\theta(t)-\theta^h(t)\right\|_Q^2\\
&\quad\le\mathcal{S}\left(t,v^h,\dot{u}(t)\right)+\mathcal{S}_\varrho^h+\mathcal{S}_\ell^h+\mathcal{S}_\chi^h+\mathcal{S}^h,
\end{aligned}\tag{105}$$

where

$$\begin{aligned}
\mathcal{S}^h=&\,\mathfrak{c}\left(\dot{u}^h(t)-\dot{u}(t),v^h-w(t)\right)+\mathfrak{a}\left(u^h(t)-u(t),v^h-\dot{u}^h(t)\right)\\
&+\mathfrak{e}\left(v^h-\dot{u}^h(t),\varphi^h(t)-\varphi(t)\right)-\mathfrak{m}\left(\theta^h(t)-\theta(t),v^h-\dot{u}^h(t)\right)\\
&\mathfrak{b}\left(\varphi(t)-\varphi^h(t),\varphi(t)-\xi^h\right)-\mathfrak{e}\left(u(t)-u^h(t),\varphi(t)-\xi^h\right)\\
&-\mathfrak{e}\left(u(t)-u^h(t),\varphi^h(t)-\varphi(t)\right)-\mathfrak{p}\left(\theta(t)-\theta^h(t),\varphi(t)-\xi^h\right)\\
&-\mathfrak{p}\left(\theta(t)-\theta^h(t),\varphi^h(t)-\varphi(t)\right)+\mathfrak{d}\left(\theta(t)-\theta^h(t),\theta(t)-\eta^h\right)\\
&-\left(\dot{\theta}(t)-\dot{\theta}^h(t),\eta^h-\theta(t)\right)-\left(\dot{\theta}(t)-\dot{\theta}^h(t),\theta(t)-\theta^h(t)\right).
\end{aligned}\tag{106}$$

Let us estimate each of the terms in (105), $(\mathrm{HP}_1)-(\mathrm{HP}_3)$, and

$$xy\le\beta x^2+\frac{1}{4\beta}y^2,\quad\forall\beta>0,\tag{107}$$

allowing us to obtain

$$\begin{aligned}
\left|\mathcal{S}^h\right|\le\alpha_h&\left\{\left\|\dot{u}(t)-\dot{u}^h(t)\right\|_V^2+\left\|u(t)-u^h(t)\right\|_V^2+\left\|\varphi(t)-\varphi^h(t)\right\|_W^2\right.\\
&+\left\|\theta(t)-\theta^h(t)\right\|_Q^2+\left\|\dot{\theta}(t)-\dot{\theta}^h(t)\right\|_Q^2+\left\|\dot{u}^h(t)-v^h\right\|_V^2\\
&\left.+\left\|\varphi^h(t)-\xi^h\right\|_W^2+\left\|\theta(t)-\eta^h\right\|_Q^2\right\}.
\end{aligned}\tag{108}$$

Using the assumptions (HP_3), (38), and (58), we find

$$\begin{aligned}\left|\mathcal{S}_\ell^h\right| \leq & \Big| -\ell\big(u(t),\varphi(t),\xi^h-\varphi(t)\big)-\ell\big(u(t),\varphi(t),\varphi(t)\\ & -\varphi^h(t)\big)+\ell\big(u^h(t),\varphi^h(t),\xi^h-\varphi(t)\big)\\ & +\ell\big(u^h(t),\varphi^h(t),\varphi(t)-\varphi^h(t)\big)\Big|\\ \leq & \Big|\int_{\Gamma_C}\psi(u(t)_\nu-g)\phi_L(\varphi(t)-\varphi_F)(\xi-\varphi(t))da\\ & +\int_{\Gamma_C}\psi(u(t)_\nu-g)\phi_L(\varphi(t)-\varphi_F)\big(\varphi-\varphi^h(t)\big)da\\ \leq & \Big|\int_{\Gamma_C}\psi(u(t)_\nu-g)\phi_L(\varphi(t)-\varphi_F)(\xi-\varphi(t))da\\ & +\int_{\Gamma_C}\psi(u(t)_\nu-g)\phi_L(\varphi(t)-\varphi_F)\big(\varphi-\varphi^h(t)\big)da\\ & +\int_{\Gamma_C}\psi\big(u^h(t)_\nu-g\big)\phi_L\big(\varphi^h(t)-\varphi_F\big)\big(\xi^h-\varphi(t)\big)da\\ & +\int_{\Gamma_C}\psi\big(u^h(t)_\nu-g\big)\phi_L\big(\varphi^h(t)-\varphi_F\big)\big(\varphi-\varphi^h(t)\big)da\Big|\\ \leq & M_\psi\int_{\Gamma_C}\left|\varphi(t)-\varphi^h(t)\right|^2da\\ & +LL_\psi\int_{\Gamma_C}\left|\varphi(t)-\varphi^h(t)\right|\left|u_\nu(t)-u_\nu^h(t)\right|da\\ \leq & M_\psi C_{s1}\left\|\varphi(t)-\varphi^h(t)\right\|_W^2+LL_\psi C_{s1}C_{s2}\left\|u(t)-u^h(t)\right\|_V\\ & \left\|\varphi(t)-\varphi^h(t)\right\|_W\\ & +M_\psi C_{s2}\left\|\varphi(t)-\varphi^h(t)\right\|_W\left\|\varphi(t)-\xi^h\right\|_{L^2(\Gamma_C)}\\ & +LL_\psi c_0\left\|u(t)-u^h(t)\right\|_V\left\|\varphi(t)-\xi^h\right\|_{L^2(\Gamma_C)}\\ \leq & \alpha_\ell\left\{\left\|\varphi(t)-\varphi^h(t)\right\|_W^2+\left\|u(t)-u^h(t)\right\|_V^2+\left\|\varphi(t)-\xi^h\right\|_W\right\}.\end{aligned}\tag{109}$$

Similarly, we find

$$\left|\mathcal{S}_\chi^h\right| \leq \alpha_\chi\left\{\left\|\theta(t)-\theta^h(t)\right\|_W^2+\left\|u(t)-u^h(t)\right\|_V^2+\left\|\theta(t)-\eta^h\right\|_Q^2\right\}.\tag{110}$$

By combining the inequality

$$\left|[x]^+-[y]^+\right| \leq |x-y|,\tag{111}$$

with (107), we observe that

$$\left|\mathcal{S}_\varrho^h\right| \leq c_\varrho\left\{\left\|u(t)-u^h(t)\right\|_V^2+\left\|\dot u(t)-\dot u^h(t)\right\|_V^2+\left\|v^h-\dot u(t)\right\|_V^2\right\}.\tag{112}$$

Thus, by (106) and (108)–(110), we have the inequality

$$\begin{aligned}&\left\|\dot u(t)-\dot u^h(t)\right\|_V^2+\left\|\varphi(t)-\varphi^h(t)\right\|_W^2+\left\|\theta(t)-\theta^h(t)\right\|_Q^2\\ &\quad\leq c_1^*\Big\{\left\|u(t)-u^h(t)\right\|_V^2+\left\|\dot\theta(t)-\dot\theta^h(t)\right\|_Q^2+\left|R\big(t;v^h,\dot u(t)\big)\right|\\ &\qquad+\left\|\dot u(t)-v^h\right\|_V^2+\left\|\varphi(t)-\xi^h\right\|_W^2+\left\|\theta(t)-\eta^h\right\|_Q^2\Big\}.\end{aligned}\tag{113}$$

Then, by (75) and (76), we have

$$\begin{aligned}u(t)-u^h(t)&=\int_0^t\big(\dot u(s)-\dot u^h(s)\big)ds+u_0-u_0^h,\\ \theta(t)-\theta^h(t)&=\int_0^t\big(\dot\theta(s)-\dot\theta^h(s)\big)ds+\theta_0-\theta_0^h,\end{aligned}\tag{114}$$

and so

$$\begin{aligned}\left\|u(t)-u^h(t)\right\|_V^2&\leq c_2^*\left\{\int_0^t\left\|\dot u(s)-\dot u^h(s)\right\|_V^2ds+\left\|u_0-u_0^h\right\|_V^2\right\},\\ \left\|\theta(t)-\theta^h(t)\right\|_Q^2&\leq c_3^*\left\{\int_0^t\left\|\dot\theta(s)-\dot\theta^h(s)\right\|_Q^2ds+\left\|\theta_0-\theta_0^h\right\|_Q^2\right\}.\end{aligned}\tag{115}$$

Consequently, from the previous inequalities and Grönwall's inequality in (113), we find (95). □

Proof of Theorem 2. To estimate the error provided by the approximation of the finite element space V^h, W^h, and Q^h, we use $\Pi^h u$, $\Pi^h\varphi$, and $\Pi^h\theta$, the standard finite element interpolation operator of u, φ and θ, respectively. We then have the interpolation error estimate [16]

$$\begin{aligned}\left\|\dot u-\Pi^h\dot u\right\|_V&\leq ch\|\dot u\|_{L^2(0,T;V)},\\ \left\|\varphi-\Pi^h\varphi\right\|_W&\leq ch|\varphi|_{L^2(0,T;W)},\\ \left\|\theta-\Pi^h\theta\right\|_Q&\leq ch|\theta|_{L^2(0,T;Q)}.\end{aligned}\tag{116}$$

We bound now the term $\mathcal{S}(.;v^h(.)\ \dot u(.))$.Using the properties of $\mathfrak{c}$, $\mathfrak{a}$, $\mathfrak{e}$, $\mathfrak{m}$, $\mathfrak{f}$, and (98), there exists positive constant c depending on $M_{\mathfrak{c}}$, $M_{\mathfrak{F}}$, $M_{\mathscr{E}}$, $M_{\mathscr{M}}$, $\mathfrak{f}_{L^2(0,T;V)}$, $\|u\|_{L^2(0,T;V)}$, $\|\varphi\|_{L^2(0,T;V)}$, and $\|\theta\|_{L^2(0,T;V)}$ such that

$$\left|\mathcal{S}\big(t,v^h,\dot u(t)\big)\right|\leq c\left\|v^h(t)-\dot u(t)\right\|_V+\left|\mathfrak{j}\big(v^h(t)\big)-\mathfrak{j}(\dot u(t))\right|.\tag{117}$$

Taking (116) in (95), we have

$$\begin{aligned}&\left\|u-u^h\right\|_{L^2(0,T;V)}+\left\|\varphi-\varphi^h\right\|_{L^2(0,T;W)}+\left\|\theta-\theta^h\right\|_{L^2(0,T;Q)}\\ &\quad\leq c+\Big\{\left\|\dot u-\Pi^h\dot u\right\|_{L^2(0,T;V)}\left\|\varphi-\Pi^h\varphi\right\|_{L^2(0,T;W)}\\ &\qquad+\left\|\theta-\Pi^h\theta\right\|_{L^2(0,T;Q)}+\left\|\dot u-\Pi^h\dot u\right\|_{L^2(0,T;V)}^{1/2}\\ &\qquad+\left\|\mathfrak{j}\big(\Pi^h\dot u\big)-\mathfrak{j}(\dot u)\right\|_{L^2(0,T;V)}^{1/2}+\left\|u_0-u_0^h\right\|_V+\left\|\theta_0-\theta_0^h\right\|_Q\Big\}.\end{aligned}\tag{118}$$

From (116), the constant c is independent of h, u, φ, and θ. This implies

$$\begin{gathered}\left\|\dot{u}-\Pi^h\dot{u}\right\|_V \longrightarrow 0,\\ \left\|\varphi-\Pi^h\varphi\right\|_W \longrightarrow 0\\ \left\|\theta-\Pi^h\theta\right\|_Q\\ \text{as } h \longrightarrow 0.\end{gathered} \tag{119}$$

From (HP$_4$) and (53), it follows that,

$$\left\|\mathfrak{j}\left(\Pi^h\dot{u}\right)-\mathfrak{j}(\dot{u})\right\|_{L^2(0,T;V)} \longrightarrow 0. \tag{120}$$

Finally, we find the result (78). □

3.3. Proof of Theorem 3. Add (80) with $v^h = v_n^h$ and (64) with $v = w_n^{hk}$ at $t = t_n$, we find

$$\begin{aligned}&\mathfrak{c}\left(w_n^{hk}, v_n^h - w_n^{hk}\right) + \mathfrak{c}\left(w_n, w_n^{hk} - w_n\right) + \mathfrak{a}\left(u_n^{hk}, v_n^h - w_n^{hk}\right)\\ &\quad + \mathfrak{a}\left(u_n, w_n^{hk} - w_n\right) + \mathfrak{e}\left(v_n^h - w_n^{hk}, \varphi_{n-1}^{hk}\right) + \mathfrak{e}\left(w_n^{hk} - w_n, \varphi_n\right)\\ &\quad - \mathfrak{m}\left(\theta_{n-1}^{hk}, v_n^h - w_n^{hk}\right) - \mathfrak{m}\left(\theta_n, w_n^{hk} - w_n\right)\\ &\quad + \frac{1}{\varepsilon}\left[\varrho\left(u_n^{hk}, v_n^h - w_n^{hk}\right) + \varrho\left(u_n, w_n^{hk} - w_n\right)\right] + \mathfrak{j}\left(v_n^h\right) - \mathfrak{j}\left(w_n^{hk}\right)\\ &\quad + \mathfrak{j}\left(w_n^{hk}\right) - \mathfrak{j}(w_n) \geq \left(\mathfrak{f}_n, v_n^h - w_n^{hk}\right) + \left(\mathfrak{f}_n, w_n^{hk} - w_n\right).\end{aligned} \tag{121}$$

This equality,

$$\begin{aligned}\mathfrak{c}\left(w_n - w_n^{hk}, w_n - w_n^{hk}\right) &= \mathfrak{c}\left(w_n^{hk}, w_n^{hk} - v_n^h\right) + \mathfrak{c}\left(w_n, w_n - v_n^h\right)\\ &\quad - \mathfrak{c}\left(w_n^{hk}, w_n - v_n^h\right) + \mathfrak{c}\left(w_n, v_n^h - w_n^{kh}\right),\end{aligned} \tag{122}$$

allows us to obtain

$$\begin{aligned}\mathfrak{c}\left(w_n - w_n^{hk}, w_n - w_n^{hk}\right) &\leq R\left(v_n^h, w_n\right) + \mathcal{S}_\varrho^{hk}\\ &\quad + \mathfrak{c}\left(w_n^{hk} - w_n, v_n^h - w_n\right)\\ &\quad + \mathfrak{a}\left(u_n^{hk} - u_n, v_n^h - w_n^{hk}\right)\\ &\quad + \mathfrak{e}\left(v_n^h - w_n^{hk}, \varphi_{n-1}^{hk} - \varphi_n\right)\\ &\quad - \mathfrak{m}\left(\theta_{n-1}^{hk} - \theta_n, v_n^h - w_n^{hk}\right),\end{aligned} \tag{123}$$

where

$$\begin{aligned}\mathcal{S}_n\left(v_n^h, w_n\right) &= \mathfrak{c}\left(w_n, v_n^h - w_n\right) + \mathfrak{a}\left(u_n, v_n^h - w_n\right)\\ &\quad + \mathfrak{e}\left(v_n^h - w_n, \varphi_n\right) - \mathfrak{m}\left(\theta_n, v_n^h - w_n\right) + \mathfrak{j}\left(v_n^h\right)\\ &\quad - \mathfrak{j}(w_n) - \left(\mathfrak{f}_n, v_n^h - w_n\right),\end{aligned} \tag{124}$$

$$\mathcal{S}_\varrho^{hk} = \frac{1}{\varepsilon}\left[\varrho\left(u_n^{hk}, v_n^h - w_n^{hk}\right) + \varrho\left(u_n, w_n^{hk} - w_n\right) - \varrho\left(u_n, v_n^h - w_n\right)\right]. \tag{125}$$

For all $n \geq 1$, we subtract (65) from (81) at $t = t_n$; we obtain that

$$\begin{aligned}&\mathfrak{b}\left(\varphi_n - \varphi_n^{hk}, \xi^h\right) - \mathfrak{e}\left(u_n - u_n^{hk}, \xi^h\right) - \mathfrak{p}\left(\theta_n - \theta_n^{hk}, \xi^h\right)\\ &\quad + \ell\left(u_n, \varphi_n, \xi^h\right) - \ell\left(u_n^{hk}, \varphi_n^{hk}, \xi^h\right) = 0.\end{aligned} \tag{126}$$

We replace ξ^h by $(\varphi_n - \varphi_n^{hk})$ and $(\varphi_n - \xi^h)$ in (126); we find that

$$\begin{aligned}\mathfrak{b}\left(\varphi_n - \varphi_n^{hk}, \varphi_n - \varphi_n^{hk}\right) &\leq \mathcal{S}_\ell^{hk} + \mathfrak{b}\left(\varphi_n - \varphi_n^{hk}, \varphi_n - \xi^h\right)\\ &\quad + \mathfrak{e}\left(u_n - u_n^{hk}, \xi^h - \varphi_n^{hk}\right)\\ &\quad + \mathfrak{p}\left(\theta_n - \theta_n^{hk}, \xi^h - \varphi_n^{hk}\right),\end{aligned} \tag{127}$$

where

$$\begin{aligned}\mathcal{S}_\ell^{hk} &= \ell\left(u_n, \varphi_n, \varphi_n - \xi^h\right) - \ell\left(u_n^{hk}, \varphi_n^{hk}, \varphi_n - \xi^h\right)\\ &\quad - \ell\left(u_n, \varphi_n, \varphi_n - \varphi_n^{hk}\right) + \ell\left(u_n^{hk}, \varphi_n^{hk}, \varphi_n - \varphi_n^{hk}\right).\end{aligned} \tag{128}$$

For all $n \geq 1$ and similar to (127) and (128), we have

$$\begin{aligned}\mathfrak{d}\left(\theta_n - \theta_n^{hk}, \theta_n - \theta_n^{hk}\right) &\leq \mathcal{S}_\chi^{hk} + \mathfrak{d}\left(\theta_n - \theta_n^{hk}, \theta_n - \eta^h\right)\\ &\quad + \left(\delta\theta_n, \theta_n - \eta^h\right) - \left(\delta\theta_n, \theta_n - \theta_n^{hk}\right)\\ &\quad - \left(\delta\theta_n^{hk}, \theta_n - \eta^h\right) + \left(\delta\theta_n^{hk}, \theta_n - \theta_n^{hk}\right),\end{aligned} \tag{129}$$

where $\delta\theta_n = 1/k_n(\theta_n - \theta_{n-1})$ and $\delta\theta_n^{hk} = 1/k_n(\theta_n^{hk} - \theta_{n-1}^{hk})$, and

$$\begin{aligned}\mathcal{S}_\chi^{hk} &= \chi\left(u_n, \theta_n, \theta_n - \eta^h\right) - \chi\left(u_n^{hk}, \theta_n^{hk}, \theta_n - \eta^h\right)\\ &\quad + \chi\left(u_n^{hk}, \theta_n^{hk}, \theta_n - \theta_n^{hk}\right) - \chi\left(u_n, \theta_n, \theta_n - \theta_n^{hk}\right).\end{aligned} \tag{130}$$

Adding (124), (128), and (130), we find

$$\begin{aligned}&\mathfrak{c}\left(w_n - w_n^{hk}, w_n - w_n^{hk}\right) + \mathfrak{b}\left(\varphi_n - \varphi_n^{hk}, \varphi_n - \varphi_n^{hk}\right)\\ &\quad + \mathfrak{d}\left(\theta_n - \theta_n^{hk}, \theta_n - \theta_n^{hk}\right) \leq \mathcal{S}_n\left(v_n^h, w_n\right) + \mathcal{S}_\varrho^{hk} + \mathcal{S}_\ell^{hk} + \mathcal{S}_\chi^{hk}\\ &\quad + c\left(w_n^{hk} - w_n, v_n^h - w_n\right) + \mathfrak{a}\left(u_n^{hk} - u_n, v_n^h - w_n^{hk}\right)\\ &\quad + \mathfrak{e}\left(v_n^h - w_n^{hk}, \varphi_{n-1}^{hk} - \varphi_n\right) - \mathfrak{m}\left(\theta_{n-1}^{hk} - \theta_n, v_n^h - w_n^{hk}\right)\\ &\quad + \mathfrak{b}\left(\varphi_n - \varphi_n^{hk}, \varphi_n - \xi^h\right) + \mathfrak{d}\left(\theta_n - \theta_n^{hk}, \theta_n - \eta^h\right)\\ &\quad + \mathfrak{e}\left(u_n - u_n^{hk}, \xi^h - \varphi_n^{hk}\right) + \mathfrak{p}\left(\theta_n - \theta_n^{hk}, \xi^h - \varphi_n^{hk}\right)\\ &\quad + \left(\delta\theta_n, \theta_n - \eta^h\right) - \left(\delta\theta_n, \theta_n - \theta_n^{hk}\right)\\ &\quad - \left(\delta\theta_n^{hk}, \theta_n - \eta^h\right) + \left(\delta\theta_n^{hk}, \theta_n - \theta_n^{hk}\right).\end{aligned} \tag{131}$$

The ellipticity of operators $\mathfrak{c}$, $\mathfrak{b}$, $\mathfrak{d}$, and the previous inequality follows that

$$m_C\left\|w_n - w_n^{hk}\right\|_V^2 + m_\beta\left\|\varphi_n - \varphi_n^{hk}\right\|_W^2 + m_K\left\|\theta_n - \theta_n^{hk}\right\|_V^2 \leq \mathcal{S}_n\left(v_n^h, w_n\right) + \mathcal{S}_\varrho^{hk} + \mathcal{S}_\ell^{hk} + \mathcal{S}_\chi^{hk} + \mathcal{S}^{hk}, \tag{132}$$

where

$$\begin{aligned}\mathcal{S}^{hk} &= \mathfrak{c}\left(w_n^{hk} - w_n, v_n^h - w_n\right) + \mathfrak{a}\left(u_n^{hk} - u_n, v_n^h - w_n^{hk}\right)\\ &+ \mathfrak{e}\left(v_n^h - w_n^{hk}, \varphi_{n-1}^{hk} - \varphi_n\right) - \mathfrak{m}\left(\theta_{n-1}^{hk} - \theta_n, v_n^h - w_n^{hk}\right)\\ &+ \mathfrak{b}\left(\varphi_n - \varphi_n^{hk}, \varphi_n - \xi^h\right) + \mathfrak{e}\left(u_n - u_n^{hk}, \xi^h - \varphi_n^{hk}\right)\\ &+ \mathfrak{p}\left(\theta_n - \theta_n^{hk}, \xi^h - \varphi_n^{hk}\right) + \mathfrak{d}\left(\theta_n - \theta_n^{hk}, \theta_n - \eta^h\right)\\ &+\left(\delta\theta_n, \theta_n - \eta^h\right) - \left(\delta\theta_n, \theta_n - \theta_n^{hk}\right) - \left(\delta\theta_n^{hk}, \theta_n - \eta^h\right)\\ &+\left(\delta\theta_n^{hk}, \theta_n - \theta_n^{hk}\right).\end{aligned} \tag{133}$$

Now, we estimate each of the terms in (132). Thanks to $(\mathrm{HP}_1) - (\mathrm{HP}_3)$, (107), (117), (111),and

$$\left\|v_n^h - w_n^{hk}\right\| \leq \left\|v_n^h - w_n\right\| + \left\|w_n - w_n^{hk}\right\|, \tag{134}$$

we get

$$\begin{aligned}\left|\mathcal{S}^{hk}\right| \leq \alpha_{hk}\Big\{ &\left\|w_n - w_n^{hk}\right\|_V^2 + \left\|u_n - u_n^{hk}\right\|_V^2 \left\|\varphi_n - \varphi_n^{hk}\right\|_W^2\\ &+\left\|\theta_n - \theta_n^{hk}\right\|_Q^2 + \left\|\varphi_n - \varphi_{n-1}^{hk}\right\|_W^2 + \left\|\theta_n - \theta_{n-1}^{hk}\right\|_Q^2\\ &+\left\|\theta_{n-1} - \theta_{n-1}^{hk}\right\|_Q^2 + \left\|v_n^h - w_n\right\|_V^2 + \left\|\xi^h - \varphi_n\right\|_W^2\\ &+\left\|\theta_n - \eta^h\right\|_Q^2\Big\},\end{aligned} \tag{135}$$

$$\left|\mathcal{S}_\varrho^{hk}\right| \leq c_\varrho'\left\{\left\|u_n - u_n^{hk}\right\|_V^2 + \left\|w_n - w_n^{hk}\right\|_V^2 + \left\|v_n^h - w_n\right\|_V^2\right\}, \tag{136}$$

$$\left|\mathcal{S}_\ell^{hk}\right| \leq \alpha_\ell\left\{\left\|\varphi_n - \varphi_n^{hk}\right\|_W^2 + \left\|u_n - u_n^{hk}\right\|_V^2 + \left\|\varphi_n - \xi^h\right\|_W^2\right\}, \tag{137}$$

$$\left|\mathcal{S}_\chi^{hk}\right| \leq \alpha_\chi\left\{\left\|\theta_n - \theta_n^{hk}\right\|_Q^2 + \left\|u_n - u_n^{hk}\right\|_V^2 + \left\|\theta_n - \eta^h\right\|_Q^2\right\}. \tag{138}$$

Combining (132) and (135)–(138), we have the following estimate:

$$\begin{aligned}\left\{\left\|w_n - w_n^{hk}\right\|_V + \left\|\varphi_n - \varphi_n^{hk}\right\|_W + \left\|\theta_n - \theta_n^{hk}\right\|_Q\right\} &\leq c\Big\{\left\|\varphi_n - \varphi_{n-1}^{hk}\right\|_W\\ &+\left\|\theta_n - \theta_{n-1}^{hk}\right\|_Q + \left\|\theta_{n-1} - \theta_{n-1}^{hk}\right\|_Q + \left|\mathcal{S}_n\left(v_n^h, w_n\right)\right|^{1/2}\\ &+\left\|v_n^h - w_n\right\|_V + \left\|\xi^h - \varphi_n\right\|_W + \left\|\eta^h - \theta_n\right\|_Q\Big\}.\end{aligned} \tag{139}$$

We estimate now the terms $\|\varphi_n - \varphi_{n-1}^{hk}\|$, $\|\theta_n - \theta_{n-1}^{hk}\|$, and $\|\theta_{n-1} - \theta_{n-1}^{hk}\|$, for the first term; we have

$$\begin{aligned}\varphi_{n-1}^{hk} - \varphi_n &= \sum_{j=1}^{n-1} k_j\delta\varphi_j^{hk} + \varphi_0^h - \int_0^{t_n}\delta\varphi(s)ds - \varphi_0\\ &= \sum_{j=1}^{n-1} k_j\left(\delta\varphi_j^{hk} - \delta\varphi_j\right) + \varphi_0^h - \varphi_0\\ &+ \sum_{j=1}^{n-1}\left(\delta\varphi_j k_j - \int_{t_{j-1}}^{t_j}\delta\varphi(s)ds\right) - \int_{t_{n-1}}^{t_n}\delta\varphi(s)ds,\end{aligned} \tag{140}$$

where $\delta\varphi_j = 1/k_j(\varphi_j - \varphi_{j-1})$ and $\delta\varphi_j^{hk} = 1/k_j(\varphi_j^{hk} - \varphi_{j-1}^{hk})$, and

$$\begin{aligned}&\left\|\sum_{j=1}^{n-1}\left(\delta\varphi_j k_j - \int_{t_{j-1}}^{t_j}\delta\varphi(s)ds\right)\right\|_W\\ &= \left\|\sum_{j=1}^{n-1}\left(\int_{t_{j-1}}^{t_j}\left(\delta\varphi_j - \delta\varphi(s)\right)ds\right)\right\|_W\\ &\leq \sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}\left\|\delta\varphi_j - \delta\varphi(s)\right\|_W ds = I_k(\delta\varphi),\end{aligned} \tag{141}$$

$$\left\|\int_{t_{n-1}}^{t_n}\delta\varphi(s)ds\right\| \leq \int_{t_{n-1}}^{t_n}\|\delta\varphi(s)\|_W ds \leq k\|\delta\varphi\|_{C(0,T;W)}. \tag{142}$$

Thus,

$$\left\|\varphi_{n-1}^{hk} - \varphi_n\right\|_W \leq \sum_{j=1}^{n-1} k_j\left\|\delta\varphi_j^{kh} - \delta\varphi_j\right\|_W + \left\|\varphi_0^h - \varphi_0\right\|_W + I_k(\delta\varphi) + k\|\varphi\|_{C(0,T;W)}. \tag{143}$$

Similar to (143), we deduce that

$$\left\|\theta_{n-1}^{hk} - \theta_n\right\|_Q \leq \sum_{j=1}^{n-1} k_j\left\|\delta\theta_j^{kh} - \delta\theta_j\right\|_Q + \left\|\theta_0^h - \theta_0\right\|_Q + I_k(\delta\theta) + k\|\theta\|_{C^1(0,T;Q)}, \tag{144}$$

$$\left\|\theta_{n-1}^{hk} - \theta_{n-1}\right\|_Q \leq \sum_{j=1}^{n-1} k_j\left\|\delta\theta_j^{kh} - \delta\theta_j\right\|_Q + \left\|\theta_0^h - \theta_0\right\|_Q + I_k(\delta\theta). \tag{145}$$

To proceed, we need the following discrete version of the Grönwall's inequality presented in [12].

Lemma 5. *Assuming that $\{g_n\}_{n=1}^N$ and $\{e_n\}_{n=1}^N$ are the two sequences of nonnegative numbers satisfying*

$$e_n \leq cg_n + c\sum_{j=1}^{n-1} k_j e_j. \tag{146}$$

Then,

$$e_n \le c\left(g_n + \sum_{j=1}^{n-1} k_j g_j\right), \quad n = 1, \ldots, N. \tag{147}$$

Therefore,

$$\max_{1\le n\le N} e_n \le c \max_{1\le n\le N} g_n. \tag{148}$$

Also, the following result.

Lemma 6. *Let us define the map by*

$$I_k(v) = \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} v_j - v(s)_X ds. \tag{149}$$

Then, $I_k(v)$ converges to zero as $k \longrightarrow 0$, for all $v \in C(0,T;X)$.

Proof. Since $v \in C(0,T;X)$, $t \longmapsto v(t)$ is uniformly continuous on $[0,T]$. Thus, for any $\varepsilon > 0$, there exists a $k_0 > 0$ such that if $k < k_0$, we have

$$v(t) - v(s)_X < \frac{\varepsilon}{T} \forall s, \quad t \in [0,T],\ |t-s| < k. \tag{150}$$

Then, we have

$$I_k(v) \le \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \frac{\varepsilon}{T} ds = \varepsilon. \tag{151}$$

Hence, $I_k(v)$ converges to zero as $k \longrightarrow 0$. □

We now combine the inequalities (139) and (143)–(145) and applying the Lemma 5, we obtain the following estimate:

$$\begin{aligned}
&\max_{1\le n\le N}\left\{\left\|w_n - w_n^{hk}\right\|_V + \left\|u_n - u_n^{hk}\right\|_V + \left\|\varphi_n - \varphi_n^{hk}\right\|_W + \left\|\theta_n - \theta_n^{hk}\right\|_Q\right\} \\
&\le \max_{1\le n\le N} \inf_{\substack{v^h\in V^h\\ \xi^h\in W^h\\ \eta^h\in Q^h}} c\left\{\left|\mathcal{S}_n\left(v_n^h, w_n\right)\right|^{1/2} + \left\|v_n^h - w_n\right\|_Q + \left\|\xi^h - \varphi_n\right\|_W \right.\\
&\quad \left. + \left\|\eta^h - \theta_n\right\|_Q\right\} + c\left\{\left\|\varphi_0 - \varphi_0^h\right\|_W + \left\|\theta_0 - \theta_0^h\right\|_Q\right\} + c\left\{I_k(\delta\varphi)\right.\\
&\quad \left. + I_k(\delta\varphi) + k\|\varphi\|_{C^1(0,T;W)} + k\|\theta\|_{C^1(0,T;Q)}\right\}.
\end{aligned} \tag{152}$$

Proof of Theorem 3. We take $v_n^h = \Pi^h w_n$, $\xi_n^h = \Pi^h \varphi_n$, and $\eta_n^h = \Pi^h \theta_n$ in (152).

The convergence result follows from Lemma 6 together with an argument similar to the proof of Theorem 2. □

4. Conclusion

The paper presents a numerical analysis of the contact process between a thermo-electro-viscoelastic body and a conductive foundation. A spatially semidiscrete scheme and a fully discrete scheme to approximate the contact problem were derived. We can also study this problem with more general friction laws. The numerical simulation with the same algorithm is an interesting direction for future research.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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