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Automorphisms of Σ_{n+1}-invariant trilinear forms

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Abstract. Examination of automorphism groups of forms is undertaken by many authors. Sometimes the description of such groups is a difficult task. It turns out that a representation of a form as a sum of powers of linear forms may be very helpful, especially when this representation is unique. We show this in the case of Σ_{n+1}-invariant symmetric trilinear form Θ_n considered by Egawa and Suzuki.

Key words: symmetric trilinear form, automorphism group, unique representation, sum of powers of linear forms.

Any d-linear symmetric form $\Theta\colon V^d \longrightarrow K$ on the K-vector space V determines the form (homogeneous polynomial) of degree d defined by

$$f_\Theta(X_1, \dots, X_n) := \Theta\left(\sum_{i=1}^{n} X_i e_i, \dots, \sum_{i=1}^{n} X_i e_i\right),$$

where $(e_1, \dots, e_n)$ is the basis of V. In the sequel suppose that K is a field of characteristic 0. Then, by the polarization formula, the correspondence $\Theta \longmapsto f_\Theta$ is bijective. An automorphism φ of Θ is any automorphism φ of V such that

$$\Theta\big(\varphi(\alpha_1), \dots, \varphi(\alpha_d)\big) = \Theta(\alpha_1, \dots, \alpha_d), \quad \text{for } \alpha_1, \dots, \alpha_d \in V.$$

The automorphisms of Θ form a group $\mathrm{Aut}(\Theta)$. Of course, $\varphi \in \mathrm{Aut}(\Theta)$ if and only if $\overline{\varphi} \in \mathrm{Aut}(f_\Theta)$, that is,

$$f_\Theta\big(\overline{\varphi}(X_1, \dots, X_n)\big) = f_\Theta(X_1, \dots, X_n),$$

where

$$\overline{\varphi}(X_1, \dots, X_n) = (Y_1, \dots, Y_n) \Longleftrightarrow \varphi\left(\sum_{i=1}^{n} X_i e_i\right) = \sum_{i=1}^{n} Y_i e_i.$$

It means that $\mathrm{Aut}(\Theta) \cong \mathrm{Aut}(f_\Theta)$.

It is known that the space $F_{n,d}(K)$ of forms over K of degree d in n variables is spanned by d-th powers of linear forms (see [5, Proposition 2.11]).

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Therefore any form $f \in F_{n,d}(K)$ over an algebraically closed field K can be written in the following way

$$f = l_1^d + \cdots + l_r^d, \quad \text{where } l_j^d = (\alpha_{1j}X_1 + \cdots + \alpha_{nj}X_n)^d,\ \alpha_{ij} \in K. \quad (1)$$

It is obvious that any automorphism of V that permutes the summands in (1) belongs to $\mathrm{Aut}(f)$. Thus, if we know the representation (1) of f, then we get certain information about automorphisms of f. It can happen that when r is fixed the representation (1) of f is unique (that is, the linear forms $l_1, \ldots, l_r$ are unique up to reordering and multiplying by d-th roots of unity). In such a case we can derive from (1) the complete information about $\mathrm{Aut}(f)$. Uniqueness of the representation of forms as a sum of powers of linear forms was discussed by many authors in the previous century. However, usually generic forms were considered. Nice exposition of the results presents the book by A. Iarrobino and V. Kanev [4]. Information on unique representation of special forms one can find among others in [5], [1] and [2]. The influence of a given representaion (1) of f on the structure of $\mathrm{Aut}(f)$ will be examined in [6]. In this paper we show how it works in one special case.

Egawa and Suzuki [3] considered a trilinear form constructed in the following way. Let Σ_{n+1} be the symmetric group on the set $\{0, 1, \ldots, n\}$ for $n \geq 2$ and let $V = \langle e_1, \ldots, e_n \rangle$ be the natural n-dimensional irreducible Σ_{n+1}-module over the complex number field $\mathbb{C}$. That means that $(e_1, \ldots, e_n)$ is a basis of V and Σ_{n+1} acts on $\{e_0, e_1, \ldots, e_n\}$ in a standard way, where $e_0 = -(e_1 + \cdots + e_n)$. A Σ_{n+1}-invariant symmetric trilinear form Θ_n on V was defined by

$$\begin{aligned}
&\Theta_n(e_j, e_j, e_j) = n(n-1), && 1 \leq j \leq n,\\
&\Theta_n(e_j, e_j, e_k) = -(n-1), && 1 \leq j, k \leq n,\ j \neq k,\\
&\Theta_n(e_j, e_k, e_l) = 2, && 1 \leq j, k, l \leq n,\ j \neq k \neq l \neq j.
\end{aligned}$$

They proved that if Θ is an arbitrary nonzero Σ_{n+1}-invariant symmetric trilinear form, then $\Theta = a\Theta_n$, $0 \neq a \in \mathbb{C}$. The main result of their paper was the following theorem.

Theorem ([3, Theorem 2]) *If $n = 2$ or $n \geq 4$, then $\mathrm{Aut}(\Theta_n) \cong \mu_3 \times \Sigma_{n+1}$, where μ_3 is the group of complex 3rd roots of unity*

The proof of this theorem was quite long. It took several pages, used a few lemmas and required considering separately the case of even and odd

n. The aim of this paper is to show for $n \geq 4$ a very short proof using a representation of f_{Θ_n} as a sum of third powers of linear forms. Observe that

$$\begin{aligned} f_{\Theta_n}(X_1, \dots, X_n) &= \Theta_n\Big(\sum_{i=1}^{n} X_i e_i, \sum_{i=1}^{n} X_i e_i, \sum_{i=1}^{n} X_i e_i\Big) \\ &= n(n-1)\sum_{i=1}^{n} X_i^3 - 3(n-1)\sum_{\substack{i,j=1\\ i\neq j}}^{n} X_i^2 X_j + 12 \sum_{\substack{i,j,k=1\\ i\neq j\neq k\neq i}}^{n} X_i X_j X_k \\ &= -\frac{1}{n+1}\big((X_1+X_2+\dots+X_n)^3 + (-nX_1+X_2+\dots+X_n)^3 \\ &\quad + (X_1-nX_2+\dots+X_n)^3 + \dots + (X_1+\dots-nX_n)^3\big). \end{aligned}$$

Proof. Suppose $n \geq 4$. Notice that if we apply the nonsingular linear substitution

$$X_i \longmapsto \frac{-1}{\sqrt[3]{n+1}}(X_1+\cdots X_{i-1} - nX_i + X_{i+1} + \dots + X_n), \qquad i = 1, \dots, n$$

to the form f_{Θ_n}, then we get the form

$$g_n(X_1, \dots, X_n) := (-X_1 - \dots - X_n)^3 + X_1^3 + \dots + X_n^3.$$

Thus it suffices to consider g_n instead of f_{Θ_n}. It can be readily verified that g_n is nondegenerate which means that $(0, \dots, 0)$ is the only common zero of the partial derivatives $\partial^2 g_n/\partial X_i \partial X_j$, for $i, j = 1, \dots, n$.
Now we shall show that g_n is indecomposable, that is, g_n can not be transformed by a nonsingular linear substitution to a form $g(X_1, \dots, X_k) + h(X_{k+1}, \dots, X_n)$, for some forms $g \in F_{k,3}(K)$, $h \in F_{n-k,3}(K)$ and $k \in \{1, \dots, n-1\}$.

Suppose that g_n is decomposable and for some nonsingular linear substitution φ

$$\begin{aligned} f(X_1, \dots, X_n) &:= g_n\big(\varphi(X_1, \dots, X_n)\big) \\ &= g(X_1, \dots, X_k) + h(X_{k+1}, \dots, X_n), \\ &\qquad 1 \leq k \leq n-1. \end{aligned}$$

Applying the hessian H to the sides of the above equality we have

$$H_f(X_1, \dots, X_n) = H_g(X_1, \dots, X_k) H_h(X_{k+1}, \dots, X_n). \tag{2}$$

Now taking into account that H is a covariant of weight 2 we get

$$H_f(X_1, \dots, X_n) = \det(\varphi)^2 H_{g_n}\big(\varphi(X_1, \dots, X_n)\big)$$

which together with (2) means that H_{g_n} is a reducible polynomial. However, this is not true, because

$$\begin{aligned} H_{g_n}(X_1, \dots, X_n) = 6^n \Big(&- X_1^2 \sum_{i=2}^{n} X_2 \cdots X_{i-1} X_{i+1} \cdots X_n \\ &- X_1(X_2 + \cdots + X_n) \sum_{i=2}^{n} X_2 \cdots X_{i-1} X_{i+1} \cdots X_n \\ &- (X_2 + \cdots + X_n)(X_2 \cdots X_n)\Big) \end{aligned}$$

and, by Eisenstein criterion, H_{g_n} is an irreducible polynomial.
Let

$$\begin{aligned} &l_0(X_1, \dots, X_n) = -(X_1 + \cdots + X_n), \\ &\qquad l_i(X_1, \dots, X_n) = X_i, \ i = 1, \dots, n. \end{aligned}$$

By [2, Theorem 3.4] that says

If a nondegenerate and indecomposable form of degree $d \geq 3$ in $n \geq 4$ variables is a sum of $n+1$ d-th powers of linear forms, then the linear forms are unique up to reordering and multiplying by d-th roots of unity,

the above representation of g_n as a sum of $n+1$ third powers of linear forms $l_0, l_1, \dots, l_n$ is unique and every automorphism permutes the summands in this representation. Consider the group homomorphism

$$\begin{aligned} &\Phi\colon \operatorname{Aut}(g_n) \longrightarrow \Sigma_{n+1}, \\ &\qquad \Phi(\varphi) = \sigma \iff l_i^3\big(\varphi(X_1, \dots, X_n)\big) = l_{\sigma(i)}^3(X_1, \dots, X_n). \end{aligned}$$

Notice that cycles $(0, i)$ and $(1, \dots, n)$ belong to $\operatorname{im}\Phi$, so Φ is an epimorphism. Moreover, $\ker\Phi = \mu_3 \operatorname{id}_V$. In this way we get the exact sequence

$$0 \longrightarrow \mu_3 \operatorname{id}_V \longrightarrow \operatorname{Aut}(g_n) \longrightarrow \Sigma_{n+1} \longrightarrow 0$$

which splits. Since $\mu_3 \operatorname{id}_V$ is contained in the center of $\operatorname{Aut}(g_n)$ we have

$$\operatorname{Aut}(g_n) \cong \mu_3 \times \Sigma_{n+1}.$$

□

Remark The proof presented above works over any field K of characteristic 0 which contains a primitive third root of unity. We have not to worry about the coefficient $-1/\sqrt[3]{n+1}$ used in the proof, because at the beginnig we could have considered the form $-(n+1)f_{\Theta_n}$ instead of f_{Θ_n}. In case K lacks a primitive third root of unity we can easily reorganize the proof to get $\operatorname{Aut} f_{\Theta_n} \cong \Sigma_{n+1}$.

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