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Absolute and ordinary Köthe-Toeplitz duals of some generalized sets of difference sequences

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Abstract. In this paper we determine the ordinary Köthe-Toeplitz duals of the sets $\Delta^m \ell_\infty(p)$ and $\Delta^m c(p)$ and the absolute Köthe-Toeplitz duals of the sets $\Delta^m c_0(p)$ and $\Delta^m c(p)$ defined by Et and Başarır [8].

Key words: difference sequences, α-, β- and γ-duals.

1. Introduction

Let ω be the space of all complex sequences $x = (x_k)_{k=1}^\infty$. Throughout the paper $p = (p_k)_{k=1}^\infty$ shall always be an arbitrary sequence of positive reals.

The sequence spaces $\ell_\infty(p)$, $c(p)$, $c_0(p)$ and $\ell(p)$ have been studied by various authors: (cf. [2], [3], [5], and [7]).

Ahmad and Mursaleen [1] defined the sequence spaces

$$\Delta\ell_\infty(p) = \{x \in \omega : \Delta x \in \ell_\infty(p)\},$$
$$\Delta c(p) = \{x \in \omega : \Delta x \in c(p)\},$$
$$\Delta c_0(p) = \{x \in \omega : \Delta x \in c_0(p)\}$$

where $\Delta x = (x_k - x_{k+1})$. Malkowsky [6] also studied the sequence spaces.

After then Et and Başarır [8] defined the sequence spaces

$$\Delta^m\ell_\infty(p) = \{x \in \omega : \Delta^m x \in \ell_\infty(p)\},$$
$$\Delta^m c(p) = \{x \in \omega : \Delta^m x \in c(p)\},$$
$$\Delta^m c_0(p) = \{x \in \omega : \Delta^m x \in c_0(p)\}$$

where m is a positive integer, $\Delta^0 x = (x_k)$, $\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$, $\Delta^m x_k = \sum_{v=0}^m (-1)^v \binom{m}{v} x_{k+v}$. Furthermore Et and Başarır [8] determined the absolute Köthe-Toeplitz duals of the set $\Delta^m \ell_\infty(p)$.

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The operator

$$D : \Delta^m \ell_\infty(p) \to \Delta^m \ell_\infty(p)$$

defined by $Dx = (0, 0, \ldots, x_{m+1}, x_{m+2}, \ldots)$, where $x = (x_1, x_2, x_3, \ldots)$ is a linear operator on $\Delta^m \ell_\infty(p)$. Furthermore the set

$$\begin{aligned} D[\Delta^m \ell_\infty(p)] &= D\Delta^m \ell_\infty(p) \\ &= \{x = (x_k) : x \in \Delta^m \ell_\infty(p),\ x_1 = x_2 = \cdots = x_m = 0\} \end{aligned}$$

is a subspace of $\Delta^m \ell_\infty(p)$.

Now let us define

$$\begin{aligned} \Delta^m &: D\Delta^m \ell_\infty(p) \to \ell_\infty(p) \\ \Delta^m(x) &= y = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}). \end{aligned} \tag{1}$$

It can be shown that Δ_r^m is a linear bijection.

2. Köthe-Toeplitz duals

In this section we determine β- and γ-duals of $\Delta^m \ell_\infty(p)$ and $\Delta^m c(p)$. Also we give α-dual of $\Delta^m c_0(p)$ and $\Delta^m c(p)$.

Definition 2.1 Let X be a sequence space and define

$$X^\alpha = \left\{ a \in \omega : \sum_{k=1}^\infty |a_k x_k| < \infty, \quad \text{for all } \ x \in X \right\},$$

$$X^\beta = \left\{ a \in \omega : \sum_{k=1}^\infty a_k x_k \quad \text{converges, for all } \ x \in X \right\}$$

$$X^\gamma = \left\{ a \in \omega : \sup_n \left| \sum_{k=1}^n a_k x_k \right| < \infty, \quad \text{for all } \ x \in X \right\}.$$

Then X^α, X^β, X^γ are called α-, β-, γ-dual spaces of X, respectively. It is easy to show that $\emptyset \subset X^\alpha \subset X^\beta \subset X^\gamma$. If $X \subset Y$, then $Y^\eta \subset X^\eta$ for $\eta = \alpha, \beta, \gamma$. We shall write $X^{\alpha\alpha} = (X^\alpha)^\alpha$.

Theorem 2.2 *For every strictly positive sequence $p = (p_k)$, we have*

(i) $[\Delta^m c_0(p)]^\alpha = D_0^\alpha(p)$,

(ii) $[\Delta^m c_0(p)]^{\alpha\alpha} = D_0^{\alpha\alpha}(p)$

where

$$D_0^\alpha(p) = \bigcup_{N=2}^{\infty} \left\{ a \in \omega : \sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{-1/p_j} < \infty \right\}$$

and

$$D_0^{\alpha\alpha}(p) = \bigcap_{N=2}^{\infty} \left\{ a \in \omega : \sup_{k \geq m+1} |a_k| \left[\sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{-1/p_j} \right]^{-1} < \infty \right\}.$$

(We adopt the usual convention that $\sum_{j=1}^{k-m} \binom{k-j-1}{m-1} y_j = 0$ $(k - m < 1)$ *for arbitrary* y_j*.)*

Proof. (i) Let $a \in D_0^\alpha(p)$ and $x \in \Delta^m c_0(p)$. Then there is an integer k_0 such that $\sup_{k>k_0} |\Delta^m x_k|^{p_k} \leq N^{-1}$, where N is the number in $D_0^\alpha(p)$. We put $M = \max_{1\leq k\leq k_0} |\Delta^m x_k|^{p_k}$, $n = \min_{1\leq k\leq k_0} p_k$, $L = (M+1)N$ and define the sequence y by $y_k = x_k . L^{-1/n}$ $(k = 1, 2, \ldots)$. Then it is easy to see that $\sup_k |\Delta^m y_k|^{p_k} \leq N^{-1}$.

Since $\sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{-1/p_j} > \sum_{j=1}^{m} \binom{k-j-1}{m-j} N^{-1/p_j}$ for arbitrary $N > 1$ $(k = 2m, 2m+1, \ldots)$, $a \in D_0^\alpha(p)$ implies

$$\sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{m} \binom{k-j-1}{m-j} |\Delta^{m-j} y_j| < \infty.$$

Then

$$\begin{aligned}
\sum_{k=1}^{\infty} |a_k x_k| &= L^{1/n} \sum_{k=1}^{\infty} |a_k y_k| \\
&= L^{1/n} \sum_{k=1}^{\infty} |a_k| \Bigg(\Bigg| \sum_{j=1}^{k-m} (-1)^m \binom{k-j-1}{m-1} \Delta^m y_j \\
&\qquad + \sum_{j=1}^{m} (-1)^{m-j} \binom{k-j-1}{m-j} \Delta^{m-j} y_j \Bigg| \Bigg) \\
&\leq L^{1/n} \sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{-1/p_j}
\end{aligned}$$

$$+ L^{1/n} \sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{m} \binom{k-j-1}{m-j} |\Delta^{m-j} y_j|$$
$$< \infty.$$

So we have $a \in [\Delta^m c_0(p)]^\alpha$. Therefore $D_0^\alpha(p) \subset [\Delta^m c_0(p)]^\alpha$.

Conversely, let $a \notin D_0^\alpha(p)$. Then we can determine a strictly increasing sequence $(k(s))$ of integers such that $k(1) = 1$ and

$$M_s^m = \sum_{k=k(s)}^{k(s+1)-1} |a_k| \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} (s+1)^{-1/p_j} > 1 \quad (s = 1, 2, \ldots).$$

We define the sequence x by

$$x_k = \sum_{l=1}^{s-1} \sum_{j=k(l)}^{k(l+1)-1} \binom{k-j-1}{m-1} (l+1)^{-1/p_j}$$
$$+ \sum_{j=k(s)}^{k-m} \binom{k-j-1}{m-1} (s+1)^{-1/p_j}$$
$$(k(s) \leq k \leq k(s+1) - 1;\ s = 1, 2, \ldots).$$

Then it is easy to see that $|\Delta^m x_k|^{p_k} = \frac{1}{s+1}$ $(k(s) \leq k \leq k(s+1) - 1;$ $s = 1, 2, \ldots)$ hence $x \in \Delta^m c_0(p)$, and $\sum_{k=1}^{\infty} |a_k x_k| \geq \sum_{s=1}^{\infty} = \infty$, i.e. $a \notin [\Delta^m c_0(p)]^\alpha$.

(ii) For $N = 2, 3, \ldots$, we put

$$E_N^1 = \left\{ a \in \omega : \sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{-1/p_j} < \infty \right\},$$

$$F_N^1 = \left\{ a \in \omega : \sup_{k \geq m+1} |a_k| \left[\sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{-1/p_j} \right]^{-1} < \infty \right\}.$$

By a well known result (cf. [3, Lemma 4 (iv)]), we have to show $F_N^1 = (E_N^1)^\alpha$ $(N = 2, 3, \ldots)$. The proof of this is standard and therefore omitted. □

Theorem 2.3 *For every strictly positive sequence $p = (p_k)$, we have*

$$[\Delta^m c(p)]^\alpha = D^\alpha(p) = D_0^\alpha(p) \cap \left\{ a \in \omega : \sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} < \infty \right\}.$$

Proof. Let $a \in D^\alpha(p)$ and $x \in \Delta^m c(p)$. Then there is a complex number l such that $|\Delta^m x_k - l|^{p_k} \to 0$ $(k \to \infty)$. We define $y = (y_k)$ by $y_k = x_k + l(-1)^{m+1}\sum_{j=1}^{k-m}\binom{k-j-1}{m-1}$ $(k = 1, 2, \ldots)$. Then $y \in \Delta^m c_0(p)$ and

$$\sum_{k=1}^{\infty}|a_k x_k| \le \sum_{k=1}^{\infty}|a_k|\sum_{j=1}^{k-m}\binom{k-j-1}{m-1}|\Delta^m y_j| + \sum_{k=1}^{\infty}|a_k|\sum_{j=1}^{m}\binom{k-j-1}{m-j}|\Delta^{m-j} y_j| + |l|\sum_{k=1}^{\infty}|a_k|\sum_{j=1}^{k-m}\binom{k-j-1}{m-1} < \infty$$

by Theorem 2.2 (i) and since $a \in D^\alpha(p)$.

Now let $a \in [\Delta^m c(p)]^\alpha \subset [\Delta^m c_0(p)]^\alpha = D_0^\alpha(p)$ by Theorem 2.2 (i). Since the sequence x defined by $x_k = (-1)^m\sum_{j=1}^{k-m}\binom{k-j-1}{m-1}$ $(k = 1, 2, \ldots)$ is in $\Delta^m c(p)$ we have

$$\sum_{k=1}^{\infty}|a_k|\sum_{j=1}^{k-m}\binom{k-j-1}{m-1} < \infty.$$

□

Theorem 2.4 *For every strictly positive sequence $p = (p_k)$, we have*

(i) $[D\Delta^m \ell_\infty(p)]^\beta = D_\infty^\beta(p)$,

(ii) $[D\Delta^m \ell_\infty(p)]^\gamma = D_\infty^\gamma(p)$

where

$$D_\infty^\beta(p) = \bigcap_{N>1}\left\{a \in \omega : \sum_{k=1}^{\infty} a_k \sum_{j=1}^{k-m}\binom{k-j-1}{m-1} N^{1/p_j} \quad \textit{converges and} \quad \sum_{k=1}^{\infty}|R_k|\sum_{j=1}^{k-m+1}\binom{k-j-1}{m-2} N^{1/p_j} < \infty\right\},$$

$$D_\infty^\gamma(p) = \bigcap_{N>1}\left\{a \in \omega : \sup_n\left|\sum_{k=1}^{n} a_k \sum_{j=1}^{k-m}\binom{k-j-1}{m-1} N^{1/p_j}\right| < \infty, \quad \sum_{k=1}^{\infty}|R_k|\sum_{j=1}^{k-m+1}\binom{k-j-1}{m-2} N^{1/p_j} < \infty\right\}$$

and $R_k = \sum_{v=k+1}^{\infty} a_v$ $(k = 1, 2, \ldots)$.

Proof. (i) If $x \in D\Delta^m \ell_\infty(p)$ then there exists and only one $y = (y_k) \in \ell_\infty(p)$ such that

$$x_k = \sum_{j=1}^{k-m} (-1)^m \binom{k-j-1}{m-1} y_j$$

for sufficiently large k, for instance $k > m$ by (1). Then there is an integer $N > \max\{1, \sup_k |\Delta^m x_k|^{p_k}\}$. Let $a \in D_\infty^\beta(p)$, and suppose that $\binom{-1}{-1} = 1$ (in some literature it is assumed that $\binom{r}{k} = 0$ for $k < 0$). Then we may write

$$\sum_{k=1}^{n} a_k x_k = \sum_{k=1}^{n} a_k \left(\sum_{j=1}^{k-m} (-1)^m \binom{k-j-1}{m-1} y_j \right)$$
$$= (-1)^m \sum_{k=1}^{n-m} R_{k+m-1} \sum_{j=1}^{k} \binom{k+m-j-2}{m-2} y_j - R_n x_n. \quad (2)$$

Since $\sum_{k=1}^{\infty} |R_{k+m-1}| \sum_{j=1}^{k} \binom{k+m-j-2}{m-2} N^{1/p_j} < \infty$, the series $\sum_{k=1}^{\infty} R_{k+m-1} \sum_{j=1}^{k} \binom{k+m-j-2}{m-2} y_j$ is absolutely convergent. Morever by Corollary 2 in [4], the convergence of $\sum_{k=1}^{\infty} a_k \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j}$ implies $\lim_{n\to\infty} R_n \sum_{j=1}^{n-m} \binom{n-j-1}{m-1} N^{1/p_j} = 0$. Hence $\sum_{k=1}^{\infty} a_k x_k$ is convergent for all $x \in D\Delta^m \ell_\infty(p)$, so $a \in [D\Delta^m \ell_\infty(p)]^\beta$.

Conversely let $a \in [D\Delta^m \ell_\infty(p)]^\beta$. Then $\sum_{k=1}^{\infty} a_k x_k$ is convergent for each $x \in D\Delta^m \ell_\infty(p)$. If we take the sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} 0, & k \le m \\ \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j}, & k > m \end{cases}$$

then we have

$$\sum_{k=1}^{\infty} a_k \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j} = \sum_{k=1}^{\infty} a_k x_k < \infty$$

Thus the series $\sum_{k=1}^{\infty} a_k \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j}$ is convergent. This implies that $\lim_{n\to\infty} R_n \sum_{j=1}^{n-m} \binom{n-j-1}{m-1} N^{1/p_j} = 0$ by Corollary 2 in [4].

Now let $a \in [D\Delta^m \ell_\infty(p)]^\beta - D_\infty^\beta(p)$. Then $\sum_{k=1}^{\infty} |R_k| \sum_{j=1}^{k-m+1} \binom{k-j-1}{m-2} N^{1/p_j}$ is divergent, that is,

$$\sum_{k=1}^{\infty} |R_k| \sum_{j=1}^{k-m+1} \binom{k-j-1}{m-2} N^{1/p_j} = \infty.$$

We define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} 0, & k \le m \\ \displaystyle\sum_{v=1}^{k-1} \mathrm{sgn} R_v \sum_{j=1}^{v-m+1} \binom{v-j-1}{m-2} N^{1/p_j}, & k > m \end{cases}$$

where $a_k > 0$ for all k or $a_k < 0$ for all k. It is trivial that $x = (x_k) \in D\Delta^m \ell_\infty(p)$. Then we may write for $n > m$

$$\sum_{k=1}^{n} a_k x_k = -\sum_{k=1}^{n-m} R_{k+m-1} \Delta x_{k+m-1} - R_n x_n.$$

Since

$$\left(\left(\sum_{v=1}^{n-1} \mathrm{sgn} R_v \sum_{j=1}^{v-m+1} \binom{v-j-1}{m-2} N^{1/p_j}\right)\left(\sum_{j=1}^{n-m} \binom{n-j-1}{m-1} N^{1/p_j}\right)^{-1}\right) \in \ell_\infty,$$

now letting $n \to \infty$ we get

$$\sum_{k=1}^{\infty} a_k x_k = \sum_{k=1}^{\infty} |R_{k+m-1}| \sum_{j=1}^{k} \binom{k+m-j-2}{m-2} N^{1/p_j} = \infty.$$

This contradicts to $a \in [D\Delta^m \ell_\infty(p)]^\beta$. Hence $a \in D_\infty^\beta(p)$.

(ii) can be proved by the same way as above, using Corollary 1 in [4]. □

Lemma 2.5 $[D\Delta^m \ell_\infty(p)]^\eta = [D\Delta^m c(p)]^\eta$ *for* $\eta = \beta$ *or* γ.

Proof is trivial.

Lemma 2.6 (i) $[\Delta^m \ell_\infty(p)]^\eta = [D\Delta^m \ell_\infty(p)]^\eta$,

(ii) $[\Delta^m c(p)]^\eta = [D\Delta^m c(p)]^\eta$ *for* $\eta = \beta$ *or* γ.

Proof is omitted.

Corollary 2.7 *Let* X *stand for* ℓ_∞ *or* c. *Then*

$$\text{i)}\quad [\Delta^m X(p)]^\beta = \bigcap_{N>1} \Big\{ a \in \omega : \textstyle\sum_{k=1}^\infty a_k \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j} \text{ converges}$$
$$\text{and } \textstyle\sum_{k=1}^\infty |R_k| \sum_{j=1}^{k-m+1} \binom{k-j-1}{m-2} N^{1/p_j} < \infty \Big\},$$

$$\text{ii)}\quad [\Delta^m X(p)]^\gamma = \bigcap_{N>1} \Big\{ a \in \omega : \sup_n \Big| \textstyle\sum_{k=1}^n a_k \sum_{j=1}^{k-m} \binom{k-j-1}{m-1} N^{1/p_j} \Big| < \infty,$$
$$\textstyle\sum_{k=1}^\infty |R_k| \sum_{j=1}^{k-m+1} \binom{k-j-1}{m-2} N^{1/p_j} < \infty \Big\}$$

where $R_k = \sum_{v=k+1}^\infty a_v$.

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