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Toeplitz operators and Carleson measures on parabolic Bergman spaces

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Abstract. Let $\boldsymbol{b}_\alpha^p$ be the parabolic Bergman space, which is the Banach space of all L^p-solutions of the parabolic equation $(\partial/\partial t + (-\Delta)^\alpha)u = 0$ on the upper half space $\boldsymbol{R}_+^{n+1}$ with $0 < \alpha \leq 1$. We discuss the relation of Toeplitz operators to Carleson measures.

Key words: Carleson measure, Toeplitz operator, heat equation, parabolic operator of fractional order, Bergman space.

1. Introduction

Let $\boldsymbol{R}_+^{n+1}$ be the upper half space of the $(n+1)$-dimensional Euclidean space $(n \geq 1)$. We denote by $X = (x, t)$ a point in $\boldsymbol{R}_+^{n+1} = \boldsymbol{R}^n \times (0, \infty)$, and by $L^{(\alpha)}$ the α-parabolic operator on $\boldsymbol{R}_+^{n+1}$:

$$L^{(\alpha)} := \frac{\partial}{\partial t} + (-\Delta)^\alpha,$$

where $\Delta := \partial_{x_1}^2 + \cdots + \partial_{x_n}^2$ is the Laplacian on the x-space $\boldsymbol{R}^n$ and $0 < \alpha \leq 1$. For $0 < p \leq \infty$, we denote by

$$L^p(V) := \left\{ f;\ \text{Borel measurable on } \boldsymbol{R}_+^{n+1}, \right.$$
$$\left. \|f\|_{L^p(V)} := \left(\int |f|^p dV \right)^{1/p} < \infty \right\}$$

the usual Lebesgue space, where V is the Lebesgue measure on $\boldsymbol{R}_+^{n+1}$ and $\|\cdot\|_{L^\infty(V)}$ is considered as the essential supremum norm. We consider the parabolic Bergman space and the Bloch space on the upper half space:

$$\begin{aligned} \boldsymbol{b}_\alpha^p &:= \{u \in C(\boldsymbol{R}_+^{n+1}); L^{(\alpha)}u = 0,\ \|u\|_{L^p(V)} < \infty\}, \\ \mathcal{B}_\alpha &:= \{u \in C^1(\boldsymbol{R}_+^{n+1}); L^{(\alpha)}u = 0,\ \|u\|_{\mathcal{B}_\alpha} < \infty\}, \end{aligned}$$

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where $0 < p \leq \infty$, and

$$\|u\|_{\mathcal{B}_\alpha} := |u(0,1)| + \sup_{(x,t) \in \boldsymbol{R}^{n+1}_+} \{t^{1/(2\alpha)} |\nabla_x u(x,t)| + t|\partial_t u(x,t)|\}. \quad (1)$$

Here, "$L^{(\alpha)} u = 0$" means that u is $L^{(\alpha)}$-harmonic on $\boldsymbol{R}^{n+1}_+$, which is defined later (see also [2]). The orthogonal projection from $L^2(V)$ to $\boldsymbol{b}^2_\alpha$ is an integral operator by a kernel R_α, called the α-parabolic Bergman kernel (see [3]). Then for a positive measure μ on the upper half space $\boldsymbol{R}^{n+1}_+$, we can discuss the Toeplitz operator, defined by

$$(T_\mu u)(X) := \int R_\alpha(X, Y) u(Y) d\mu(Y). \quad (2)$$

In [5], authors treat the case where μ is absolutely continuous with respect to the Lebesgue measure and discuss the condition that T_μ be bounded on $\boldsymbol{b}^2_\alpha$, relating with the condition that μ be a Carleson type measure. In this paper, we generalize this result to consider a condition that T_μ be a bounded operator from $\boldsymbol{b}^p_\alpha$ to $\boldsymbol{b}^q_\alpha$ and from $\boldsymbol{b}^p_\alpha$ to $\mathcal{B}_\alpha/\boldsymbol{R}$, where $1 \leq p \leq q < \infty$. Here we remark that $\mathcal{B}_\alpha/\boldsymbol{R}$ can be identified with the dual space of $\boldsymbol{b}^1_\alpha$ (see [3, Theorem 8.4]), which corresponds to the case $q = \infty$, where $\boldsymbol{R}$ is the space of all constant functions, and then

$$\|u\|_{\mathcal{B}_\alpha/\boldsymbol{R}} = \sup_{(x,t) \in \boldsymbol{R}^{n+1}_+} \{t^{1/(2\alpha)} |\nabla_x u(x,t)| + t|\partial_t u(x,t)|\}. \quad (3)$$

In [1], B.R. Choe, H. Koo and H. Yi discuss the Toeplitz operators for the harmonic Bergman spaces on the upper half space, which corresponds to our case for $\alpha = 1/2$ (see [3, Corollary 4.4] and [4, Section 3]).

Now we shall state the results with some definitions.

Definition 1 Let μ be a positive Borel measure on $\boldsymbol{R}^{n+1}_+$ and τ be a positive number. We say that μ is a τ-Carleson measure (with respect to $L^{(\alpha)}$) if there exists a constant $C > 0$ such that

$$\mu(Q^{(\alpha)}(X)) \leq C t^{(n/(2\alpha)+1)\tau} \quad (4)$$

holds for all $X = (x, t) \in \boldsymbol{R}^{n+1}_+ = \boldsymbol{R}^n \times (0, \infty)$. Here $Q^{(\alpha)}(X)$ is an α-parabolic Carleson box, defined by

$$Q^{(\alpha)}(X) := \{(y_1, \ldots, y_n, s); t \leq s \leq 2t, \\ |y_j - x_j| \leq 2^{-1} t^{1/2\alpha},\ j = 1, \ldots, n\}.$$

Our first result is the boundedness of the inclusion of $\boldsymbol{b}_\alpha^p$ into $L^q(\mu)$.

Theorem 1 *Let $1 \le p \le q < \infty$ and $\mu \ge 0$ be a Borel measure on $\boldsymbol{R}_+^{n+1}$. Then μ is a q/p-Carleson measure with respect to $L^{(\alpha)}$ if and only if there exists a constant $C > 0$ such that the inequality*

$$\left(\int |u(X)|^q d\mu(X) \right)^{1/q} \le C \left(\int |u(X)|^p dV(X) \right)^{1/p}, \tag{5}$$

i.e., $\|u\|_{L^q(\mu)} \le C\|u\|_{L^p(V)}$ holds for all $u \in \boldsymbol{b}_\alpha^p$.

In order to characterize the boundedness of Toeplitz operators, we introduce some auxiliary functions.

Definition 2 Let μ be a positive Borel measure on $\boldsymbol{R}_+^{n+1}$. For $Y = (y,\, s) \in \boldsymbol{R}_+^{n+1}$, we put

$$\hat{\mu}_\alpha(Y) := \frac{\mu(Q^{(\alpha)}(Y))}{V(Q^{(\alpha)}(Y))},$$
$$\tilde{\mu}_\alpha(Y) := \frac{\int R_\alpha(X,Y)^2 d\mu(X)}{\int R_\alpha(X,\, Y)^2 dV(X)},$$

where R_α is the α-parabolic Bergman kernel (see § 2). We call $\hat{\mu}_\alpha$ the averaging function of μ and call $\tilde{\mu}_\alpha$ the Berezin transformation of μ. Note that

$$V(Q^{(\alpha)}(Y)) = s^{n/(2\alpha)+1} \quad \text{and} \quad \int R_\alpha(X,\, Y)^2 dV(X) = Cs^{-(n/(2\alpha)+1)}$$

with some constant $C > 0$ independent of $Y \in \boldsymbol{R}_+^{n+1}$.

We also use a modified kernel defined by

$$R_\alpha^m(X,\, Y) = R_\alpha^m(x,\, t; y,\, s) := c_m s^m \frac{\partial^m}{\partial s^m} R_\alpha(x,\, t; y,\, s).$$

Here m is an nonnegative integer and $c_m = (-2)^m/m!$. Note that $R_\alpha^0 = R_\alpha$.

To state our main result, we use $\mathcal{E}_m$, the vector space generated by $\{R_\alpha^m(\,\cdot\,,\, Y); Y \in \boldsymbol{R}_+^{n+1}\}$. Remark that if $m \ge 1$ and $1 \le p < \infty$, then $\mathcal{E}_m$ is dense in $\boldsymbol{b}_\alpha^p$ (cf. [3, Lemma 8.2]).

Theorem 2 *Let $1 \leq p < \infty$ and $1 < q \leq \infty$ with $p \leq q$ and $1/p - 1/q < 1$. Assume that a positive Borel measure μ on $\boldsymbol{R}_+^{n+1}$ satisfies*

$$\int |R_\alpha^m(X, \cdot)| d\mu(X) < \infty, \quad V\text{-a.e.} \tag{6}$$

for some integer $m \geq 1$. Then the following statements are equivalent:

(I) (a) *When $1 < q < \infty$, the Toeplitz operator $T_\mu \colon \boldsymbol{b}_\alpha^p \to \boldsymbol{b}_\alpha^q$ is bounded, i.e., for every $u \in \boldsymbol{b}_\alpha^p$, $\int |R_\alpha(\,\cdot\,, Y) u(Y)| d\mu(Y) < \infty$, V-a.e. and*

$$\|T_\mu u\|_{L^q(V)} \leq C_1 \|u\|_{L^p(V)}$$

with some constant $C_1 > 0$;

(b) *When $q = \infty$, $T_\mu \colon \boldsymbol{b}_\alpha^p \to \mathcal{B}_\alpha / \boldsymbol{R}$ is bounded, which here means that there exists a constant C_1 such that for every $u \in \mathcal{E}_m$*

$$\|T_\mu u\|_{\mathcal{B}_\alpha / \boldsymbol{R}} \leq C_1 \|u\|_{L^p(V)};$$

(II) *μ is a τ-Carleson measure with respect to $L^{(\alpha)}$, where $\tau = 1 + 1/p - 1/q$, i.e., there exists a constant $C_2 > 0$ such that for all $X = (x, t) \in \boldsymbol{R}_+^{n+1}$,*

$$\hat{\mu}_\alpha(X) \leq C_2 t^{(n/(2\alpha)+1)(1/p-1/q)};$$

(III) *There exists a constant $C_3 > 0$ such that for all $X = (x, t) \in \boldsymbol{R}_+^{n+1}$,*

$$\tilde{\mu}_\alpha(X) \leq C_3 t^{(n/(2\alpha)+1)(1/p-1/q)}.$$

Remark 1 If we replace $\tilde{\mu}_\alpha$ by a modified Berezin transformation $\tilde{\mu}_{\alpha,1}$ in the statement (III), Theorem 2 remains true for the case $p = 1$ and $q = \infty$. For the definition of $\tilde{\mu}_{\alpha,1}(X)$, see Section 5 below.

Remark 2 In the above theorem, if $\mu \geq 0$ satisfies

$$\int (1 + t + |x|^{2\alpha})^{-\eta} d\mu(x, t) < \infty$$

for some η, then (6) holds for $m \geq \eta + n/(2\alpha) + 1$ (see Lemma 2 below). The condition (6) is used only when we show (I) implies (II). In (b) of (I), since $\mathcal{E}_m$ is dense in $\boldsymbol{b}_\alpha^p$, it can be considered that the Toeplitz operator T_μ is extended on $\boldsymbol{b}_\alpha^p$.

Throughout this paper, C will denote a positive constant whose value is not necessary the same at each occurrence; it may vary even within a line.

2. Preliminaries

First, we give the definition of $L^{(\alpha)}$-harmonic functions. For an open set D in $\boldsymbol{R}^{n+1}$, let $C_K^\infty(D)$ denote the set of all infinitely differentiable functions with compact support on D. In order to define $L^{(\alpha)}$-harmonic functions, we shall recall how the adjoint operator $\tilde{L}^{(\alpha)} = -\partial/\partial t + (-\Delta)^\alpha$ acts on $C_K^\infty(\boldsymbol{R}^{n+1})$. For $0 < \alpha < 1$, $(-\Delta)^\alpha$ is the convolution operator defined by $-c_{n,\alpha}\text{p.f.}|x|^{-n-2\alpha}$, where

$$c_{n,\alpha} = -4^\alpha \pi^{-n/2}\Gamma((n+2\alpha)/2)/\Gamma(-\alpha) > 0$$

and $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$. Hence for $\varphi \in C_K^\infty(\boldsymbol{R}^{n+1})$,

$$\begin{aligned}\tilde{L}^{(\alpha)}\varphi(x,\, t) = &-\frac{\partial}{\partial t}\varphi(x,\, t)\\ &- c_{n,\alpha}\lim_{\delta\downarrow 0}\int_{|y|>\delta}(\varphi(x+y,\, t) - \varphi(x,\, t))|y|^{-n-2\alpha}dy.\end{aligned}$$

It is easily seen that if $\text{supp}(\varphi)$, the support of φ, is contained in $\{|x| < r,\ t_1 < t < t_2\}$, then

$$|\tilde{L}^{(\alpha)}\varphi(x,\, t)| \le 2^{n+2\alpha}c_{n,\alpha}\left(\sup_{t_1<s<t_2}\int_{\boldsymbol{R}^n}|\varphi(y,\, s)|dy\right)\cdot |x|^{-n-2\alpha} \tag{7}$$

for $(x,\, t)$ with $|x| \ge 2r$. For an open set D in $\boldsymbol{R}^{n+1}$, we put

$$s(D) := \{(x,\, t) \in \boldsymbol{R}^{n+1};\, (y,\, t) \in D \text{ for some } y \in \boldsymbol{R}^n\}.$$

Since $\text{supp}(\tilde{L}^{(\alpha)}\varphi)$ may lie in $s(D)$ even if $\text{supp}(\varphi) \subset D$, we can define the $L^{(\alpha)}$-harmonicity on D only for functions defined on $s(D)$.

Definition 3 A function u is said to be $L^{(\alpha)}$-harmonic on an open set D, if u is defined on $s(D)$ and satisfies the following conditions:
(a) u is a Borel measurable function on $s(D)$,
(b) u is continuous on D,
(c) for every $\varphi \in C_K^\infty(D)$, $\iint_{s(D)}|u\tilde{L}^{(\alpha)}\varphi|dxdt < \infty$ and $\iint_{s(D)}u\tilde{L}^{(\alpha)}\varphi dxdt = 0$.

Remark 3 When $0 < \alpha < 1$, the inequality (7) implies that the integrability condition in (c) of Definition 3 is equivalent to the following: for any closed strip $[t_1, t_2] \times \boldsymbol{R}^n \subset s(D)$

$$\int_{t_1}^{t_2} \int_{\boldsymbol{R}^n} |u(x, t)|(1 + |x|)^{-n-2\alpha} dxdt < \infty.$$

Next, we introduce the fundamental solution $W^{(\alpha)}$ of $L^{(\alpha)}$, defined by

$$W^{(\alpha)}(x, t) = \begin{cases} (2\pi)^{-n} \displaystyle\int_{\boldsymbol{R}^n} \exp(-t|\xi|^{2\alpha} + \sqrt{-1}\, x \cdot \xi) d\xi & t > 0 \\ 0 & t \leq 0, \end{cases}$$

and give some properties and estimates necessary for our discussions. When $\alpha = 1$ or $\alpha = 1/2$, we know the explicit form. In fact, for $t > 0$,

$$W^{(1)}(x, t) = (4\pi t)^{-n/2} e^{-|x|^2/(4t)}$$

and

$$W^{(1/2)}(x, t) = a_n \frac{t}{\bigl(t^2 + |x|^2\bigr)^{(n+1)/2}},$$

where $a_n = \Gamma\bigl((n+1)/2\bigr)/\pi^{(n+1)/2}$. The following homogeneity of $W^{(\alpha)}$ is useful:

$$\partial_x^\beta \partial_t^k W^{(\alpha)}(x, t) = t^{-((n+|\beta|)/(2\alpha)+k)} (\partial_x^\beta \partial_t^k W^{(\alpha)})(t^{-1/(2\alpha)} x, 1), \tag{8}$$

where $\beta = (\beta_1, \ldots, \beta_n)$ be a multi-index and $k \geq 0$ be an integer.

Lemma 1 *Let $\beta = (\beta_1, \ldots, \beta_n)$ be a multi-index of nonnegative integers and $k \geq 0$ be an integer. Then there exists a constant $C > 0$ such that*

$$|\partial_x^\beta \partial_t^k W^{(\alpha)}(x, t)| \leq C\bigl(t + |x|^{2\alpha}\bigr)^{-(n+|\beta|)/(2\alpha)-k}$$

for all $(x, t) \in \boldsymbol{R}_+^{n+1}$.

Proof. Quite the same argument as in the proof of [3, Lemma 3.1] gives us an estimate

$$|\partial_x^\beta \partial_t^k W^{(\alpha)}(x, 1)| \leq C|x|^{-n-|\beta|-2\alpha k}$$

instead of (3.5) in [3]. Then by the homogeneity property (8) of $W^{(\alpha)}$, when $t \leq |x|^{2\alpha}$,

$$\begin{aligned} |\partial_x^\beta \partial_t^k W^{(\alpha)}(x, t)| &= t^{-((n+|\beta|)/(2\alpha)+k)} \bigl|(\partial_x^\beta \partial_t^k W^{(\alpha)})(t^{-1/(2\alpha)} x, 1)\bigr| \\ &\leq C t^{-((n+|\beta|)/(2\alpha)+k)} t^{((n+|\beta|)/(2\alpha)+k)} |x|^{-n-|\beta|-2\alpha k} \end{aligned}$$

$$\leq C\big(t+|x|^{2\alpha}\big)^{-((n+|\beta|)/(2\alpha)+k)}$$

and when $|x|^{2\alpha}\leq t$,

$$\begin{aligned}|\partial_x^\beta\partial_t^k W^{(\alpha)}(x,\,t)| &= t^{-((n+|\beta|)/(2\alpha)+k)}|(\partial_x^\beta\partial_t^k W^{(\alpha)})(t^{-1/(2\alpha)}x,\,1)|\\ &\leq Ct^{-((n+|\beta|)/(2\alpha)+k)}\\ &\leq C\big(t+|x|^{2\alpha}\big)^{-((n+|\beta|)/(2\alpha)+k)},\end{aligned}$$

which give the lemma. □

We recall some properties of a modified α-parabolic Bergman kernel R_α^m, which is given by

$$R_\alpha^m(x,\,t;y,\,s)=\frac{(-2)^{m+1}}{m!}s^m\partial_t^{m+1}W^{(\alpha)}(x-y,\,t+s).$$

This kernel has the reproducing property, i.e., for $m\geq 0,\ p\geq 1$ and for every $u\in\boldsymbol{b}_\alpha^p$,

$$R_\alpha^m u := \int R_\alpha^m(\,\cdot\,,\,Y)u(Y)dV(Y)=u \tag{9}$$

(see [3] for $n\geq 2$ and [4] for $n=1$). Lemma 1 gives the following estimate for R_α^m.

Lemma 2 *For an integer $m\geq 0$, there exists a constant $C>0$ such that*

$$|R_\alpha^m(x,\,t;y,\,s)|\leq Cs^m\big(t+s+|x-y|^{2\alpha}\big)^{-(n/(2\alpha)+1)-m}.$$

Later, we also use the following estimates.

Lemma 3 *Let $0<p\leq\infty$. If $m>\big(n/(2\alpha)+1\big)\big(1/p-1\big)$, then we have*

$$\|R_\alpha^m(\,\cdot\,,\,Y)\|_{L^p(V)}=Cs^{(n/(2\alpha)+1)(1/p-1)}$$

with some constant $C>0$ independent of $Y=(y,\,s)\in\boldsymbol{R}_+^{n+1}$.

Proof. This follows from [3, Lemma 3.2], where the condition $p\geq 1$ is assumed but it is not necessary. □

Lemma 4 ([5, Corollary 1]) *Let $m\geq 0$ be an integer. Then there exist constants $C>0$ and $\rho>0$ such that*

$$|R_\alpha^m(X,\,Y)|\geq Cs^{-(n/(2\alpha)+1)}=CV\big(Q^{(\alpha)}(Y_\rho)\big)^{-1}$$

for all $Y=(y,\,s)\in\boldsymbol{R}_+^{n+1}$ and all $X\in Q^{(\alpha)}(Y_\rho)$, where $Y_\rho:=(y,\,\rho s)$.

Lemma 5 *Let $\gamma,\, \eta \in \boldsymbol{R}$. If $0 < 1+\gamma < -\eta - n/(2\alpha)$, then*

$$\int t^{\gamma}(t+s+|x-y|^{2\alpha})^{\eta}dV(x,\,t) = Cs^{\gamma+\eta+n/(2\alpha)+1}$$

with some constant $C > 0$ independent of $(y,\, s) \in \boldsymbol{R}^{n+1}_{+}$.

3. A characterization of Carleson measures

Carleson measures are characterized by some norm inequalities.

Proposition 1 *Let μ be a positive Borel measure on $\boldsymbol{R}^{n+1}_{+}$ and let $0 < p,\ q < \infty$. For an nonnegative integer m with $m > \bigl(n/(2\alpha)+1\bigr)\bigl(1/p-1\bigr)$, there exists $C > 0$ such that*

$$\Bigl(\int |R^m_\alpha(X,\,Y)|^q d\mu(X)\Bigr)^{1/q} \le C\Bigl(\int |R^m_\alpha(X,\,Y)|^p dV(X)\Bigr)^{1/p} \tag{10}$$

for all $Y \in \boldsymbol{R}^{n+1}_{+}$. Then μ is a q/p-Carleson measure.

Proof. For every $Y = (y,\, s) \in \boldsymbol{R}^{n+1}_{+}$, by Lemmas 3 and 4, we have

$$\begin{aligned} s^{(n/(2\alpha)+1)(1/p-1)q} &= C\Bigl(\int |R^m_\alpha(X,\,Y)|^p dV(X)\Bigr)^{q/p} \\ &\ge C\int |R^m_\alpha(X,\,Y)|^q d\mu(X) \\ &\ge C\int_{Q^{(\alpha)}(Y_\rho)} |R^m_\alpha(X,\,Y)|^q d\mu(X) \\ &\ge C\int_{Q^{(\alpha)}(Y_\rho)} s^{-(n/(2\alpha)+1)q} d\mu(X) \\ &= Cs^{-(n/(2\alpha)+1)q}\mu\bigl(Q^{(\alpha)}(Y_\rho)\bigr). \end{aligned}$$

Hence

$$\mu\bigl(Q^{(\alpha)}(Y)\bigr) \le C\Bigl(\frac{s}{\rho}\Bigr)^{(n/(2\alpha)+1)(q/p)},$$

which implies that μ is a q/p-Carleson measure. □

As for the converse assertion, we see the following proposition.

Proposition 2 *Let $0 < p,\ q < \infty$ with $q/p > n/(n+2\alpha)$ and let m be a nonnegative integer such that $m > \big(n/(2\alpha)+1\big)\big(1/p-1\big)$. Assume that μ is a q/p-Carleson measure on $\boldsymbol{R}^{n+1}_+$, i.e.,*

$$\mu\big(Q^{(\alpha)}(X)\big) \le Ct^{(n/(2\alpha)+1)(q/p-1)}V\big(Q^{(\alpha)}(X)\big) \tag{11}$$

for all $X=(x,\,t)\in\boldsymbol{R}^{n+1}_+$ with some constant $C>0$. Then there exists another constant $C>0$ such that

$$\left(\int |R^m_\alpha(X,\,Y)|^q d\mu(X)\right)^{1/q} \le C\left(\int |R^m_\alpha(X,\,Y)|^p dV(X)\right)^{1/p}$$

for all $Y\in\boldsymbol{R}^{n+1}_+$.

Proof. We use a Whitney type decomposition. For $Y=(y,\,s)$ $=(y_1,\,\ldots,\,y_n,\,s)\in\boldsymbol{R}^{n+1}_+$ and $\nu=(\beta,\,k)=(\beta_1,\,\ldots,\,\beta_n,\,k)\in\boldsymbol{Z}^{n+1}$, we put

$$t_\nu := 2^k s, \quad x_\nu := y+(2^k s)^{1/(2\alpha)}\Big(\frac{2\beta_1+1}{2},\,\ldots,\,\frac{2\beta_n+1}{2}\Big)$$

and

$$\begin{aligned} Q_\nu &:= Q^{(\alpha)}(x_\nu,\,t_\nu) \\ &= \{(x,\,t);\beta_j(2^k s)^{1/(2\alpha)} \le x_j-y_j \le (\beta_j+1)(2^k s)^{1/(2\alpha)} \\ &\qquad\qquad (j=1,\,\ldots,\,n),\ 2^k s\le t\le 2^{k+1}s\}. \end{aligned}$$

Then there exists a constant $C>1$, independent of $Y=(y,\,s)$ and ν, such that

$$\begin{aligned} C^{-1}(t+s+|x-y|^{2\alpha}) &\le (t_\nu+s+|x_\nu-y|^{2\alpha}) \\ &\le C(t+s+|x-y|^{2\alpha}) \end{aligned}$$

for every $(x,\,t)\in Q_\nu$. Hence by Lemmas 2, 3, and 5, and (11), we have

$$\begin{aligned} &\int |R^m_\alpha(X,\,Y)|^q d\mu(X) \\ &\le Cs^{qm}\sum_{\nu\in\boldsymbol{Z}^{n+1}}\int_{Q_\nu}(t+s+|x-y|^{2\alpha})^{-(n/(2\alpha)+1+m)q}d\mu(x,t) \\ &\le Cs^{qm}\sum_{\nu\in\boldsymbol{Z}^{n+1}}(t_\nu+s+|x_\nu-y|^{2\alpha})^{-(n/(2\alpha)+1+m)q}\mu(Q_\nu) \\ &\le Cs^{qm}\sum_{\nu\in\boldsymbol{Z}^{n+1}}(t_\nu+s+|x_\nu-y|^{2\alpha})^{-(n/(2\alpha)+1+m)q} \end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad \times t_\nu^{(n/(2\alpha)+1)(q/p-1)} V(Q_\nu) \\
&\le C s^{qm} \int t^{(n/(2\alpha)+1)(q/p-1)} \\
&\qquad\qquad\qquad \times (t+s+|x-y|^{2\alpha})^{-(n/(2\alpha)+1+m)q} dV(x,\, t) \\
&= C s^{(n/(2\alpha)+1)(1/p-1)q} \\
&= C\Big(\int |R_\alpha^m(X,\, Y)|^p dV(X) \Big)^{q/p}.
\end{aligned}
$$

□

4. Proof of Theorem 1

In this section, we complete the proof of Theorem 1.

Proof of Theorem 1. Let $1 \le p \le q < \infty$ and take an integer m with $m > \big(n/(2\alpha)+1\big)\big(1/p-1\big)$. Since $R_\alpha^m(\,\cdot\,,\, Y) \in \boldsymbol{b}_\alpha^p$, (5) gives (10), and hence the "if" part follows from Proposition 1.

To prove the "only if" part, we denote by p' the exponent conjugate to p. Then, by the Hölder inequality and [3, Lemma 6.2],

$$
\begin{aligned}
|u(X)| &= \Big| \int s^{-1/(p'q)} s^{1/(p'q)} u(Y) R_\alpha^m(X,\, Y) dV(Y) \Big| \\
&\le \Big(\int s^{-1/q} |R_\alpha^m(X,\, Y)| dV(Y) \Big)^{1/p'} \\
&\quad \times \Big(\int s^{p/(p'q)} |u(Y)|^p |R_\alpha^m(X,\, Y)| dV(Y) \Big)^{1/p} \\
&= C t^{-1/(p'q)} \Big(\int s^{p/(p'q)} |u(Y)|^p |R_\alpha^m(X,\, Y)| dV(Y) \Big)^{1/p}.
\end{aligned}
$$

Here we use the convention "$a^{1/\infty} = 1$". Since $q/p \ge 1$, the Minkowski inequality yields

$$
\begin{aligned}
&\Big(\int |u(X)|^q d\mu(X) \Big)^{p/q} \\
&\le C\Big[\int \Big(\int s^{p/(p'q)} |u(Y)|^p |R_\alpha^m(X,\, Y)| dV(Y) \Big)^{q/p} t^{-1/p'} d\mu(X) \Big]^{p/q}
\end{aligned}
$$

$$\leq C \int s^{p/(p'q)} |u(Y)|^p \left[\int |R_\alpha^m(X, Y)|^{q/p} t^{-1/p'} d\mu(X) \right]^{p/q} dV(Y).$$

As in the proof of Proposition 2, we also obtain

$$\begin{aligned}
&\int |R_\alpha^m(X, Y)|^{q/p} t^{-1/p'} d\mu(X) \\
&\leq C s^{qm/p} \sum_{\nu \in \boldsymbol{Z}^{n+1}} \int_{Q_\nu} t^{-1/p'} \\
&\qquad\qquad \times (t + s + |x - y|^{2\alpha})^{-(n/(2\alpha)+1+m)(q/p)} d\mu(x, t) \\
&\leq C s^{qm/p} \sum_{\nu \in \boldsymbol{Z}^{n+1}} t_\nu^{-1/p'} \\
&\qquad\qquad \times (t_\nu + s + |x_\nu - y|^{2\alpha})^{-(n/(2\alpha)+1+m)(q/p)} \mu(Q_\nu) \\
&\leq C s^{qm/p} \sum_{\nu \in \boldsymbol{Z}^{n+1}} t^{-1/p'} (t_\nu + s + |x_\nu - y|^{2\alpha})^{-(n/(2\alpha)+1+m)(q/p)} \\
&\qquad\qquad \times t_\nu^{(n/(2\alpha)+1)(q/p-1)} V(Q_\nu) \\
&\leq C s^{qm/p} \int t^{-1/p' + (n/(2\alpha)+1)(q/p-1)} \\
&\qquad\qquad \times (t + s + |x - y|^{2\alpha})^{-(n/(2\alpha)+1+m)(q/p)} dV(x, t) \\
&= C s^{-1/p'},
\end{aligned}$$

where the last equality follows from Lemma 5. Hence we have

$$\begin{aligned}
\left(\int |u(X)|^q d\mu(X) \right)^{p/q} &\leq C \int s^{p/(p'q)} |u(Y)|^p s^{-p/(p'q)} dV(Y) \\
&= \|u\|^p_{L^p(V)}.
\end{aligned}$$

□

We note two remarks, which follow from the proof of Theorem 1.

Remark 4 Assume that μ is a q/p-Carleson measure. If $1 < p \leq q < \infty$, then

$$\|v\|_{L^q(\mu)} \leq C \|f\|_{L^p(V)},$$

where we put $v(X) := \int |f(Y) R_\alpha(X, Y)| dV(Y)$ for $f \in L^p(V)$.

Remark 5 The norm of the inclusion $\iota_\mu\colon \boldsymbol{b}^p_\alpha \to L^q(\mu)$ is estimated by a weighted supremum norm of the averaging function $\hat{\mu}_\alpha$, i.e., there exists a constant $C \geq 1$ such that for every $\mu \geq 0$

$$\frac{1}{C}\|\hat{\mu}_\alpha\|_\tau \leq \|\iota_\mu\|^q_{p,q} \leq C\|\hat{\mu}_\alpha\|_\tau,$$

where $\tau = q/p$ and

$$\|\iota_\mu\|_{p,q} := \sup_{u\in \boldsymbol{b}^p_\alpha} \frac{\|u\|_{L^q(\mu)}}{\|u\|_{L^p(V)}} \quad \text{and}$$
$$\|\hat{\mu}_\alpha\|_\tau := \sup_{X=(x,\,t)\in \boldsymbol{R}^{n+1}_+} \hat{\mu}_\alpha(X)t^{(n/(2\alpha)+1)(1-\tau)}.$$

5. An estimate of Toeplitz operators

In this section we consider the relation between Carleson measures and bounded Toeplitz operators. We begin with the following proposition.

Proposition 3 *Let $0 < p < \infty,\ 1 < q \leq \infty$ and let μ be a positive Borel measure on $\boldsymbol{R}^{n+1}_+$. Put $\tau = 1+1/p-1/q$. If μ is a τ-Carleson measure with respect to $L^{(\alpha)}$, then for every nonnegative integer $m > \big(n/(2\alpha)+1\big)\big(1/p-1\big)$, there exists a constant $C > 0$ such that the following assertions hold.*

(a) *If $0 < p < 1+2\alpha/n$ and $1 < q \leq \infty$, then $\|T_\mu R^m_\alpha(\,\cdot\,, Y)\|_{L^q(V)} \leq C\|R^m_\alpha(\,\cdot\,, Y)\|_{L^p(V)}$ for every $Y \in \boldsymbol{R}^{n+1}_+$.*

(b) *If $1 \leq p < \infty$, $1 < q < \infty$ and $p \leq q$, then for every $u \in \boldsymbol{b}^p_\alpha$ and every $X \in \boldsymbol{R}^{n+1}_+$, $\int\big|R_\alpha(X, Y)u(Y)\big|d\mu(Y) < \infty$ and $\|T_\mu u\|_{L^q(V)} \leq C\|u\|_{L^p(V)}$. In particular, $\|T_\mu R^m_\alpha(\,\cdot\,, Y)\|_{L^q(V)} \leq C\|R^m_\alpha(\,\cdot\,, Y)\|_{L^p(V)}$ for every $Y \in \boldsymbol{R}^{n+1}_+$.*

(c) *If $1 \leq p < \infty$ and $q = \infty$, then for every $u \in \mathcal{E}_m$ and every $X \in \boldsymbol{R}^{n+1}_+$, $\int\big|R_\alpha(X, Y)u(Y)\big|d\mu(Y) < \infty$ and $\|T_\mu u\|_{\mathcal{B}_\alpha/\boldsymbol{R}} \leq C\|u\|_{L^p(V)}$. In particular, $\|T_\mu R^m_\alpha(\,\cdot\,, Y)\|_{\mathcal{B}_\alpha/\boldsymbol{R}} \leq C\|R^m_\alpha(\,\cdot\,, Y)\|_{L^p(V)}$ for every $Y \in \boldsymbol{R}^{n+1}_+$.*

Proof. We write $X = (x,\, t),\ Y = (y,\, s)$ and $Z = (z,\, r)$. By assumption, there exists a constant $C > 0$ such that for all $(x,\, t) \in \boldsymbol{R}^{n+1}_+$,

$$\hat{\mu}_\alpha(x,\, t) \leq Ct^{(n/(2\alpha)+1)(1/p-1/q)}.$$

Case (a): The assertion follows from a direct calculation. In fact, in the similar manner as in the proof of Proposition 2, by the Minkowski inequality and Lemmas 3 and 5, we have

$$\begin{aligned}
&\|T_\mu R_\alpha^m(\,\cdot\,, Y)\|_{L^q(V)} \\
&\le \left\| \int \bigl|R_\alpha(\,\cdot\,, Z) R_\alpha^m(Z, Y)\bigr| d\mu(Z) \right\|_{L^q(V)} \qquad (12)\\
&\le \int \|R_\alpha(\,\cdot\,, Z)\|_{L^q(V)} \bigl|R_\alpha^m(Z, Y)\bigr| d\mu(Z) \\
&= C \int r^{(n/(2\alpha)+1)(1/q-1)} |R_\alpha^m(Z, Y)| d\mu(Z) \\
&\le C \sum_{\nu \in \boldsymbol{Z}^{n+1}} \int_{Q_\nu} r^{(n/(2\alpha)+1)(1/q-1)} s^m \\
&\qquad\qquad \times (s + r + |z - y|^{2\alpha})^{-(n/(2\alpha)+1)-m} d\mu(Z) \\
&\le C \sum_{\nu \in \boldsymbol{Z}^{n+1}} r_\nu^{(n/(2\alpha)+1)(1/q-1)} s^m \\
&\qquad\qquad \times (s + r_\nu + |z_\nu - y|^{2\alpha})^{-(n/(2\alpha)+1)-m} \mu(Q_\nu^{(\alpha)}) \\
&\le C \sum_{\nu \in \boldsymbol{Z}^{n+1}} r_\nu^{(n/(2\alpha)+1)(1/q-1)} s^m (s + r_\nu + |z_\nu - y|^{2\alpha})^{-(n/(2\alpha)+1)-m} \\
&\qquad\qquad \times r_\nu^{(n/(2\alpha)+1)(1/p-1/q)} V(Q_\nu^{(\alpha)}) \\
&\le C s^m \int r^{(n/(2\alpha)+1)(1/p-1)} (s + r + |y - z|^{2\alpha})^{-(n/(2\alpha)+1)-m} dV(Z) \\
&= C s^{(n/(2\alpha)+1)(1/p-1)} \\
&= C \|R_\alpha^m(\,\cdot\,, Y)\|_{L^p(V)}.
\end{aligned}$$

Case (b): Denote by q' the exponent conjugate to q and take $u \in \boldsymbol{b}_\alpha^p$, $u_1 \in \boldsymbol{b}_\alpha^{q'}$ arbitrarily. Then

$$\frac{1}{p\tau} + \frac{1}{q'\tau} = 1.$$

Since $\tau = (p\tau)/p$ and $\tau = (q'\tau)/q'$, Theorem 1 and Remark 5 give that

$$\|u\|_{L^{p\tau}(\mu)} \le C\|u\|_{L^p(V)} \quad \text{and} \quad \|v\|_{L^{q'\tau}(\mu)} \le C\|u_1\|_{L^{q'}(V)}, \qquad (13)$$

where

$$v := \int \bigl|u_1(X) R_\alpha(X, \cdot)\bigr| dV(X).$$

These inequalities assure the following integrability:

$$\begin{aligned}\iint \big|u_1(X)R_\alpha(X, W)u(W)\big|dV(X)d\mu(W) &= \int v(W)|u(W)|d\mu(W)\\ &\le \|v\|_{L^{q'\tau}(\mu)}\|u\|_{L^{p\tau}(\mu)} \le C\|u_1\|_{L^{q'}(V)}\|u\|_{L^p(V)} < \infty.\end{aligned}$$

Therefore the Fubini theorem shows that

$$\int T_\mu u(X)u_1(X)dV(X) = \int u(W)u_1(W)d\mu(W) \tag{14}$$

and hence (13) gives

$$\begin{aligned}\left|\int T_\mu u \cdot u_1 dV\right| = \left|\int u\, u_1 d\mu\right| &\le \|u\|_{L^{p\tau}(\mu)}\|u_1\|_{L^{q'\tau}(\mu)}\\ &\le C\|u\|_{L^p(V)}\|u_1\|_{L^{q'}(V)}.\end{aligned}$$

This implies that there exists $w \in \boldsymbol{b}_\alpha^q$ with $\|w\|_{L^q(V)} \le C\|u\|_{L^p(V)}$ such that

$$\int T_\mu u(X)u_1(X)dV(X) = \int w(X)u_1(X)dV(X)$$

for all $u_1 \in \boldsymbol{b}_\alpha^{q'}$, because of the duality $\big(\boldsymbol{b}_\alpha^{q'}\big)' \simeq \boldsymbol{b}_\alpha^q$. For each $X \in \boldsymbol{R}_+^{n+1}$, taking $u_1 := R_\alpha(\,\cdot\,, X) \in \boldsymbol{b}_\alpha^{q'}$, we have

$$\begin{aligned}T_\mu u(X) &= \int u(W)R_\alpha(X, W)d\mu(W)\\ &= \int T_\mu u \cdot u_1 dV = \int w \cdot u_1 dV = w(X)\end{aligned} \tag{15}$$

by (14) and the reproducing property (9). This shows

$$\|T_\mu u\|_{L^q(V)} \le C\|u\|_{L^p(V)}.$$

Case (c): If $p \ge 1 + (2\alpha/n)$, then we can choose $1 < p_1 < 1 + (2\alpha/n)$ and $1 < q_1 \le \infty$ such that

$$\frac{1}{p_1\tau} + \frac{1}{q_1'\tau} = 1, \tag{16}$$

where q_1' denotes the exponent conjugate to q_1. If $1 \le p < 1 + (2\alpha/n)$, we put $p_1 := p$ and $q_1 := q$. Then (16) also holds. We take $u \in \mathcal{E}_m$ and $v \in \mathcal{E}_1$ arbitrarily. Then for each $Y \in \boldsymbol{R}_+^{n+1}$, by (12) in the proof of Case (a) above, we have

$$\begin{aligned}\|T_\mu R_\alpha^m(\,\cdot\,,Y)\|_{L^{q_1}(V)} &\le \| \int \big|R_\alpha(\,\cdot\,,W)R_\alpha^m(W,Y)\big|d\mu(W)\|_{L^{q_1}(V)}\\ &\le C\|R_\alpha^m(\,\cdot\,,Y)\|_{L^{p_1}(V)}<\infty, \qquad (17)\end{aligned}$$

which implies $T_\mu u\in \boldsymbol{b}_\alpha^{q_1}$ and

$$\begin{aligned}&\iint \big|v(X)R_\alpha(X,W)u(W)\big|d\mu(W)dV(X)\\ &\qquad\le \left\|\int\big|R_\alpha(\,\cdot\,,W)\,u(W)\big|\,d\mu(W)\right\|_{L^{q_1}(V)}\|v\|_{L^{q_1'}(V)}<\infty.\end{aligned}$$

Since $\tau=p\tau/p$, $\tau=\tau/1$ and $(p\tau)^{-1}+\tau^{-1}=1$, the Fubini theorem and Theorem 1 show that

$$\begin{aligned}\left|\int T_\mu u(X)v(X)dV(X)\right| &= \left|\int u(Y)v(Y)d\mu(Y)\right|\\ &\le \|u\|_{L^{p\tau}(\mu)}\|v\|_{L^\tau(\mu)}\le C\|u\|_{L^p(V)}\|v\|_{L^1(V)}.\end{aligned}$$

Since $\mathcal{E}_1$ is dense in $\boldsymbol{b}_\alpha^1$ and $(\boldsymbol{b}_\alpha^1)'\simeq \mathcal{B}_\alpha/\boldsymbol{R}$, there exists $w\in\mathcal{B}_\alpha$ with $\|w\|_{\mathcal{B}_\alpha/\boldsymbol{R}}\le C\|u\|_{L^p(V)}$ such that

$$\int T_\mu u(X)R_\alpha^1(X,Z)dV(X)=\int w(X)R_\alpha^1(X,Z)dV(X) \qquad (18)$$

for all $Z\in\boldsymbol{R}_+^{n+1}$ by [3, Lemma 8.3 and Theorem 8.4], where

$$\|w\|_{\mathcal{B}_\alpha/\boldsymbol{R}}=\sup_{(x,t)\in\boldsymbol{R}_+^{n+1}}\{t^{1/(2\alpha)}|\nabla_x w(x,t)|+t|\partial_t w(x,t)|\}.$$

Then by [3, Theorem 7.9] we have

$$w(X)=w(X_0)-2\int\big(R_\alpha(X,Z)-R_\alpha(X_0,Z)\big)r\partial_r w(z,r)dV(Z),$$

where $X_0=(0,1)$ and $Z=(z,r)$, so that

$$\begin{aligned}\partial_t w(X) &= -2\frac{\partial}{\partial t}\left(\int\big(R_\alpha(X,Z)-R_\alpha(X_0,Z)\big)r\partial_r w(z,r)\,dV(Z)\right)\\ &= -2\int r\partial_r w(z,r)\partial_t R_\alpha(X,Z)dV(Z)\\ &= -\frac{2}{t}\int r\partial_r w(z,r)R_\alpha^1(Z,X)dV(Z)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{t}\int w(Z)R^1_\alpha(Z,\, X)dV(Z)\\
&= \frac{1}{t}\int T_\mu u(Z)R^1_\alpha(Z,\, X)dV(Z)\\
&= \frac{\partial}{\partial t}\int T_\mu u(Z)R_\alpha(X,\, Z)dV(Z)\\
&= \partial_t T_\mu u(X)
\end{aligned}
$$

by [3, Lemma 8.3] and (18). Therefore the $L^{(\alpha)}$-harmonic function

$$\tilde{w}(x,\, t) := w(x,\, t) - T_\mu u(x,\, t)$$

is independent of t. Now remarking that

$$\left|\frac{\partial w}{\partial x_i}(x,\, t)\right| \le \|w\|_{\mathcal{B}_\alpha} t^{-1/(2\alpha)} \quad \text{and}$$

$$\left|\frac{\partial T_\mu u}{\partial x_i}(x,\, t)\right| \le \|T_\mu u\|_{L^{q_1}(V)} t^{-1/(2\alpha)-(n/(2\alpha)+1)(1/q_1)}$$

by the definition of the parabolic Bloch norm (1) and [3, Theorem 5.4], where $1 \le i \le n$, we have

$$\frac{\partial \tilde{w}}{\partial x_i}(x,\, t) = \lim_{t\to+\infty}\left(\frac{\partial w}{\partial x_i}(x,\, t) - \frac{\partial T_\mu u}{\partial x_i}(x,\, t)\right) = 0.$$

Hence $\tilde{w}$ is constant, which means $w = T_\mu u$ in $\mathcal{B}_\alpha/\boldsymbol{R}$, so that $\|T_\mu u\|_{\mathcal{B}_\alpha/\boldsymbol{R}} = \|w\|_{\mathcal{B}_\alpha/\boldsymbol{R}}$. □

To discuss the converse assertion, we use a modified Berezin transformation of a measure $\mu \ge 0$. For an integer $m \ge 0$, we put

$$\tilde{\mu}_{\alpha,m}(Y) := \frac{\int R^m_\alpha(X,\, Y)^2 d\mu(X)}{\int R^m_\alpha(X,\, Y)^2 dV(X)}.$$

Note that $\tilde{\mu}_{\alpha,0} = \tilde{\mu}_\alpha$. The averaging function and modified Berezin transformations are comparable to each other in the following sense.

Lemma 6 *Let $m \ge 0$ be an integer, $-1 < \eta < n/(2\alpha)+1+2m$, and $\mu \ge 0$ be a Borel measure on $\boldsymbol{R}^{n+1}_+$. Then we have the following estimates.*

(i) *$\hat{\mu}_\alpha(y,\, \rho s) \le C\tilde{\mu}_{\alpha,m}(y,\, s)$ on $\boldsymbol{R}^{n+1}_+$ for some constant $C > 0$, where $\rho > 0$ is a constant in* Lemma 4.

(ii) *$\hat{\mu}_\alpha(y,\, s) \le Cs^\eta$ on $\boldsymbol{R}^{n+1}_+$ for some constant $C > 0$ if and only if $\tilde{\mu}_{\alpha,m}(y,\, s) \le Cs^\eta$ on $\boldsymbol{R}^{n+1}_+$ for some constant $C > 0$.*

Proof. To show (i), take $Y = (y, s) \in \boldsymbol{R}^{n+1}_+$ arbitrarily. From Lemmas 3 and 4, it follows that

$$\begin{aligned}\tilde{\mu}_{\alpha,m}(Y) &= Cs^{n/(2\alpha)+1} \int R^m_\alpha(X, Y)^2 d\mu(X) \\ &\geq CV\big(Q^{(\alpha)}(Y_\rho)\big) \int_{Q^{(\alpha)}(Y_\rho)} R^m_\alpha(X, Y)^2 d\mu(X) \\ &\geq CV\big(Q^{(\alpha)}(Y_\rho)\big)^{-1} \mu\big(Q^{(\alpha)}(Y_\rho)\big) \\ &= C\hat{\mu}_\alpha(Y_\rho).\end{aligned}$$

For (ii), the "if" part follows from (i). Conversely, as in the proof of Proposition 2, by the aide of Whitney type decomposition, the first inequality in (ii) gives

$$\int R^m_\alpha(X, Y)^2 d\mu(X) \leq Cs^{\eta-(n/(2\alpha)+1)}$$

and hence the "only if" part follows. □

The main result of this section is the following proposition.

Proposition 4 *Let $0 < p < \infty$, $1 \leq q \leq \infty$ and let μ be a positive Borel measure on $\boldsymbol{R}^{n+1}_+$ satisfying*

$$\int |R^m_\alpha(X, \cdot)| d\mu(X) < \infty, \quad V\text{-a.e.}$$

for some integer $m \geq 1$. Assume further that $m > \big(n/(2\alpha) + 1\big)\big(1/p - 1\big)$ and there exists a constant $C > 0$ such that for every $Y \in \boldsymbol{R}^{n+1}_+$,

$$\begin{cases} \|T_\mu R^m_\alpha(\,\cdot\,, Y)\|_{L^q(V)} \leq C\|R^m_\alpha(\,\cdot\,, Y)\|_{L^p(V)} & \text{when } 1 \leq q < \infty, \\ \|T_\mu R^m_\alpha(\,\cdot\,, Y)\|_{\mathcal{B}_\alpha/\boldsymbol{R}} \leq C\|R^m_\alpha(\,\cdot\,, Y)\|_{L^p(V)} & \text{when } \quad q = \infty. \end{cases}$$

Then there exists a constant $C > 0$ such that for all $Y = (y, s) \in \boldsymbol{R}^{n+1}_+$,

$$\tilde{\mu}_{\alpha,m}(Y) \leq Cs^{(n/(2\alpha)+1)(1/p-1/q)}.$$

In particular μ is a τ-Carleson measure with $\tau = 1 + 1/p - 1/q$.

Proof. Let $Y = (y, s) \in \boldsymbol{R}^{n+1}_+$ be fixed such that $R^m_\alpha(\,\cdot\,, Y) \in L^1(\mu)$. Write $u := R^m_\alpha(\,\cdot\,, Y)$ and $u_\delta(x, t) := u(x, t + \delta)$ for $\delta > 0$. Then we remark that

u is $L^{(\alpha)}$-harmonic and

$$u \in L^p(V) \cap L^1(V) \cap L^\infty(V).$$

Since, writing $Z = (z, r)$, by Lemma 3 we have

$$\begin{aligned}
&\int \Big(\int \big| u_\delta(X) R_\alpha\big(X, (z, r+\delta)\big) \big| dV(X) \Big) |u(Z)| d\mu(Z) \\
&\le \int \|u_\delta\|_{L^2(V)} \big\| R_\alpha\big(\,\cdot\,, (z, r+\delta)\big) \big\|_{L^2(V)} |u(Z)| d\mu(Z) \\
&\le C(s\delta)^{(-1/2)(n/(2\alpha)+1)} \int |u| d\mu < \infty,
\end{aligned}$$

the Fubini theorem implies

$$\begin{aligned}
&\int \Big(\int u_\delta(X) R_\alpha\big(X, (z, r+\delta)\big) dV(X) \Big) u(Z) d\mu(Z) \\
&\qquad\qquad = \int u_\delta(X) \Big(\int R_\alpha\big((x, t+\delta), Z\big) u(Z) d\mu(Z) \Big) dV(X).
\end{aligned}$$

Hence

$$\int u_\delta(z, r+\delta) u(Z) d\mu(Z) = \int u(x, t+\delta) T_\mu u(x, t+\delta) dV(x, t),$$

the right hand side of which converges to $\int u T_\mu u dV$ as δ tends to 0. In fact, when $1 \le q < \infty$,

$$\begin{aligned}
\int |u T_\mu u| dV &\le \|u\|_{L^{q'}(V)} \|T_\mu u\|_{L^q(V)} \\
&\le C \|u\|_{L^{q'}(V)} \|u\|_{L^p(V)} < \infty. \qquad (19)
\end{aligned}$$

For $q = \infty$, since $m \ge 1$, we also see $\int |u T_\mu u| dV < \infty$ by [3, Proposition 7.2] and Lemma 2. Hence [3, Lemma 8.3] shows

$$\Big| \int u T_\mu u dV \Big| \le 2\|u\|_{L^1(V)} \|T_\mu u\|_{\mathcal{B}_\alpha / \boldsymbol{R}} \le C \|u\|_{L^1(V)} \|u\|_{L^p(V)}. \qquad (20)$$

Moreover, since

$$|u_\delta(z, r+\delta) u(Z)| \le C |u(Z)|,$$

which is in $L^1(\mu)$, the Lebesgue dominated convergence theorem gives

$$\int uT_\mu u dV = \lim_{\delta \to 0} \int u_\delta(z, r+\delta)u(Z)d\mu(Z) = \int u^2 d\mu.$$

Therefore by (19), (20) and Lemma reflem3

$$\int R_\alpha^m(Z, Y)^2 d\mu(Z) = \int u^2 d\mu = \int uT_\mu u dV$$
$$\leq Cs^{(n/(2\alpha)+1)(1/p-1/q-1)}.$$

Since $\int R_\alpha^m(Z, Y)^2 dV(Z) = Cs^{-(n/(2\alpha)+1)}$, we have

$$\tilde{\mu}_{\alpha,m}(Y) \leq Cs^{(n/(2\alpha)+1)(1/p-1/q)}$$

for V-a.e. Y. By the Fatou lemma, this inequality holds everywhere. Lemma 6 (i) shows that $\hat{\mu}_\alpha$ satisfies the same inequality, so that μ is a τ-Carleson measure. □

6. Proof of Theorem 2

In this section, we complete the proof of Theorem 2.

Proof of Theroem 2. Proposition 4 shows (I) implies (II). By Lemma 6, (II) and (III) are equivalent. The implication (II) ⇒ (I) follows from (b) or (c) in Proposition 3 according as $1 < q < \infty$ or $q = \infty$. □

Finally, we give two remarks concerning Theorem 2.

Remark 6 In (a) of (I) where $1 < q < \infty$, as a result, for every $u \in \boldsymbol{b}_\alpha^p$, $T_\mu u(X)$ can be well-defined by the integral (2) for all $X \in \boldsymbol{R}_+^{n+1}$, and hence $T_\mu u \in \boldsymbol{b}_\alpha^q$. This follows from the proof of Case (b) in Proposition 3.

Remark 7 The constants C_1, C_2 and C_3 in Theorem 2 are comparable to each other. In particular, the operator norm of the Toeplitz operator T_μ is controlled by a weighted supremum norm of $\tilde{\mu}_\alpha$, i.e., there exists a constant $C > 0$ independent of μ such that

$$\frac{1}{C}\|\tilde{\mu}_\alpha\|_\tau \leq \|T_\mu\|_{p,q} \leq C\|\tilde{\mu}_\alpha\|_\tau$$

where $\tau = 1 + 1/p - 1/q$,

$$\|T_\mu\|_{p,q} := \sup_{u \in \boldsymbol{b}^p_\alpha} \frac{\|T_\mu u\|_{L^q(V)}}{\|u\|_{L^p(V)}} \quad (1 < q < \infty) \quad \text{and}$$

$$\|T_\mu\|_{p,\infty} := \sup_{u \in \boldsymbol{b}^p_\alpha} \frac{\|T_\mu u\|_{\mathcal{B}_\alpha/\boldsymbol{R}}}{\|u\|_{L^p(V)}}$$

and where

$$\|\tilde{\mu}_\alpha\|_\tau := \sup_{X=(x,t) \in \boldsymbol{R}^{n+1}_+} \tilde{\mu}_\alpha(X) t^{(n/(2\alpha)+1)(1-\tau)}.$$

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