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Semi-implicit schemes with multilevel wavelet-like incremental unknowns for solving reaction diffusion equation

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Abstract. Our aim in this paper is to present two types of semi-implicit schemes based on multilevel wavelet-like incremental unknowns (WIU) for solving a one-dimensional reaction-diffusion equation with a polynomial growth nonlinearity. The stability of schemes is proved which also shows the advantage over explicit and implicit schemes in the same conceptual framework of multilevel WIU. Numerical examples are provided to test the efficiency of the new schemes.

Key words: Wavelet-like incremental unknowns, reaction-diffusion equation, nonlinear Galerkin method, semi-implicit schemes.

1. Introduction

A method with incremental unknowns has been developed as a means to approximate inertial manifolds when finite differences are used. They play evidently an important role in the study of the long time behavior of the solutions of partial differential equations and in fact they produce a new and different efficient concept in finite differences which are fundamental and useful in the field of numerical solution of partial differential equations (see e.g. [5,10,12]).

Much effort about the method with incremental unknowns has been devoted in the past to the approximation of the linear elliptic equations and also some nonlinear differential equations or even dissipative evolution equations, among them are Navier-Stokes equations in dimension two, Kuramoto-Sivashinsky equations and convection-diffusion equations etc. (See also [2,6,7,9,13] and the references therein.)

Wavelet-like incremental unknowns deserve special stress because they enjoy the L^2 orthogonality property between different levels of unknowns.

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This makes the method with multilevel wavelet-like incremental unknowns particularly appropriate for the approximation of evolution equation (see e.g. [3,4]). The purpose of this paper is to establish two semi-implicit schemes with multilevel WIU methods for some one-dimensional reaction diffusion equation, especially for an equation with a polynomial growth nonlinearity of arbitrary order.

The article is organized as follows. In Section 2 we present the reaction-diffusion equation and its finite difference discretization. Then in Section 3 we recall the definition of the WIU and the multilevel discretization in space. Two types of new semi-implicit schemes based on multilevel WIU are established and some numerical results are shown in Section 4. Finally in Section 5 we develop the stability study of the schemes.

2. Equation and discretization

In general case, we denote by Ω an open bounded set of $\mathbf{R}^n$ with boundary $\Gamma = \partial\Omega$. Consider the following initial-boundary value problem involving a scalar function $u = u(x, t)$; u satisfies

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \triangle u + g(x,\, u) = 0, & \text{in } \Omega, \\ u(x,\, 0) = u_0(x), & \text{in } \Omega. \end{cases} \tag{2.1}$$

together with one of the following boundary conditions.

(i) Dirichlet type boundary condition

$$u|_\Gamma = 0. \tag{2.2a}$$

(ii) Neumann type boundary condition

$$\left.\frac{\partial u}{\partial \mathbf{n}}\right|_\Gamma = 0. \tag{2.2b}$$

(iii) Periodic type boundary condition

$$\Omega = (0,\, 1)^n, \quad u \text{ is } \Omega\text{-periodic}. \tag{2.2c}$$

Here the function $g\colon \Omega \times \mathbf{R} \to \mathbf{R}$ is measurable in x and of class C^1 in s satisfying

$$\begin{cases} \textit{There exists } q > 2 \textit{ and } \gamma_i > 0 \ (i = 0,\, 1,\, 2,\, 3) \textit{ such that} \\ \gamma_1 |s|^q - \gamma_0 \le g(x,\, s)s \le \gamma_2 |s|^q + \gamma_3,\ \forall s \in \mathbf{R}^+,\ \textit{a.e. } x \in \Omega. \end{cases} \tag{2.3}$$

and

$$\begin{cases} \textit{There exists } \gamma_4 > 0 \textit{ such that} \\ g'_s(x,\, s) \geq -\gamma_4, \quad \forall s \in \mathbf{R}^+,\ \textit{a.e. } x \in \Omega. \end{cases} \tag{2.4}$$

Remark 1 The Chafee-Infante equation which reads

$$\frac{\partial u}{\partial t} - \triangle u + \alpha u^3 - \beta u = 0, \quad (\alpha,\, \beta > 0) \tag{2.5}$$

is a simple example of (2.1).

Our attention should be paid to a special one-dimensional example which is a reaction-diffusion equation with initial-boundary value conditions

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\triangle u + g(u) = 0, & \text{in } \Omega = (0,\, 1), \\ u(x,\, 0) = u_0(x), & \text{in } \Omega, \\ u|_\Gamma = 0. \end{cases} \tag{2.6}$$

where $g(s) = \sum_{j=0}^{2q-1} b_j s^j$, $b_{2q-1} > 0$. (See, e.g., [4,8].)

Under suitable condition, we know there exist two constants c_1, $c_2 > 0$ such that

$$g(s)s \geq \frac{1}{2} b_{2q-1} s^{2q} - c_1, \tag{2.7}$$

$$g(s)^2 \leq 2b_{2q-1} s^{4q-2} + c_2. \tag{2.8}$$

If the spatial variable x of the equation is discretized by finite difference with mesh size $h_d = 1/(2^d N+1)$, where $N \in \mathbf{N}$, $\mathbf{N}$ denoting positive integer, we have

$$\frac{\partial U_d}{\partial t} + \nu A_d U_d + g(U_d) = 0, \tag{2.9}$$

where U_d is the vector of approximate values of u at the grid points, $U_d \in \mathbf{R}^{2^d N}$, and A_d is a regular matrix of order $2^d N$. Denoting by u_i^d the discretized step function with nodal values: $u_i^d = U_d(i) \approx u(ih_d)$, $i = 1, 2, \ldots, 2^d N$, we have for the convection term with central difference scheme the equality

$$A_d u_i^d = \frac{-1}{h_d^2}(u_{i+1}^d - 2u_i^d + u_{i-1}^d).$$

Ordering u_i^d, $i = 1, 2, \ldots, 2^d N$ in its natural way, we see that A_d is tridiagonal.

3. Multilevel wavelet-like incremental unknowns

Recalling [3,4], we introduce the wavelet-like incremental unknowns into equation (2.9).

We separate evenly the unknowns into two parts according to the grid (u_{2i}^d corresponding to coarse grid, or u_{2i-1}^d to complementary grid), and the first separation of new variables is obtained by defining

$$\begin{cases} y_{2i}^d = \dfrac{u_{2i-1}^d + u_{2i}^d}{2}, \\ z_{2i-1}^d = \dfrac{u_{2i-1}^d - u_{2i}^d}{2}, \end{cases} \qquad i = 1, 2, \ldots, 2^{d-1}N.$$

Inversely, we have

$$\begin{cases} u_{2i}^d = y_{2i}^d - z_{2i-1}^d, \\ u_{2i-1}^d = z_{2i-1}^d + y_{2i}^d, \end{cases} \qquad i = 1, 2, \ldots, 2^{d-1}N. \tag{3.1}$$

We reorder U_d into $\widetilde{U}_d$ by letting $\widetilde{U}_d = (u_2^d, u_4^d, \ldots, u_{2^dN}^d, u_1^d, u_3^d, \ldots, u_{2^dN-1}^d)^T$ and denote

$$\overline{U}_d = \begin{pmatrix} Y_d \\ Z_d \end{pmatrix} = (y_2^d, y_4^d, \ldots, y_{2^dN}^d, z_1^d, z_3^d, \ldots, z_{2^dN-1}^d)^T.$$

We see that $U_d = P_d\widetilde{U}_d$, $\widetilde{U}_d = S_d\overline{U}_d$. Here we know that P_d is a permutation matrix of order 2^dN, and that S_d is a transfer matrix of order 2^dN with the following form

$$S_d = \begin{pmatrix} I_{d-1} & -I_{d-1} \\ I_{d-1} & I_{d-1} \end{pmatrix}.$$

where I_{d-1} is the identity matrix of order $2^{d-1}N$. We can easily see that $S_d^{-1} = (1/2)S^T$. Substituting $U_d = P_dS_d\overline{U}_d$ into the finite difference equation (2.9) and multiplying the equation by $(P_dS_d)^T$, we have

$$\frac{\partial (P_dS_d)^T P_dS_d\overline{U}_d}{\partial t} + \nu (P_dS_d)^T A_d P_dS_d\overline{U}_d + (P_dS_d)^T g(P_dS_d\overline{U}_d) = 0.$$

Thanks to the observations that $P_d^TP_d = I_d$, $S_d^TS_d = 2I_d$, P_d^T and g

commute, we obtain

$$2\frac{\partial \overline{U}_d}{\partial t} + \nu (P_d S_d)^T A_d P_d S_d \overline{U}_d + (P_d S_d)^T g(P_d S_d \overline{U}_d) = 0, \tag{3.2}$$

which expresses the finite difference scheme obtained when 2-level wavelet-like incremental unknowns are used.

The next level of wavelet-like incremental unknowns on Y_d can be introduced by repeating the same procedure. We now separate Y_d into two parts and denote that

$$\overline{Y}_d = \begin{pmatrix} Y_{d-1} \\ Z_{d-1} \end{pmatrix} = (y_4^{d-1}, y_8^{d-1}, \ldots, y_{2^d N}^{d-1}, z_2^{d-1}, z_4^{d-1}, \ldots, z_{2^d N-2}^{d-1})^T.$$

Similar to (3.1) we define

$$\begin{cases} y_{2i}^d = y_{4i}^{d-1} - z_{4i-2}^{d-1}, \\ y_{4i-2}^d = z_{4i-2}^{d-1} + y_{4i}^{d-1}, \end{cases} \quad i = 1, 2, \ldots, 2^{d-2}N. \tag{3.3}$$

Therefore, we obtain the equality $Y_d = P_{d-1} S_{d-1} \overline{Y}_d$, where P_{d-1} is a permutation matrix of order $2^{d-1}N$ and

$$S_{d-1} = \begin{pmatrix} I_{d-2} & -I_{d-2} \\ I_{d-2} & I_{d-2} \end{pmatrix}.$$

As usual, we should set

$$\overline{U}_{d-1} = \begin{pmatrix} Y_{d-1} \\ Z_{d-1}, \\ \frac{1}{\sqrt{2}} Z_d \end{pmatrix}, \quad \widetilde{P}_{d-1} = \begin{pmatrix} P_{d-1} & 0 \\ 0 & I_{d-1} \end{pmatrix},$$

$$\widetilde{S}_{d-1} = \begin{pmatrix} S_{d-1} & 0 \\ 0 & \sqrt{2} I_{d-1} \end{pmatrix},$$

such that $\widetilde{P}_{d-1}^T \widetilde{P}_{d-1} = I_d$, $\widetilde{S}_{d-1}^T \widetilde{S}_{d-1} = 2I_d$ and $\overline{U}_d = \widetilde{P}_{d-1} \widetilde{S}_{d-1} \overline{U}_{d-1}$. Substituting $\overline{U}_d = \widetilde{P}_{d-1} \widetilde{S}_{d-1} \overline{U}_{d-1}$ into (3.2) and multiplying the equation by $(\widetilde{P}_{d-1} \widetilde{S}_{d-1})^T$, we can also obtain

$$2^2 \frac{\partial \overline{U}_{d-1}}{\partial t} + \nu S^T A_d S \overline{U}_{d-1} + S^T g(S \overline{U}_{d-1}) = 0,$$

here $S = P_d S_d \widetilde{P}_{d-1} \widetilde{S}_{d-1}$.

Repeating the same procedure d times, we obtain finally the finite difference scheme when $(d+1)$-level wavelet-like incremental unknowns are used.

$$2^d \frac{\partial \overline{U}_0}{\partial t} + \nu S^T A_d S \overline{U}_0 + S^T g(S\overline{U}_0) = 0, \tag{3.4}$$

where $S = \widetilde{P}_d \widetilde{S}_d \cdots \widetilde{P}_1 \widetilde{S}_1$. P_l are permutation matrices with similar structure but different order, S_l are transfer matrixes with similar structure but different order, and

$$\overline{U}_0 = \begin{pmatrix} Y_0 \\ Z \end{pmatrix} = \begin{pmatrix} Y_0 \\ Z_0 \\ \frac{1}{\sqrt{2}} Z_1 \\ \vdots \\ \frac{1}{\sqrt{2}^d} Z_d \end{pmatrix}, \quad \widetilde{P}_l = \begin{pmatrix} P_l & 0 \\ 0 & I_k \end{pmatrix},$$

$$\widetilde{S}_l = \begin{pmatrix} S_l & 0 \\ 0 & \sqrt{2} I_k \end{pmatrix} = \begin{pmatrix} I_{l-1} & -I_{l-1} & 0 \\ I_{l-1} & I_{l-1} & 0 \\ 0 & 0 & \sqrt{2} I_k \end{pmatrix},$$

where $k = (2^d - 2^l)N$, $l = d, d-1, \ldots, 1$.

4. Schemes and numerical computation

We now propose some schemes based on the utilization of the incremental unknowns introduced above.

The equation (3.4) with $d = 1$ has the form

$$2\frac{\partial \overline{U}_d}{\partial t} + \nu S_d^T P_d^T A_d P_d S_d \overline{U}_d + S_d^T g(S_d \overline{U}_d) = 0.$$

According to simple computation we can find

$$S_d^T g(S_d \overline{U}_d) = \begin{pmatrix} 2g(Y_d) + O(|Z_d|^2) \\ O(|Z_d|) \end{pmatrix}. \tag{4.1}$$

By neglecting the terms $O(|Z_d|^2)$ and $O(|Z_d|)$, we obtain

$$2\frac{\partial \overline{U}_d}{\partial t} + \nu S_d^T P_d^T A_d P_d S_d \overline{U}_d + 2\begin{pmatrix} g(Y_d) \\ 0 \end{pmatrix} = 0.$$

The form of the equation (3.4) with $d = 2$ is

$$2^2 \frac{\partial \overline{U}_{d-1}}{\partial t} + \nu S^T A_d S \overline{U}_{d-1} + \widetilde{S}_{d-1}^T \widetilde{P}_{d-1}^T S_d^T g(S_d \overline{U}_d) = 0.$$

Using the same approximation as in (4.1), we can set the approximate equation to be

$$2^2 \frac{\partial \overline{U}_{d-1}}{\partial t} + \nu S^T A_d S \overline{U}_{d-1} + 2^2 \begin{pmatrix} g(Y_{d-1}) \\ 0 \end{pmatrix} = 0.$$

This technique is called a nonlinear Galerkin method [4,12].

Finally, with the use of the nonlinear Galerkin method, the $(d+1)$-level incremental unknowns equation (3.4) is approximated by

$$2^d \frac{\partial}{\partial t} \begin{pmatrix} Y_0 \\ Z \end{pmatrix} + \nu S^T A_d S \overline{U}_0 + 2^d \begin{pmatrix} g(Y_0) \\ 0 \end{pmatrix} = 0. \tag{4.2}$$

As for time discretization of (4.2), we now propose two new schemes which are based on fully discretized explicit and implicit schemes when forward differences and backward differences in time are used. The results are two kinds of semi-implicit schemes.

Scheme I Semi-implicit scheme with Y-implicit and Z-explicit components

$$\frac{2^d}{\tau} \begin{pmatrix} Y_0^{n+1} - Y_0^n \\ Z^{n+1} - Z^n \end{pmatrix} + \nu S^T A_d S \begin{pmatrix} Y_0^{n+1} \\ Z^n \end{pmatrix} + 2^d \begin{pmatrix} g(Y_0^n) \\ 0 \end{pmatrix} = 0.$$

Scheme II Semi-implicit scheme with Y-explicit and Z-implicit components

$$\frac{2^d}{\tau} \begin{pmatrix} Y_0^{n+1} - Y_0^n \\ Z^{n+1} - Z^n \end{pmatrix} + \nu S^T A_d S \begin{pmatrix} Y_0^n \\ Z^{n+1} \end{pmatrix} + 2^d \begin{pmatrix} g(Y_0^n) \\ 0 \end{pmatrix} = 0.$$

New schemes generalize obviously the schemes presented in [4]. Stability conditions for the new schemes will be given in the next section. The effective implementation of the semi-implicit schemes requires about half computation of that of the implicit scheme. Especially, the product of $S^T A_d S$ with a vector can be obtained without giving the explicit form of S.

Now let us show some numerical results in numerical experiments which even include the ones for certain two-dimensional problems.

Table 1. $(\tau = 1/8 \cdot 2^d/(2+2^d) \cdot h^2)$

k $(t = k\tau)$	Exact Sol.	Explicit	Implicit	Semi-Implicit(II)
1	0.0355	0.0354	0.0354	0.0354
5	0.1690	0.1690	0.1689	0.1690
10	0.3602	0.3603	0.3603	0.3603
15	0.5827	0.5852	0.5852	0.5852
20	0.8416	0.8493	0.8493	0.8493
25	1.1429	1.1525	1.1524	1.1524
30	1.4935	1.5012	1.5010	1.5010

Table 2. $(\tau = 1/3000)$

k $(t = k\tau)$	Exact Sol.	Explicit	Implicit	Semi-Implicit(II)
1	0.0481	0.0842	0.0479	0.0482
5	0.1690	0.1690	0.1689	0.1690
10	0.3602	0.3603	0.3603	0.3603
15	0.5827	0.5852	0.5852	0.5852
20	0.8416	0.8493	0.8493	0.8493
25	1.1429	1.1525	1.1524	1.1524
30	1.4935	1.5012	1.5010	1.5010

For one-dimensional problem, the test equation is given below

$$\begin{cases} \dfrac{\partial u}{\partial t} - 2\dfrac{\partial^2 u}{\partial x^2} + 1 + u = 0 & 0 < x < 1, \\ u = e^t - 1 & x = 0, \\ u = e^{1+t} - 1 & x = 1, \\ u = e^x - 1 & t = 0. \end{cases}$$

It is easy to check that the exact solution is $u = e^{x+t} - 1$. The numerical experiment is carried out by taking $N = 4$, $d = 3$, and $h = 1/33$. (See Table 1.)

The second experiment is carried out by taking $\tau = 1/3000$. (Note that the approximate values obtained by the explicit schemes are multiplied by 10^{-8} in order to compare with the other values.) (See Table 2.)

Remark 2 We can give also some examples for the semi-implicit schemes to solve two-dimensional equations. The nonlinear equation gives $g(u) = -u + u^3$ with a right-hand side function $f(x, y, t) = (\sin(\pi x)\sin(\pi y)e^{-t})^3$. Here we show two figures of the approximate computation of the solutions

Table 3. Comparison of CPU Time and Error

Scheme	$(d=2)$ CPU Time	Error	$(d=3)$ CPU Time	Error
Explicit	402	2.3E-3	403	1.05E-2
Scheme I	302	2.5E-3	319	1.03E-2
Implicit	65	4.3E-3	76	1.21E-2
Scheme II	68	2.3E-3	80	1.07E-2

Fig. 1. Case $d=2$ $\tau=0.01$ for approx sol.

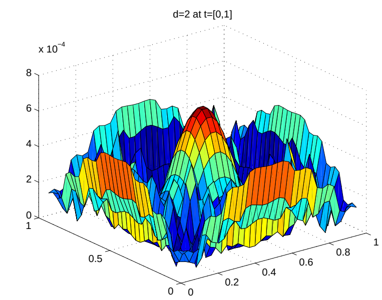


Fig. 2. Case $d=3$ $\tau=0.01$ for approx sol.

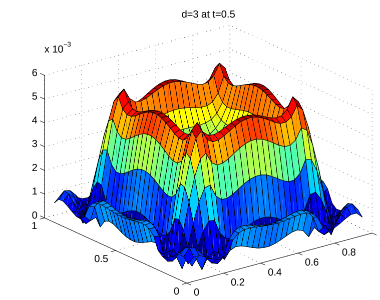


Fig. 3. Case $d=2$ $\tau=0.01$ for error.

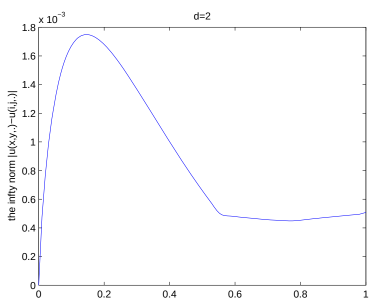
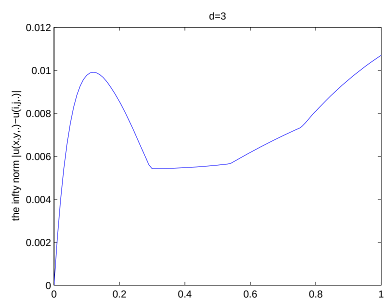


Fig. 4. Case $d=3$ $\tau=0.01$ for error.



(see Fig. 1 and Fig. 2) and two figures of error curves (see Fig. 3 and Fig. 4). Table 3 gives comparison of the CPU time (in second) and the maximum error for the 4 schemes. More details will be given elsewhere.

5. Stability analysis

Let $\mathcal{V}_{h_d}$ be the function space spanned by the basis functions w_{h_d,ih_d}, $i=1, 2, \ldots, 2^d N$, and

$$w_{h_d,ih_d} = \begin{cases} 1, & ih_d \le x < (i+1)h_d, \\ 0, & \text{otherwise.} \end{cases}$$

$u_{h_d}(x)$ be a step function in $\mathcal{V}_{h_d}$, and $u_{h_d}(x) = \sum_{i=1}^{2^d N} u_{h_d}(ih_d) w_{h_d,\,ih_d}$, $x \in \Omega$. We introduce the finite difference operator

$$\nabla_{h_d}\phi(x) = \frac{1}{h_d}(\phi(x+h_d) - \phi(x)),$$

and endow $\mathcal{V}_{h_d}$ with the scalar product $((u_{h_d},\, v_{h_d}))_{h_d} = (\nabla_{h_d} u_{h_d},\, \nabla_{h_d} v_{h_d})$, where $(\,\cdot\,,\,\cdot\,)$ is the scalar product in $L^2(\Omega)$. We set $\|\cdot\|_{h_d} = \{((\,\cdot\,,\,\cdot\,))_{h_d}\}^{1/2}$ and observe that $\|\cdot\|_{h_d}$ and $|\cdot|$ are Hilbert norms on $\mathcal{V}_{h_d}$. Using the space $\mathcal{V}_{h_d}$, we can write the finite difference discretization schemes (2.9) in variational form as

$$\Big(\frac{\partial u_{h_d}}{\partial t},\, \tilde{u}\Big) + \nu((u_{h_d},\, \tilde{u}))_{h_d} + (g(u_{h_d}),\, \tilde{u}) = 0, \quad \forall \tilde{u} \in \mathcal{V}_{h_d}.$$

If we choose $\tilde{u} = w_{h_d,\,ih_d}$, we can recover (2.9). We now separate space $\mathcal{V}_{h_d}$ into two spaces $\mathcal{Y}_{h_d}$ and $\mathcal{Z}_{h_d}$ according to the definition of wavelet-like incremental unknowns.

Let $\mathcal{Y}_{h_d}$ be the function space spanned by the basis functions $\psi_{2h_d,\,2ih_d}$, $i = 1, 2, \ldots, 2^{d-1}N$, and

$$\psi_{2h_d,\,2ih_d} = \begin{cases} 1, & (2i-1)h_d \le x < (2i+1)h_d, \\ 0, & \text{otherwise.} \end{cases}$$

$$y_{h_d}(x) = \sum_{i=1}^{2^{d-1}N} y_{h_d}(2ih_d)\psi_{2h_d,2ih_d}, \quad x \in \Omega,\ \forall y_{h_d} \in \mathcal{Y}_{h_d}.$$

Let $\mathcal{Z}_{h_d}$ be the function space spanned by the basis functions $\chi_{2h_d,\,(2i-1)h_d}$, $i = 1, 2, \ldots, 2^{d-1}N$, and

$$\begin{aligned} \chi_{2h_d,\,(2i-1)h_d} &= w_{h_d,\,(2i-1)h_d} - w_{h_d,\,2ih_d} \\ &= \begin{cases} 1, & (2i-1)h_d \le x < 2ih_d, \\ -1, & 2ih_d \le x < (2i+1)h_d. \end{cases} \end{aligned}$$

$$z_{h_d}(x) = \sum_{i=1}^{2^{d-1}N} z_{h_d}((2i-1)h_d)\chi_{2h_d,(2i-1)h_d}, \quad x \in \Omega,\ \forall z_{h_d} \in \mathcal{Z}_{h_d}.$$

By decomposition, we have

$$\mathcal{V}_{h_d} = \mathcal{Y}_{h_d} \oplus \mathcal{Z}_{h_d}. \tag{5.1}$$

So, the approximate solution $u_{h_d} \in \mathcal{V}_{h_d}$ is separated into two parts

$$u_{h_d} = y_{h_d} + z_{h_d}, \quad y_{h_d} \in \mathcal{Y}_{h_d}, \quad z_{h_d} \in \mathcal{Z}_{h_d}.$$

We can easily recover (3.1) by above decomposition.
With (5.1), we see that (3.2) is identical to

$$\begin{aligned}
&\Big(\frac{\partial y_{h_d}}{\partial t}, \tilde{y}\Big) + \nu((y_{h_d} + z_{h_d}, \tilde{y}))_{h_d} + (g(y_{h_d} + z_{h_d}), \tilde{y}) = 0, \ \forall\ \tilde{y} \in \mathcal{Y}_{h_d}, \\
&\Big(\frac{\partial z_{h_d}}{\partial t}, \tilde{z}\Big) + \nu((y_{h_d} + z_{h_d}, \tilde{z}))_{h_d} + (g(y_{h_d} + z_{h_d}), \tilde{z}) = 0, \ \forall\ \tilde{z} \in \mathcal{Z}_{h_d}.
\end{aligned}$$

The method with multilevel incremental unknowns can be recovered in a similar fashion. We decompose $\mathcal{Y}_{h_l}$ into $\mathcal{Y}_{h_l} = \mathcal{Y}_{h_{l-1}} \oplus \mathcal{Z}_{h_{l-1}}$, $l = d, d-1, \ldots, 1$. Therefore, for any function $u_{h_d} \in \mathcal{V}_{h_d}$, we can write it as $u_{h_d} = y_{h_0} + z$, with $y_{h_0} \in \mathcal{Y}_{h_0}$, and $z \in \mathcal{Z} = \mathcal{Z}_{h_0} \oplus \mathcal{Z}_{h_1} \oplus \cdots \oplus \mathcal{Z}_{h_d}$. The function space $\mathcal{Y}_{h_0}$ is of course spanned by the step functions with step size $h_0 = 2^d h_d$, and the orthogonality between $\mathcal{Y}_{h_0}$ and $\mathcal{Z}$ holds true

$$(y, z) = 0, \quad \forall\ y \in \mathcal{Y}_{h_0}, \quad \forall\ z \in \mathcal{Z}.$$

Therefore, equation (4.2) is identical to

$$\begin{aligned}
&\Big(\frac{\partial y_{h_0}}{\partial t}, \tilde{y}\Big) + \nu((y_{h_0} + z, \tilde{y}))_{h_d} + (g(y_{h_0}), \tilde{y}) = 0, \quad \forall\ \tilde{y} \in \mathcal{Y}_{h_0}, \\
&\Big(\frac{\partial z}{\partial t}, \tilde{z}\Big) + \nu((y_{h_0} + z, \tilde{z}))_{h_d} = 0, \quad \forall\ \tilde{z} \in \mathcal{Z}.
\end{aligned}$$

Before presenting the stability theory, let us introduce some lemmas. (See, e.g., [3,4,11])

Lemma 1 *For every function $u_h \in \mathcal{V}_h$,*

$$\sqrt{2}|u_h| \le \|u_h\|_h \le \frac{1}{S_1(h)}|u_h|, \quad \textit{with } S_1(h) = \frac{h}{2}.$$

Lemma 2 *For every function $y_{h_0} \in \mathcal{Y}_{h_0}$,*

$$\begin{aligned}
&S_2(h_0)|y_{h_0}|_\infty^2 \le |y_{h_0}|^2, \quad \textit{with } S_2(h_0) = h_0, \\
&\bar{S}_1(h_0, h_d)\|y_{h_0}\|_{h_d} \le |y_{h_0}|, \quad \textit{with } \bar{S}_1(h_0, h_d) = \frac{1}{2}\sqrt{h_0 h_d},
\end{aligned}$$

where $|y_{h_0}|_\infty$ is the maximum (L^∞) norm of y_{h_0}.

Lemma 3 *For given time step τ, let us define a q-mesh ratio by*

$$r(q) = \max\Big\{\frac{\tau}{h_d^2},\, \frac{\tau}{h_d^{q-1}}\Big\},$$

then we have $r(q) = \tau/h_d^{q-1}$ $(q < 3)$ or τ/h_d^2 $(q \geq 3)$.

Thus, we have the stability theorem for Scheme I.

Theorem 1 *Assuming that $\tau \leq K_0$ for some K_0 fixed, we set*

$$B_0 = |u_{h_d}^0|^2 + \frac{1}{2\nu}(2c_1 + c_2K_0)|\Omega|.$$

If the q-mesh ratio $r(q)$ satisfies

$$r(q) \leq \min\Big\{\frac{1}{4\nu},\, \frac{2^{d(q-1)}}{2b_{2q-1}B_0^{q-1}}\Big\},$$

then we have the following estimate for any $n \geq 0$: $|u_{h_d}^n|^2 = |y_{h_0}^n|^2 + |z^n|^2 \leq B_0$.

Proof. We write the semi-implicit scheme in its variational form

$$\begin{cases} (y_{h_0}^{n+1} - y_{h_0}^n,\, \tilde{y}) + \tau\nu((y_{h_0}^{n+1} + z^n,\, \tilde{y}))_{h_d} + \tau(g(y_{h_0}^n),\, \tilde{y}) = 0, \\ (z^{n+1} - z^n,\, \tilde{z}) + \tau\nu((y_{h_0}^{n+1} + z^n,\, \tilde{z}))_{h_d} = 0. \end{cases} \tag{5.2}$$

Let $\tilde{y} = 2y_{h_0}^{n+1}, \tilde{z} = 2z^{n+1}$, we have

$$\begin{aligned} &2(y_{h_0}^{n+1} - y_{h_0}^n,\, y_{h_0}^{n+1}) + 2\tau\nu((y_{h_0}^{n+1} + z^n,\, y_{h_0}^{n+1}))_{h_d} \\ &\qquad\qquad + 2\tau(g(y_{h_0}^n),\, y_{h_0}^{n+1}) = 0, \\ &2(z^{n+1} - z^n,\, z^{n+1}) + 2\tau\nu((y_{h_0}^{n+1} + z^n,\, z^{n+1}))_{h_d} = 0. \end{aligned}$$

Adding these two relations, since $2(a-b,\, a) = |a|^2 - |b|^2 + |a-b|^2$ we obtain

$$\begin{aligned} &|y_{h_0}^{n+1}|^2 - |y_{h_0}^n|^2 + |y_{h_0}^{n+1} - y_{h_0}^n|^2 + |z^{n+1}|^2 - |z^n|^2 + |z^{n+1} - z^n|^2 \\ &+2\tau\nu((y_{h_0}^{n+1} + z^n,\, y_{h_0}^{n+1} + z^{n+1}))_{h_d} + 2\tau(g(y_{h_0}^n),\, y_{h_0}^{n+1}) = 0, \end{aligned}$$

Denoting $A = |y_{h_0}^{n+1}|^2 - |y_{h_0}^n|^2 + |z^{n+1}|^2 - |z^n|^2$ for simplicity, we have

$$\begin{aligned} &A + |y_{h_0}^{n+1} - y_{h_0}^n|^2 + |z^{n+1} - z^n|^2 \\ &\quad + 2\tau\nu((y_{h_0}^{n+1} + z^{n+1},\, y_{h_0}^{n+1} + z^{n+1}))_{h_d} \\ &\quad - 2\tau\nu((z^{n+1} - z^n,\, y_{h_0}^{n+1} + z^{n+1}))_{h_d} \end{aligned}$$

$$+ 2\tau(g(y_{h_0}^n),\, y_{h_0}^{n+1} - y_{h_0}^n) + 2\tau(g(y_{h_0}^n),\, y_{h_0}^n) = 0.$$

Since

$$\begin{aligned}
2\tau\nu((z^{n+1} - z^n, y_{h_0}^{n+1} + z^{n+1}))_{h_d}& \\
&\leq 2\tau\nu\|y_{h_0}^{n+1} + z^{n+1}\|_{h_d}\|z^{n+1} - z^n\|_{h_d} \\
&\leq \frac{2\tau v}{S_1(h_d)}\|y_{h_0}^{n+1} + z^{n+1}\|_{h_d}|z^{n+1} - z^n| \\
&\leq \frac{\tau^2\nu^2}{S_1(h_d)^2}\|y_{h_0}^{n+1} + z^{n+1}\|_{h_d}^2 + |z^{n+1} - z^n|^2,
\end{aligned}$$

$$\begin{aligned}
2\tau(g(y_{h_0}^n),\, y_{h_0}^{n+1} - y_{h_0}^n) &\leq 2\tau|g(y_{h_0}^n)|\ |y_{h_0}^{n+1} - y_{h_0}^n| \\
&\leq \tau^2|g(y_{h_0}^n)|^2 + |y_{h_0}^{n+1} - y_{h_0}^n|^2.
\end{aligned}$$

Hence

$$\begin{aligned}
A + 2\tau\nu\Big(1 - \frac{2\tau v}{h_d^2}\Big)\|y_{h_0}^{n+1} + z^{n+1}\|_{h_d}^2& \\
&+ 2\tau(g(y_{h_0}^n),\, y_{h_0}^n) - \tau^2|g(y_{h_0}^n)|^2 \leq 0.
\end{aligned}$$

Using Lemmas 1 and 3, the condition satisfied by $r(q)$ implies

$$A + 2\tau\nu|y_{h_0}^{n+1} + z^{n+1}|^2 + 2\tau(g(y_{h_0}^n),\, y_{h_0}^n) - \tau^2|g(y_{h_0}^n)|^2 \leq 0.$$

Using inequalities (2.7), (2.8) and Lemma 2, we have

$$(g(y_{h_0}^n),\, y_{h_0}^n) = \int_\Omega g(y_{h_0}^n)y_{h_0}^n dx \geq \frac{1}{2}b_{2q-1}\int_\Omega (y_{h_0}^n)^{2p}dx - c_1|\Omega|. \quad (5.3)$$

and

$$\begin{aligned}
\tau^2|g(y_{h_0}^n)|^2 &\leq 2\tau^2 b_{2q-1}^2\int_\Omega (y_{h_0}^n)^{4q-2}dx + \tau^2 c_1|\Omega| \\
&\leq 2\tau^2 b_{2q-1}^2|y_{h_0}^n|_\infty^{2q-2}\int_\Omega (y_{h_0}^n)^{2q}dx + \tau^2 c_2|\Omega| \\
&\leq \frac{2\tau^2 b_{2q-1}^2}{2^{d(q-1)}h_d^{q-1}}|y_{h_0}^n|^{2p-2}\int_\Omega (y_{h_0}^n)^{2p}dx + \tau^2 c_2|\Omega|. \qquad (5.4)
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&A + 2\tau\nu|y_{h_0}^{n+1} + z^{n+1}|^2 \\
&+\tau b_{2q-1}\Big(1 - \frac{2\tau b_{2q-1}}{2^{d(q-1)}h_d^{q-1}}|y_{h_0}^n|^{2q-2}\Big)\int_\Omega (y_{h_0}^n)^{2q}dx \leq 2\tau c_1|\Omega| + \tau^2 c_2|\Omega|.
\end{aligned}$$

We are now ready to prove the Theorem by induction:

- $k = 0$ is obvious since $|y_{h_0}^0|^2 + |z^0|^2 \le B_0$.
- Assuming the conclusion is correct up to $k = n$, we then have $|y_{h_0}^n|^2 + |z^n|^2 \le B_0$.
- For $k = n + 1$, using the inequality satisfied by $r(q)$, we obtain

$$
\begin{aligned}
&|y_{h_0}^{n+1}|^2 + |z^{n+1}|^2 - |y_{h_0}^n|^2 - |z^n|^2 + 2\tau\nu|y_{h_0}^{n+1} + z^{n+1}|^2 \\
&\qquad\qquad\qquad\qquad \le (2c_1\tau + c_2\tau^2)|\Omega|.
\end{aligned}
$$

That is

$$
\begin{aligned}
&|y_{h_0}^{n+1}|^2 + |z^{n+1}|^2 \\
&\le \frac{1}{1+2\tau\nu}(|y_{h_0}^n|^2 + |z^n|^2) + \frac{1}{1+2\tau\nu}(2c_1\tau + c_2\tau^2)|\Omega| \\
&\le \Big(\frac{1}{1+2\tau\nu}\Big)^{n+1}(|y_{h_0}^0|^2 + |z^0|^2) + \frac{1}{1+2\tau\nu}\Big(1 + \frac{1}{1+2\tau\nu} \\
&\quad + \frac{1}{(1+2\tau\nu)^2} + \cdots + \frac{1}{(1+2\tau\nu)^n}\Big)(2c_1\tau + c_2\tau^2)|\Omega| \\
&\le \Big(\frac{1}{1+2\tau\nu}\Big)^{n+1}(|y_{h_0}^0|^2 + |z^0|^2) \\
&\quad + \frac{1}{1+2\tau\nu}\frac{1}{1 - 1/(1+2\tau\nu)}(2c_1 + c_2K_0)|\Omega| \\
&\le |y_{h_0}^0|^2 + |z^0|^2 + \frac{1}{2\nu}(2c_1 + c_2K_0)|\Omega|.
\end{aligned}
$$

Therefore, we obtain the estimate

$$
|u_{h_d}^n|^2 = |y_{h_0}^n|^2 + |z^n|^2 \le B_0.
$$

□

As for the stability condition of Scheme II, we have

Theorem 2 *Assuming that $k \le K_0$ for some K_0 fixed, we set*

$$
B_1 = |u_{h_d}^0|^2 + \frac{1}{\nu}(c_1 + c_2K_0)|\Omega|.
$$

If the q-mesh ratio $r(q)$ satisfies

$$
r(q) \le \min\Big\{\frac{2^d}{8\nu}, \frac{2^{d(q-1)}}{4b_{2q-1}B_1^{q-1}}\Big\},
$$

then we have the following estimate for any $n \geq 0$: $|u^n_{h_d}|^2 = |y^n_{h_0}|^2 + |z^n|^2 \leq B_1$.

Proof. We write the semi-implicit scheme in its variational form

$$\begin{cases} (y^{n+1}_{h_0} - y^n_{h_0},\, \tilde{y}) + \tau\nu((y^n_{h_0} + z^{n+1},\, \tilde{y}))_{h_d} + \tau(g(y^n_{h_0}), \tilde{y}) = 0, \\ (z^{n+1} - z^n,\, \tilde{z}) + \tau\nu((y^n_{h_0} + z^{n+1},\, \tilde{z}))_{h_d} = 0. \end{cases} \tag{5.5}$$

Let $\tilde{y} = 2y^{n+1}_{h_0}, \tilde{z} = 2z^{n+1}$, we have

$$\begin{aligned} &2(y^{n+1}_{h_0} - y^n_{h_0},\, y^{n+1}_{h_0}) + 2\tau\nu((y^n_{h_0} + z^{n+1},\, y^{n+1}_{h_0}))_{h_d} \\ &\qquad\qquad + 2\tau(g(y^n_{h_0}),\, y^{n+1}_{h_0}) = 0. \\ &2(z^{n+1} - z^n,\, z^{n+1}) + 2\tau\nu((y^n_{h_0} + z^{n+1},\, z^{n+1}))_{h_d} = 0. \end{aligned}$$

Adding these relations, we have

$$\begin{aligned} &|y^{n+1}_{h_0}|^2 - |y^n_{h_0}|^2 + |y^{n+1}_{h_0} - y^n_{h_0}|^2 + |z^{n+1}|^2 - |z^n|^2 + |z^{n+1} - z^n|^2 \\ &\quad + 2\tau\nu((y^n_{h_0} + z^{n+1}, y^{n+1}_{h_0} + z^{n+1}))_{h_d} + 2\tau(g(y^n_{h_0}), y^{n+1}_{h_0}) = 0. \end{aligned}$$

Denoting also $A = |y^{n+1}_{h_0}|^2 - |y^n_{h_0}|^2 + |z^{n+1}|^2 - |z^n|^2$, we have

$$\begin{aligned} &A + |y^{n+1}_{h_0} - y^n_{h_0}|^2 + |z^{n+1} - z^n|^2 \\ &\quad + 2\tau\nu((y^{n+1}_{h_0} + z^{n+1}, y^{n+1}_{h_0} + z^{n+1}))_{h_d} \\ &\quad - 2\tau\nu((y^{n+1}_{h_0} - y^n_{h_0}, y^{n+1}_{h_0} + z^{n+1}))_{h_d} \\ &\quad + 2\tau(g(y^n_{h_0}), y^{n+1}_{h_0} - y^n_{h_0}) + 2\tau(g(y^n_{h_0}), y^n_{h_0}) = 0. \end{aligned}$$

Since

$$\begin{aligned} &2\tau\nu((y^{n+1}_{h_0} - y^n_{h_0},\, y^{n+1}_{h_0} + z^{n+1}))_{h_d} \\ &\qquad\qquad \leq 2\tau\nu\|y^{n+1}_{h_0} + z^{n+1}\|_{h_d}\|y^{n+1}_{h_0} - y^n_{h_0}\|_{h_d} \\ &\qquad\qquad \leq \frac{2\tau v}{\bar{S}_1(h_0,\, h_d)}\|y^{n+1}_{h_0} + z^{n+1}\|_{h_d}|y^{n+1}_{h_0} - y^n_{h_0}| \\ &\qquad\qquad \leq \frac{2\tau^2 v^2}{\bar{S}_1(h_0,\, h_d)^2}\|y^{n+1}_{h_0} + z^{n+1}\|^2_{h_d} + \frac{1}{2}|y^{n+1}_{h_0} - y^n_{h_0}|^2, \\ &2\tau(g(y^n_{h_0}),\, y^{n+1}_{h_0} - y^n_{h_0}) \leq 2\tau|g(y^n_{h_0})|\; |y^{n+1}_{h_0} - y^n_{h_0}| \\ &\qquad\qquad \leq 2\tau^2|g(y^n_{h_0})|^2 + \frac{1}{2}|y^{n+1}_{h_0} - y^n_{h_0}|^2. \end{aligned}$$

Hence

$$A + 2\tau\nu\Big(1 - \frac{4\tau\nu}{2^d h_d^2}\Big)\|y_{h_0}^{n+1} + z^{n+1}\|_{h_d}^2 \\ + 2\tau(g(y_{h_0}^n),\, y_{h_0}^n) - 2\tau^2|g(y_{h_0}^n)|^2 \le 0.$$

Using Lemma 1 and the inequality satisfied by q-mesh ratio, we obtain

$$A + 2\tau\nu|y_{h_0}^{n+1} + z^{n+1}|^2 + 2\tau(g(y_{h_0}^n),\, y_{h_0}^n) - 2\tau^2|g(y_{h_0}^n)|^2 \le 0.$$

According to (5.3) and (5.4), we have

$$A + 2\tau\nu|y_{h_0}^{n+1} + z^{n+1}|^2 \\ + \tau b_{2p-1}\Big(1 - \frac{4\tau b_{2p-1}}{2^{d(q-1)}h_d^{q-1}}|y_{h_0}^n|^{2q-2}\Big)\int_\Omega (y_{h_0}^n)^{2q}dx \le 2\tau c_1|\Omega| + 2\tau^2 c_2|\Omega|.$$

We are now also by induction to prove the theorem:

- $k = 0$ is obvious since $|y_{h_0}^0|^2 + |z^0|^2 \le\ B_1$.
- Assuming the conclusion is correct up to $k = n$, we then have $|y_{h_0}^n|^2 + |z^n|^2 \le\ B_1$.
- For $k = n + 1$, using the condition satisfied by $r(q)$, we see that

$$|y_{h_0}^{n+1}|^2 + |z^{n+1}|^2 - |y_{h_0}^n|^2 - |z^n|^2 \\ + 2\tau\nu|y_{h_0}^{n+1} + z^{n+1}|^2 \le (2c_1\tau + 2c_2\tau^2)|\Omega|.$$

That is

$$\begin{aligned}
&|y_{h_0}^{n+1}|^2 + |z^{n+1}|^2 \\
&\le \frac{1}{1+2\tau\nu}(|y_{h_0}^n|^2 + |z^n|^2) + \frac{1}{1+2\tau v}(2c_1\tau + 2c_2\tau^2)|\Omega| \\
&\le \Big(\frac{1}{1+2\tau\nu}\Big)^{n+1}(|y_{h_0}^0|^2 + |z^0|^2) + \frac{1}{1+2\tau\nu}\Big(1 + \frac{1}{1+2\tau\nu} \\
&\quad + \frac{1}{(1+2\tau\nu)^2} + \cdots + \frac{1}{(1+2\tau\nu)^n}\Big)(2c_1\tau + 2c_2\tau^2)|\Omega| \\
&\le \Big(\frac{1}{1+2\tau\nu}\Big)^{n+1}(|y_{h_0}^0|^2 + |z^0|^2) + \frac{1}{2v}(2c_1 + 2c_2K_0)|\Omega| \\
&\le |y_{h_0}^0|^2 + |z^0|^2 + \frac{1}{\nu}(c_1 + c_2K_0)|\Omega|.
\end{aligned}$$

Therefore, we obtain the estimate

$$|u^n_{h_d}|^2 = |y^n_{h_0}|^2 + |z^n|^2 \leq \ B_1.$$

$\square$

Concluding Remarks We can compare our schemes with two ordinary schemes with WIU; namely,

Explicit Scheme:

$$\frac{2^d}{\tau}\begin{pmatrix} Y_0^{n+1} - Y_0^n \\ Z^{n+1} - Z^n \end{pmatrix} + \nu S^T A_d S \begin{pmatrix} Y_0^n \\ Z^n \end{pmatrix} + 2^d \begin{pmatrix} g(Y_0^n) \\ 0 \end{pmatrix} = 0.$$

Implicit Scheme:

$$\frac{2^d}{\tau}\begin{pmatrix} Y_0^{n+1} - Y_0^n \\ Z^{n+1} - Z^n \end{pmatrix} + \nu S^T A_d S \begin{pmatrix} Y_0^{n+1} \\ Z^{n+1} \end{pmatrix} + 2^d \begin{pmatrix} g(Y_0^n) \\ 0 \end{pmatrix} = 0.$$

1. Since we have $2^d/(2+2^d) < 1$ and $B_0 < M_0$ (see, e.g., in [4]), the bound of q-mesh ratio $r(q)$ satisfied for Scheme I in Theorem 1 is greater than that for Explicit Scheme. Therefore, Scheme I has an improved stability comparing with Explicit Scheme.
2. Even B_1 and M_0 are essentially identical, the bound of q-mesh ratio $r(q)$ satisfied for Scheme II in Theorem 2 is possibly greater than that for Explicit Scheme. The stability condition of Scheme II is at least the same better as that of Explicit Scheme.
3. When nonlinear effect is strong, i.e., the dominant conditions satisfied by q-mesh ratio $r(q)$ for Schemes I and II are $r(q) \leq 2^{d(q-1)}/2b_{2q-1}B_0^{2q-1}$ and $r(q) \leq 2^{d(q-1)}/4b_{2q-1}B_1^{q-1}$ respectively, these conditions are comparable when compared with Implicit Scheme. However, the computational process of our two semi-implicit schemes is evidently not so complex than that of Implicit Scheme.
4. The numerical results confirm our theoretical conclusion.

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