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On universal hyperbolic orbifold structures in S^3 with the Borromean rings as singularity

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Abstract. An orientable 3-orbifold is *universal* iff every closed, orientable 3-manifold is the underlying space of an orbifold structure that is an orbifold-covering of it. The first known example of universal orbifold was $\mathbf{B}_{4,4,4} = (S^3, B, 4)$ where B denotes the Borromean rings and all the isotropy groups are cyclic of order 4. The main result in this article is that the hyperbolic orbifold $\mathbf{B}_{m,2p,2q}$ is universal for every $m \geq 3$, $p \geq 2$, $q \geq 2$.

1. Introduction

In the course of the symposium "Branched Coverings, Degenerations, and Related Topics" held in Hiroshima University in March 2009 the third author was asked by Makoto Sakuma *if every hyperbolic orbifold with underlying space* S^3 *is universal*. The question came after the same author's presentation of a proof of the universality of $\mathbf{B}_{4,4,4}$, that is, the orbifold structure in S^3 whose singular set is B (the Borromean rings) and all the isotropy groups are cyclic of order 4. We will present in this paper our present knowledge of the universality of the orbifolds of the form $\mathbf{B}_{m,n,p}$. While we can prove that $\mathbf{B}_{m,2p,2q}$ is universal for every $m \geq 3$, $p \geq 2$, $q \geq 2$, we do not even know if the (hyperbolic) orbifold $\mathbf{B}_{3,3,3}$ is universal or not.

2. Preliminaries

If M and N are connected, unbounded, triangulable n-manifolds, then a function

$$f : M \to N$$

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is a *(combinatorial) branched covering of* N if there exist triangulations of M and N such that f is an open simplicial map.

Assume $n = 3$ for simplicity.

Then $f : M \to N$ is an ordinary covering when restricted to the complement of the 1-skeleton of M.

Therefore there is a minimal subset Σ of the 1-skeleton of N such that the restriction

$$f|_{(M \backslash f^{-1}(\Sigma))} : M \backslash f^{-1}(\Sigma) \to N \backslash \Sigma$$

is an ordinary covering (the *associated covering*).

We call Σ the *singular set* of $f : M \to N$.

A point $x \in f^{-1}(\Sigma)$ has *branching index* $b(x)$ if $St(x) \backslash f^{-1}(\Sigma)$ is mapped $b(x)$ to 1 onto $St(f(x)) \backslash \Sigma$, where $St(x)$ stands for the open star of x in the polyhedron M.

The part of $f^{-1}(\Sigma)$ with branching index $b > 1$ is called the *branch cover*.

The part of $f^{-1}(\Sigma)$ with branching index $b = 1$ is called the *pseudo-branch cover*.

Following Kato ([7]) a (combinatorial) orientable 3-*orbifold* is a triple (M, Σ, v), where (*the underlying space*) M is an unbounded, triangulable 3-manifold; the (*singular*) set Σ is a polyhedral graph in M; and v is an (*isotropy*) function that associates an integer > 1 to each component of $\Sigma \backslash V_\Sigma$, where V_Σ is the set of vertices of Σ of valence > 2. If x belongs to a component l of $\Sigma \backslash V_\Sigma$ we define $v(x) = v(l)$.

We assume that Σ has no isolated point.

Let (M, Σ', v') and (N, Σ, v) be two orientable orbifolds. An orbifold covering $f : (M, \Sigma', v') \to (N, \Sigma, v)$ is a covering $f : M \to N$ branched over a subset of Σ such that, $\Sigma' \subset f^{-1}(\Sigma)$ and

$$v'(x)b(x) = v(y)$$

for every $x \in f^{-1}(y)$, $y \in \Sigma \backslash V_\Sigma$.

An orbifold is *uniformizable* if it admits a non-singular orbifold covering.

An orbifold (M, Σ, v) is *locally uniformizable* iff the valence of $x \in V_\Sigma$ is 3 and the isotropies of the three components of $\Sigma \backslash V_\Sigma$ meeting at $x \in V_\Sigma$ are one of these: $(2, 2, p)$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$, $p \geq 2$. Therefore, if Σ is a link then V_Σ is empty and then (M, Σ, v) is always locally uniformizable and v is constant along every component of the link Σ. Observe that a uniformizable 3-orbifold is locally uniformizable.

An orientable 3-orbifold (M, Σ, v) is said to be *universal* iff every closed, orientable 3-manifold is the underlying space of an orbifold that is an orbifold-covering of (M, Σ, v).

Notice that if the orbifold (M, Σ, v) is universal, so is the orbifold (M, Σ, v') where v' *is a multiple of* v, that is, there exists a natural number n_l for each component l of $\Sigma \backslash V_\Sigma$ such that $v'(l) = n_l v(l)$. We therefore will say that the orbifold (M, Σ, v) is *minimal universal* iff it is universal but no orbifold (M, Σ, v') with v a multiple of v' is universal.

For a link L in S^3 and a locally constant function $v : L \to \mathbf{N}$, we denote by $\mathbf{L}_v$ the orbifold (S^3, L, v). The orbifold $\mathbf{B}_{2,2,2}$, where the link B is the Borromean rings, is not a universal 3-orbifold because it is a Euclidean orbifold. In fact, it can be seen that $\mathbf{B}_{2,n,p}$ is not a universal 3-orbifold, for any $n, p \in \mathbf{N}$.

On the other hand it was established in [2] that every closed oriented 3-manifold is a covering of S^3 branched over the Borromean rings B and such that the branching indices upon its components are $\{1,2,4\}$ upon the first and second components; and $\{2,4\}$ upon the third component. (We will often abbreviate this saying that the branching indices of the components of B are, respectively, $\{1,2,4\}$, $\{1,2,4\}$ and $\{2,4\}$). As a consequence, the orbifold $\mathbf{B}_{4,4,4}$ is a universal orbifold. (This is the first known example of universal orbifold.) Therefore $\mathbf{B}_{4,4,4}$ is minimal universal.

Notice that the orbifold $\mathbf{B}_{4,4,4}$ is hyperbolic. We will prove in the sequel that the (hyperbolic) orbifolds of the form $\mathbf{B}_{m,2p,2q}$ are universal for every $m \geq 3,\ p \geq 2,\ q \geq 2$.

3. Some universal hyperbolic orbifolds

Let $p : M \to S^3$ be a *simple* 3-fold covering branched over the link L. The meaning of *simple* is that the monodromy ω_p of the covering p is a transitive representation

$$\omega_p : \pi_1(S^3 - L) \to \Sigma_3$$

onto the symmetric group of the three numbers $\{1,2,3\}$ sending meridians to transpositions. We will define ω_p by assigning transpositions $(1,2)$, $(1,3)$ and $(2,3)$ to the overpasses of a regular projection of L. Recall that, as a consequence of the relators in a Wirtinger presentation, in every crossing the three involved transpositions are equal or pairwise different. In the pictures we use the colours $R = (a,b)$, $G = (b,c)$ and $B = (a,c)$, where $\{a,b,c\} = \{1,2,3\}$. The Montesinos moves m_M depicted in Figure 1 change the link only in the interior of the small ball depicted by a dotted line, but do not change the manifold M ([8]).

The following lemma establishes a new move by composing some m_M moves.

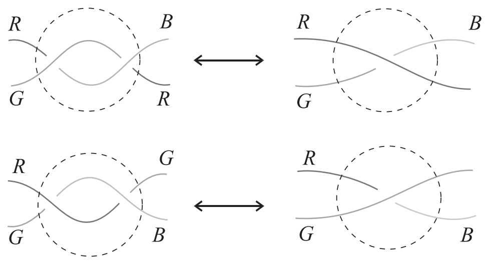

Fig. 1. Montesinos moves.

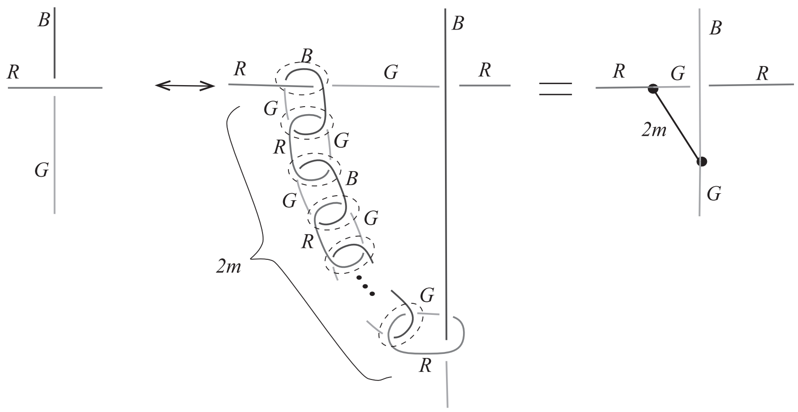

Fig. 2. The 2m-move.

LEMMA 1. *The local move depicted in Figure* 2 *and denoted here by* 2m-*move, do not change the manifold* M.

PROOF. The link in the middle of Figure 2 has $2m$ small circle components, $m \geq 1$. The case $m = 1$ was used in [3] and [1]. Applying the m_M moves of Figure 1 in each of the $2m$ small balls depicted by dotted line in Figure 2 and doing isotopy we obtain the links of Figure 3. □

The local link in the middle of Figure 2 will be denoted by the integer endowed segment on the right side of Figure 2.

LEMMA 2. *Each pn cycle* $(n \geq 1,\ p \geq 2)$ *is the product of a permutation* σ_1 *which is the product of n p-cycles and a permutation* σ_2 *which is the product of* $(n-1)$ *transpositions.*

PROOF. Write $\sigma_1 = \tau_1 \tau_2 \dots \tau_n$ where τ_i is the p-cycle $((i-1)(p-1)+1,$ $(i-1)(p-1)+2, (i-1)(p-1)+3, \dots, (i-1)(p-1)+p-1, np+1-i)$ and

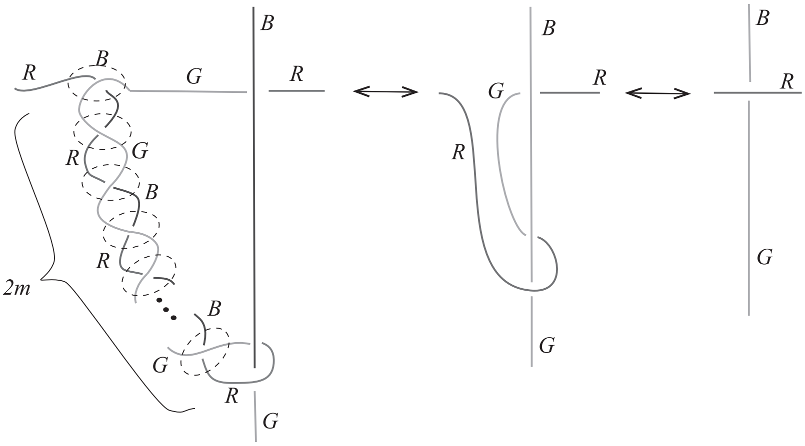

Fig. 3

$\sigma_2 = v_1 v_2 \dots v_{n-1}$ where $v_j = (np + 1 - j, j(p-1) + 1)$. This is equivalent to

$$\sigma_1 = (1, 2, \dots, p-1, np)(p, p+1, p+2, \dots, p+p-2, np-1)$$
$$\dots((n-1)(p-1)+1, (n-1)(p-1)+2, \dots, (n-1)(p-1)+p)$$
$$\sigma_2 = (np, p)(np-1, 2p-1)\dots((np-(n-2), (n-1)(p-1)+1).$$

Then

$$\sigma_1\sigma_2 = (1, 2, 3, \dots, pn). \qquad \square$$

LEMMA 3. *Given any three natural numbers $\{m, p, q\}$ such that $m \geq 1$, $p, q \geq 2$, every closed oriented 3-manifold M is a simple 3-fold covering of S^3 branched over a sublink of the link of the type depicted in Figure 7, such that the number v of vertical components is a multiple of p and the number h of horizontal components is a multiple of q.*

PROOF. The first part of the proof is as in some previous articles ([2], [3] and [1]), but we prefer to give here a self-contained proof.

It was proved in [6], [9] that every closed oriented 3-manifold M is a simple 3-fold covering of S^3 branched over a link L. As every link is a closed braid, we can assume that the diagram of L we use is a closed braid. Now we shall apply some m_M moves to L to obtain a suitable link L_1.

Using an isotopy of the type illustrated in Figure 4 we can assume every crossing has three different colours.

Using m_M moves as in Figure 1, we can assume all crossings in the braid are positive in the braided direction. At this stage still the link is a closed braid, but with all the crossings positive and 3-colored.

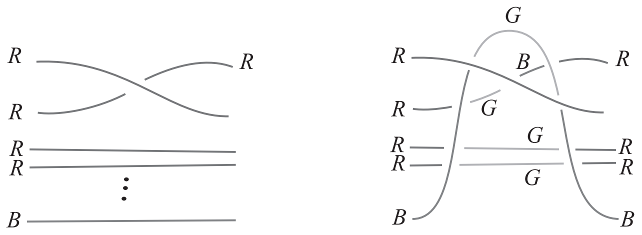

Fig. 4

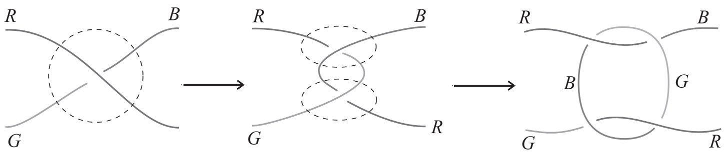

Fig. 5

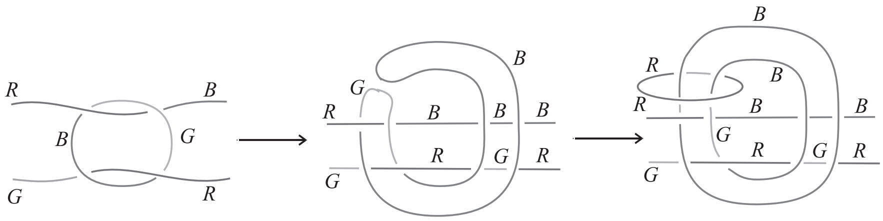

Fig. 6

We replace each crossing with a new small circle component as in Figure 5. Now we have in the link two types of components: *horizontal* components along the braid and small local circles. We change each small circle by isotopy and m_M moves as in Figure 6, to two *vertical* components and one horizontal component. Observe that the crossing where a horizontal component is over a vertical component (*horizontal crossing*) the three involved transpositions are pairwise different. Then we use the move of Lemma 1 to convert the horizontal overcrossing to a vertical overcrossing. We add trivial components to the link such that the number v of vertical components, V_i, is equal to pn, the number h of horizontal components, H_j, is equal to qk and we have a chain of $2m$ components associated to every vertical overcrossing. Let us call L_1 the new link, and $p_1 : M \to S^3$ the simple 3-fold covering of S^3 branched over the link L_1. See Figure 7. □

Lemma 4. *Given any three natural numbers* $\{r, p, q\}$ *such that* $r \geq 3$, $p, q \geq 2$, *every closed oriented* 3*-manifold* M *is a covering of* S^3 *branched*

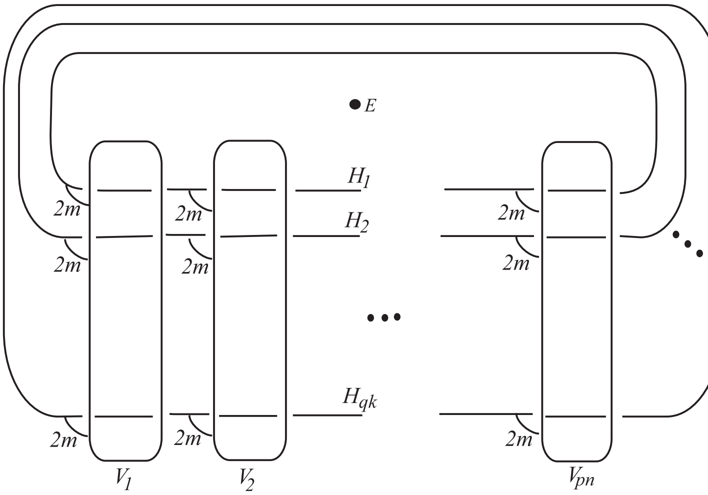

Fig. 7. The link L_1.

over the link L_4 depicted in Figure 11 *and branching indices* $\{1, 2, p, q, r\}$ *as indicated in the same figure.*

PROOF. The link L_1 can be symmetrized along the axis E of the trivial braid defined by the horizontal components such that L_1 remains invariant by a $2\pi/(pn)$–rotation around E. Instead of using the quotient of (S^3, L_1) by this $2\pi/(pn)$–rotation which produces a branched covering with branching index $v = pn$, we consider the covering $p_e : S^3 \to S^3$ branched over $E_1 \cup E_2$, two parallel unlinked copies of E contained in a neighborhood of E disjoint from L_1, with the monodromy $\omega_e : \pi_1(S^3 - (E_1 \cup E_2)) \to \Sigma_{pn}$ mapping the meridian of E_1 to σ_1 and meridian of E_2 to σ_2, where σ_1 and σ_2 are defined by Lemma 2. Then

$$p_2 = p_e \circ p_1 : M \to S^3$$

is a covering branched over the link L_2 depicted in Figure 8, such that the branching indices upon every component, but E_1, are 1 or 2. Upon the component E_1 the branching indices are 1 or p. Observe that we have reduced the number of vertical component to 1.

Next, following an analogous process as the precedent one, we can also reduce the numbers of horizontal components to 1. The link L_2 can be isotoped to be symmetric with respect to a $2\pi/(qk)$–rotation around the axis F depicted in Figure 8. We consider the trivial link $F_1 \cup F_2$ composed by two parallel copies of F contained in a neighborhood of F disjoint from L_2. The monodromy $\omega_f : \pi_1(S^3 - (F_1 \cup F_2)) \to \Sigma_{qk}$ mapping the meridian of F_1 to a

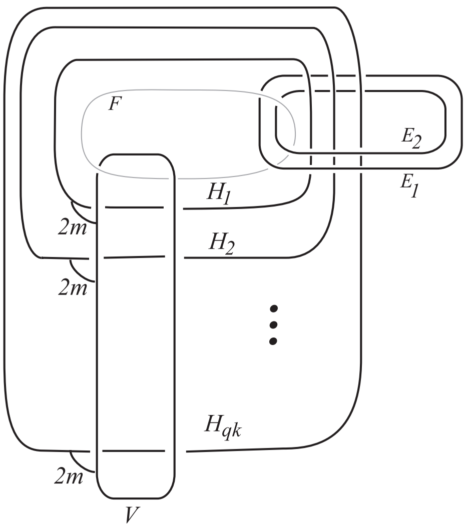

Fig. 8. The link L_2.

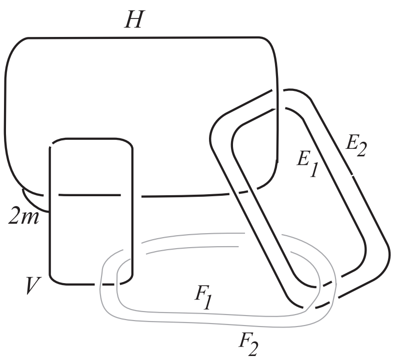

Fig. 9. The link L_3.

product of k q-cycles and the meridian of F_2 to a product of $k-1$ transpositions, as in Lemma 2, corresponds to a covering $p_f : S^3 \to S^3$ branched over $F_1 \cup F_2$. The composition

$$p_3 = p_f \circ p_2 : M \to S^3$$

is a covering branched over the link L_3 depicted in Figure 9, such that the branching indices upon the component $p_f(E_1)$, still denoted by E_1 in the figure, are $\{1, p\}$ and upon the component F_1 are $\{1, q\}$. Upon the remaining components the branching indices are $\{1, 2\}$.

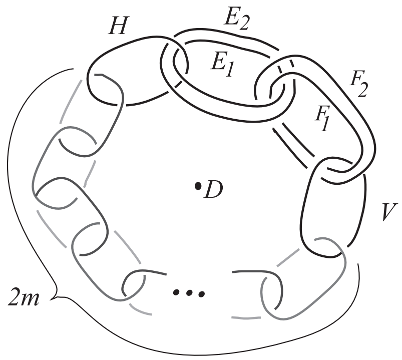

Fig. 10. The link L_3 in a symmetric projection.

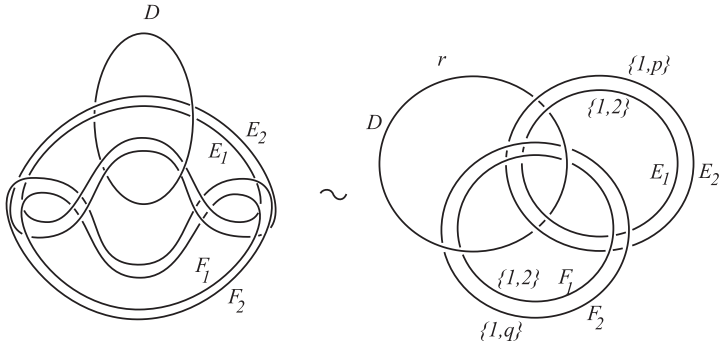

Fig. 11. The link L_4 with the branching indices.

The link L_3 is like a necklace with $2m+6$ components that can be placed as in Figure 10 such that the set of all components (but E_2 and F_2) is invariant by a $2\pi/(m+2)-$rotation around the axis D.

Let us call $p_d : S^3 \to S^3$ the quotient map of S^3 acted by the cyclic group generated by a $2\pi/(m+2)-$rotation around the axis D. It is a covering $S^3 \to S^3$ branched over the trivial knot D. The singular set L_4 for the composition

$$p_4 = p_d \circ p_3 : M \to S^3$$

consists of the union of the singular set D for p_d (upon which the branching indices are all $r = m+2 \geq 3$) and $p_d(L_3)$ where upon the two components $p_d(E_2)$ and $p_d(F_2)$ the branching indices are $\{1,2\}$; upon $p_d(E_1)$ they are $\{1,p\}$; and upon the component $p_d(F_1)$ they are $\{1,q\}$. See Figure 11, where we still denote $p_d(E_i)$ and $p_d(F_i)$ by the same labels E_i and F_i respectively, $i = 1,2$. □

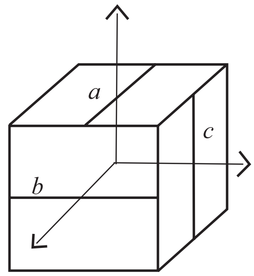

Fig. 12. The cube $\mathscr{C}$.

The next approach will be to define a branched covering $q : S^3 \to S^3$ with singular set the Borromean rings B such that the link L_4 is a suitable sublink of the preimage of B. This construction is different from the one contained in [1] but follows a similar idea.

LEMMA 5. *The homomorphism*

$$\omega_d : \pi_1(S^3 - B) = |a, b, c; abc\bar{c}\bar{b}c = b\bar{c}\bar{b}ca, bc\bar{a}\bar{c}a = c\bar{a}\bar{c}ab| \to \Sigma_5$$
$$a \to (3)(2,4)(1,5)$$
$$b \to (5)(3,4)(1,2)$$
$$c \to (1)(2)(3)(4)(5)$$

is the monodromy of a covering

$$q : S^3 \to S^3$$

branched over the Borromean rings B such that the link L_4 is contained in $q^{-1}(B)$ in such a way that D, E_1 and F_1 are in the pseudo-branch cover (the restriction of q to a neighbourhood of D, E_1 and F_1 is a homeomorphism) and E_2 and F_2 are in the branch cover.

PROOF. We shall construct the covering of S^3 defined by the homomorphism ω_d by using the Euclidean orbifold structure in S^3 with B as singular set with isotropy of order 2. This structure is the quotient of the Euclidean space E^3 by the action of the group $\hat{U}$ generated by 180 degrees rotations in the axes a, b and c displayed in the faces of a $2 \times 2 \times 2$ cube $\mathscr{C}$ centred at the origin, see Figure 12. The images by the elements of $\hat{U}$ of $\mathscr{C}$ form a tessellation of E^3 by cubes all of whose vertices have odd integer coordinates. The cube $\mathscr{C}$ is the fundamental domain for this action and we can see that $E^3/\hat{U}$ equals S^3 with B as singular set by making face identifications in $\mathscr{C}$. The result is the orbifold structure $\mathbf{B}_{2,2,2} = (S^3, B, \{2,2,2\})$. See Figure 13.

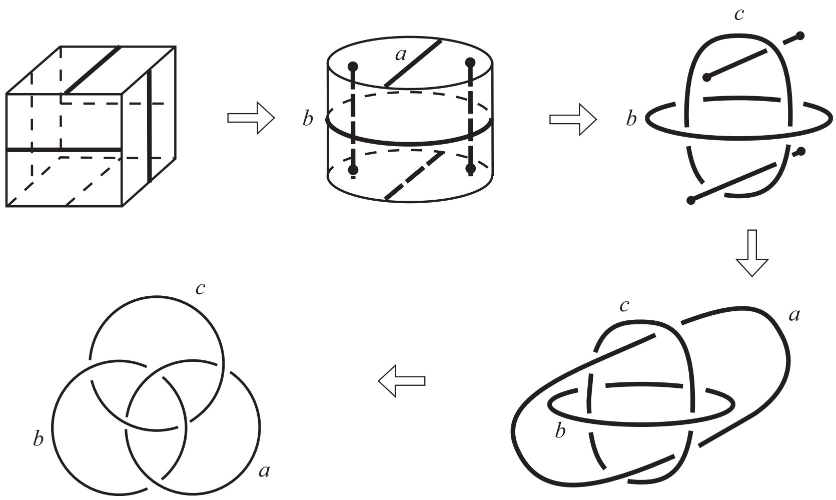

Fig. 13. The orbifold $\mathbf{B}_{2,2,2} = (S^3, B, \{2,2,2\})$.

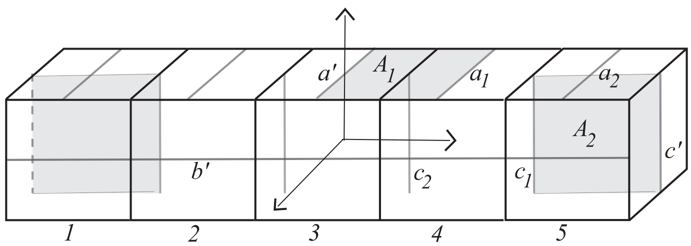

Fig. 14. The parallelepiped $\mathscr{P}$.

Consider the $2 \times 10 \times 2$ parallelepiped $\mathscr{P}$ make up by 5 cubes of the tessellation. Let $\tilde{U}$ be the group generated by 180 degrees rotation in the axes a', b' and c' displayed in the faces of $\mathscr{P}$. The parallelepiped $\mathscr{P}$ is a fundamental domain for the action of $\tilde{U}$ in E^3 and the quotient is again an Euclidean orbifold structure $\mathbf{B}_{2,2,2}$.

We are interested in the map $q: E^3/\tilde{U} \cong S^3 \to E^3/\hat{U} \cong S^3$ induced by the inclusion of $\tilde{U}$ in $\hat{U}$. It is a covering space branched over two (a and c) of the three components of B with branching indices $\{1,2\}$. Observe in the Figure 14, where the 5 copies of $\mathscr{C}$ have been numerated, that the monodromy of this covering is the wanted homomorphism ω_d. □

We have depicted in Figure 15 the link $q^{-1}(B)$, where the components a', b' and c' are the pseudo-branch cover and a_1, a_2, c_1 and c_2 form the branch cover. Observe that $L_4 = \{D, E_1, E_2, F_1, F_2\}$ is the sublink $\{b', a', a_1, c', c_1\}$.

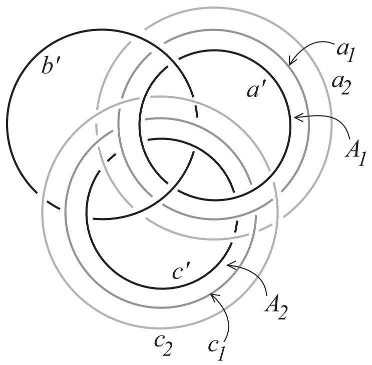

Fig. 15. The link $q^{-1}(B)$.

THEOREM 1. *Given integers* $r \geq 3$, $p \geq 2$, $q \geq 2$, *every* 3*-manifold* M^3 *is a covering* $M^3 \to S^3$ *branched over* B *such that the branching indices upon* a *are* $\{1, 2, 2p\}$*; upon* b *are* $\{1, 2, 2q\}$*; and upon* c *are all* r.

PROOF. For every 3-manifold M^3 consider the map

$$p_M = q \circ p_4 : M^3 \to S^3.$$

It is a covering branched over B, because it is a composition of branch coverings. Lemma 4 proves that the singular set of p_4 is L_4 and Lemma 5 shows that $q(L_4) \subset B$.

The branching index upon every component k of B is the product of the branching indices for k in q and the branching indices for the components $q^{-1}(k)$ in p_4. Therefore the branching indices upon a are $\{1, 2, 2p\}$; upon b they are $\{1, 2, 2q\}$; and upon c they are all r. □

COROLLARY 1. *The orbifold* $\mathbf{B}_{r,2p,2q}$, $r \geq 3$, $p \geq 2$, $q \geq 2$ *is universal.* □

REMARK 1. *Theorem* 1 *implies that there exists a hyperbolic orbifold structure in* $(M^3, p_M^{-1}(B), v)$ *such that* p_M *above is an orbifold covering over the hyperbolic orbifold* $\mathbf{B}_{r,2p,2q}$.

COROLLARY 2. *The orbifolds* $\mathbf{B}_{3,4,4}$ *and* $\mathbf{B}_{6,6,6}$ *are universals.* □

It is known that every 3-manifold has a tessellation by regular hyperbolic dodecahedra $\mathcal{D}_{\{4,4,4\}}$ with 90 degrees dihedral angles. This is a consequence of the universal condition of the orbifold $\mathbf{B}_{4,4,4}$. Here $\mathcal{D}_{\{4,4,4\}}$ is the fundamental domain in the hyperbolic space H^3 for the action of the orbifold group $\pi_1^o(\mathbf{B}_{4,4,4})$. See [4]. The following corollary contains an analogous result for the orbifolds $\mathbf{B}_{3,4,4}$ and $\mathbf{B}_{6,6,6}$.

Corollary 3. *Every closed oriented* 3*-manifold has a tessellation by a finite number of hyperbolic dodecahedra* $\mathscr{D}_{\{3,4,4\}}$*, where all the dihedral angles are of* 90 *degrees but for a pair of opposite dihedral angles which are of* 120 *degrees. The possible number of wedges around an edge with a* 90 *degrees dihedral angle is* 1*,* 2 *and* 4*. The possible number of wedges around an edge with a* 120 *degrees dihedral angle is* 1 *and* 3*.*

Every closed oriented 3*-manifold has a tessellation by a finite number of hyperbolic dodecahedra* $\mathscr{D}_{\{6,6,6\}}$*, where all the dihedral angles are of* 90 *degrees but for three pairs of disjoint opposite dihedral angles which are of* 60 *degrees. The number of wedges around an edge with a* 90 *degrees dihedral angle is always* 4*. The possible number of wedges around an edge with a* 60 *degrees dihedral angle is* 1*,* 2*,* 3 *and* 6*.* □

4. Open questions

The orbifold $\mathbf{L}_{12} = (S^3, L, v)$ where L is any non-toroidal 2-bridge knot or link and $v(L) = 12$ is universal ([5]).

Problem 1. *For each such L find the minimal universal orbifold* (S^3, L, v')*.*

As a particular case, we know that $\mathbf{K}_{12}$, where K is the figure-eight knot, is universal, but $\mathbf{K}_3$ is not universal because it is a euclidean orbifold. Is $\mathbf{K}_4$ universal (and therefore minimal universal)?

We know of some minimal universal orbifolds $\mathbf{B}_{m,n,p}$ such as $\mathbf{B}_{4,4,4}$ and $\mathbf{B}_{p,4,4}$, p any odd prime.

Problem 2. *Find the minimal universal orbifolds* $\mathbf{B}_{m,n,p}$*.*

In particular, is $\mathbf{B}_{3,3,3}$ universal (and therefore minimal universal)?

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