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A note on the sheet numbers of twist-spun knots

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ABSTRACT. The sheet number of a 2-knot is a quantity which reflects the complexity of the knotting in 4-space. The aim of this note is to determine the sheet numbers of the 2- and 3-twist-spun trefoils. For this purpose, we give a lower bound of the sheet number by the quandle cocycle invariant of a 2-knot, and an upper bound by the crossing number of a 1-knot.

1. Introduction

An n-knot is an n-sphere smoothly embedded in the Euclidian $(n+2)$-space. We have two kinds of quantities for a 2-knot K which are analogous to the crossing number of a 1-knot. The triple point number $\mathrm{t}(K)$ and the sheet number $\mathrm{sh}(K)$ are the minimal numbers of triple points and sheets for all diagrams of K. There are several studies on these invariants, for example, [9, 13, 14, 15, 19, 20, 21] for $\mathrm{t}(K)$, and [12, 16, 17, 18] for $\mathrm{sh}(K)$. In particular, we have a table of these numbers for "elementary" 2-knots as shown in the following.

K: 2-knot	$\mathrm{t}(K)$	$\mathrm{sh}(K)$
trivial 2-knot	0	1
spun trefoil	0	4
spun figure-eight knot	0	5
spun 5_2-knot	0	6
2-twist-spun trefoil	4	
3-twist-spun trefoil	6	

Moreover, it is known that

- $\mathrm{t}(K) = 0$ if and only if K is a ribbon 2-knot [21],
- $\mathrm{sh}(K) = 1$ if and only if K is a trivial 2-knot [17],

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- $\mathrm{t}(K) \geq 4$ for any non-ribbon 2-knot K [15], and
- $\mathrm{sh}(K) \geq 4$ for any non-trivial 2-knot K [17, 18].

The first four 2-knots in the above table satisfy $\mathrm{t}(K) = 0$, that is, they are ribbon 2-knots. Hence, it is natural to ask the sheet numbers of non-ribbon 2-knots, in particular, the 2- and 3-twist-spun trefoils.

In this paper, we first review the definitions of the quandle homology and cohomology groups in Section 2 and the quandle cocycle invariants of 2-knots in Section 3. In Section 4, we give a lower bound of the sheet number by using the quandle cocycle invariant (Theorem 4.5). We remark that the quandle cocycle invariant is trivial for the family of ribbon 2-knots. Section 5 is devoted to giving an upper bound of the sheet number of a twist-spun knot in terms of the crossing number of a 1-knot (Lemma 5.2 and Theorem 5.3). Combining these results, we determine the sheet numbers of the 2- and 3-twist-spun trefoils to be four and five, respectively (Theorem 6.1).

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2. Quandle (co)homology group

A non-empty set X with a binary operation $(a, b) \mapsto a * b$ is a *quandle* [10, 11] if it satisfies the following:

(i) For any element $a \in X$, it holds that $a * a = a$.

(ii) For any elements a and $b \in X$, there is a unique element $x \in X$ which satisfies $a = x * b$.

(iii) For any elements a, b, and $c \in X$, it holds that $(a * b) * c = (a * c) * (b * c)$.

DEFINITION 2.1. A quandle X is *active* if there is no distinct pair of elements a and $b \in X$ with $a * b = a$.

EXAMPLE 2.2. (i) The set $\mathbf{Z}_n = \{0, 1, \ldots, n-1\}$ with the binary operation $a * b = 2b - a$ modulo n is called the *dihedral quandle* of order n, and denoted by R_n. The quandle R_n is active if and only if n is odd.

(ii) The set $\{0, 1, 2, 3\}$ with the binary operation given in the following table is a quandle which is denoted by S_4. It is easy to see that S_4 is active.

$$\begin{array}{cccc}
0 * 0 = 0, & 0 * 1 = 2, & 0 * 2 = 3, & 0 * 3 = 1, \\
1 * 0 = 3, & 1 * 1 = 1, & 1 * 2 = 0, & 1 * 3 = 2, \\
2 * 0 = 1, & 2 * 1 = 3, & 2 * 2 = 2, & 2 * 3 = 0, \\
3 * 0 = 2, & 3 * 1 = 0, & 3 * 2 = 1, & 3 * 3 = 3.
\end{array}$$

The *associated group* of a quandle X [7, 10], denoted by $G(X)$, is the group which has a presentation

$$G(X) = \langle x \in X \mid x * y = y^{-1}xy \text{ for } x, y \in X \rangle.$$

For a quandle X, an X-*set* [7, 8] is a non-empty set S equipped with a right action $(s, g) \mapsto s \cdot g$ by the associated group $G(X)$; that is,

$$s \cdot e = s \qquad \text{and} \qquad s \cdot (gg') = (s \cdot g) \cdot g'$$

for any $s \in S$, the identity element e of $G(X)$, and any $g, g' \in G(X)$.

Example 2.3. (i) For any quandle X, the set $S = \{0\}$ with the right action $0 \cdot g = 0$ for any $g \in G(X)$ is an X-set.

(ii) For any quandle X, the set $S = \mathbf{Z}_2 = \{0, 1\}$ with the right action $0 \cdot x = 1$ and $1 \cdot x = 0$ for any generator $x \in X$ of $G(X)$ is an X-set.

Let X be a quandle, and S an X-set.

(i) The chain group $C_n^{\mathrm{R}}(X)_S$ is given by

$$C_n^{\mathrm{R}}(X)_S = \begin{cases} \mathbf{Z}[S \times X^n] & n > 0, \\ \mathbf{Z}[S] & n = 0, \\ \{0\} & n < 0, \end{cases}$$

where $\mathbf{Z}[M]$ denotes the free Abelian group generated by a set M. The boundary operation $\partial_n : C_n^{\mathrm{R}}(X)_S \to C_{n-1}^{\mathrm{R}}(X)_S$ is given by

$$\begin{aligned} \partial_n(s; x_1, \dots, x_n) = \sum_{i=1}^{n} (-1)^i \{ &(s; x_1, \dots, \hat{x}_i, \dots, x_n) \\ &- (s \cdot x_i; x_1 * x_i, \dots, x_{i-1} * x_i, \hat{x}_i, x_{i+1}, \dots, x_n) \} \end{aligned}$$

for $n > 0$, and $\partial_n = 0$ for $n \leq 0$. Then $C_*^{\mathrm{R}}(X)_S = \{C_n^{\mathrm{R}}(X)_S, \partial_n\}$ is a chain complex.

(ii) For $n \geq 2$, let $(S \times X^n)_0$ denote the subset of $S \times X^n$ whose elements are $(s; x_1, \dots, x_n)$'s with $x_i = x_{i+1}$ for some i. The chain group $D_*(X)_S$ is given by

$$D_n(X)_S = \begin{cases} \mathbf{Z}[(S \times X^n)_0] & n \geq 2, \\ \{0\} & n \leq 1. \end{cases}$$

Since $\partial_n(D_n(X)_S) \subset D_{n-1}(X)_S$, the pair $D_*(X)_S = \{D_n(X)_S, \partial_n\}$ is a subcomplex of $C_*^{\mathrm{R}}(X)_S$.

(iii) The chain complex $C_*^{\mathrm{Q}}(X)_S$ is given by $C_*^{\mathrm{R}}(X)_S / D_*(X)_S$ as a quotient.

(iv) For an Abelian group A, the chain and cochain groups

$$\begin{cases} C_*^{\mathrm{Q}}(X,A)_S = C_*^{\mathrm{Q}}(X)_S \otimes A, & \text{and} \\ C_{\mathrm{Q}}^*(X,A)_S = \mathrm{Hom}(C_*^{\mathrm{Q}}(X)_S, A), \end{cases}$$

with the coefficient A induce the homology and cohomology groups $H_*^{\mathrm{Q}}(X,A)_S$ and $H_{\mathrm{Q}}^*(X,A)_S$, respectively. They are called the *quandle homology* and cohomology groups, respectively (cf. [2, 3]).

REMARK 2.4. (i) If $A = \mathbf{Z}$, then we abbreviate A such as $H_*^{\mathrm{Q}}(X)_S$ and $H_{\mathrm{Q}}^*(X)_S$.

(ii) If $S = \{0\}$ as given in Example 2.3(i), then we abbreviate S such as $H_*^{\mathrm{Q}}(X,A)$ and $H_{\mathrm{Q}}^*(X,A)$, which are the original quandle (co)homology groups introduced in [1].

DEFINITION 2.5. (i) By definition, any n-chain $\gamma \in C_n^{\mathrm{Q}}(X)_S$ $(n \geq 1)$ can be uniquely represented by

$$\gamma = \sum m_{s;x_1,\dots,x_n}(s;x_1,\dots,x_n),$$

where $m_{s;x_1,\dots,x_n}$'s are integers and all zero except a finite number of them. The sum is taken for all $(s;x_1,\dots,x_n) \in S \times X^n$ with $x_i \neq x_{i+1}$ $(i = 1,\dots,n-1)$. The length of γ is defined by

$$\ell(\gamma) = \sum |m_{s;x_1,\dots,x_n}|.$$

(ii) For an nth homology class $[\gamma] \in H_n^{\mathrm{Q}}(X)_S$, the *length* of $[\gamma]$ is defined by

$$\ell([\gamma]) = \min\{\ell(\gamma') \mid \gamma' \in C_n^{\mathrm{Q}}(X)_S \text{ with } [\gamma'] = [\gamma]\}.$$

(iii) For an nth cohomology class $[\theta] \in H_{\mathrm{Q}}^n(X,A)_S$, the *length* of $[\theta]$ is defined by

$$\ell([\theta]) = \min\{\ell([\gamma]) \mid [\gamma] \in H_n^{\mathrm{Q}}(X)_S \text{ with } \langle[\gamma],[\theta]\rangle \neq 0\},$$

where $\langle\,,\rangle : H_n^{\mathrm{Q}}(X)_S \times H_{\mathrm{Q}}^n(X,A)_S \to A$ is the Kronecker product defined by $\langle[\gamma],[\theta]\rangle = \theta(\gamma)$ by regarding θ as a map $C_n^{\mathrm{Q}}(X)_S \to A$.

3. Quandle cocycle invariant

A *2-knot* K is a 2-sphere embedded in the Euclidian 4-space $\mathbf{R}^4$ smoothly. In this paper, we always assume that K is oriented. Many notions used in 1-knot theory can be extended to the study of 2-knots. The readers who are not familiar with 2-knot theory may refer to [4, 5], for example.

A *diagram* of a 2-knot is the projection image $\pi(K) \subset \mathbf{R}^3$ equipped with crossing information, where $\pi : \mathbf{R}^4 \to \mathbf{R}^3$ is a fixed projection, and any point on $\pi(K)$ may be assumed to be a regular point, double point, an isolated triple point, or an isolated branch point. Usually, we indicate crossing information by dividing the lower disk into two pieces near a double point, and this modification can be extended to neighborhoods of a triple point and a branch point naturally. See Figure 1.

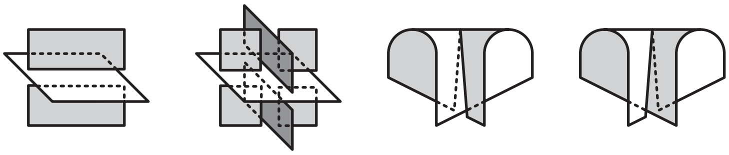


Figure 1

Any 2-knot diagram D is regarded as a disjoint union of compact, connected surfaces, each of which is called a *sheet*. We denote by $sh(D)$ and $t(D)$ the numbers of sheets and triple points of D, respectively.

The *sheet number* and *triple point number* of a 2-knot K is the minimal number of $sh(D)$'s and $t(D)$'s for all diagrams which represents (the ambient isotopy class of) K, and denoted by $\mathrm{sh}(K)$ and t(K), respectively.

Let X be a quandle, and S an X-set. A pair of maps $C = (C_1, C_2)$,

$$\begin{cases} C_1 : \{\text{the sheets of } D\} \to X, \\ C_2 : \{\text{the connected regions of } \mathbf{R}^3 \backslash \pi(K)\} \to S, \end{cases}$$

is an X_S*-coloring* for D if it satisfies the following two conditions:

(1) $C_1(H) * C_1(H') = C_1(H'')$ holds near every double point, where H and H'' are the lower sheets and H' is the upper sheet such that the orientation of H' points from H to H''.

(2) $C_2(R) \cdot C_1(H) = C_2(R')$ holds near every regular point, where R and R' are the regions adjacent to the sheet H such that the orientation of H points from R to R'.

See Figure 2. The elements $C_1(H) \in X$ and $C_2(R) \in S$ are called the *colors* of a sheet H and a region R with respect to C, respectively. An X_S-coloring $C = (C_1, C_2)$ is called *trivial* if C_1 is a constant map, and otherwise *non-trivial*. For $S = \{0\}$, we call an X_S-coloring an X*-coloring* simply.

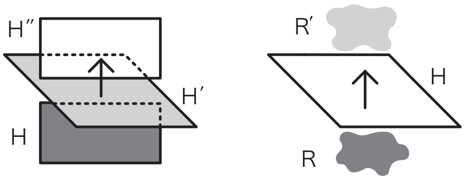


Figure 2

Let t be a triple point of a diagram D with an X_S-coloring C. Among the eight regions near t, the *specified region* R is the one such that all the orientations of the sheets adjacent to R point away from R.

The *color* of t with respect to C is the element $(s; a, b, c) \in S \times X^3$, where s is the color of the specified region R, and a, b, and c are the colors of the bottom, middle, and top sheets adjacent to R, respectively. See Figure 3.

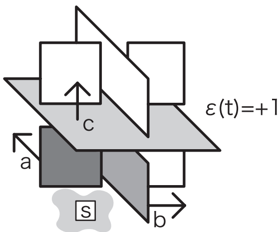


Figure 3

The *sign* of t is positive if the ordered triple of the orientations of the top, middle, and bottom sheets matches with the orientation of $\mathbf{R}^3$, and otherwise negative. We denote it by $\varepsilon(t) \in \{\pm 1\}$.

For a diagram D with an X_S-coloring C, the 3-chain $\gamma_{D,C}$ is defined by

$$\gamma_{D,C} = \sum_t \varepsilon(t) \cdot (s; a, b, c) \in C_3^{\mathrm{R}}(X)_S = \mathbf{Z}[S \times X^3],$$

where the sum is taken for all triple points t of D and $(s; a, b, c)$ is the color of t.

We remark that the 3-chain $\gamma_{D,C}$ is a 3-cycle, that is, $\partial_3(\gamma_{D,C}) = 0$. Hence, it defines a third homology class $[\gamma_{D,C}] \in H_3^{\mathrm{Q}}(X)_S$.

Theorem 3.1 (cf. [1, 3, 7]). (i) *For a diagram D of a 2-knot K, the multi-set*

$$\Psi(D) = \{[\gamma_{D,C}] \in H_3^{\mathrm{Q}}(X)_S \mid C : X_S\text{-colorings for } D\}$$

is independent of a particular choice of D.

(ii) *For a third cohomology class* $[\theta] \in H^3_{\mathrm{Q}}(X, A)_S$*, the multi-set*

$$\Phi_\theta(D) = \{\langle [\gamma_{D,C}], [\theta] \rangle \in A \mid C : X_S\text{-colorings for } D\}$$

is independent of a particular choice of D.

The multi-set $\Phi_\theta(D)$ in Theorem 3.1 is called the *quandle cocycle invariant* of K associated with $[\theta] \in H^3_{\mathrm{Q}}(X, A)_S$, and denoted by $\Phi_\theta(K)$.

Definition 3.2. Let t be a triple point of D with an X_S-coloring C, and $(s; a, b, c) \in S \times X^3$ the color of t. We say that t is *non-degenerated* with respect to C if $a \neq b \neq c$, and *degenerated* if $a = b$ or $b = c$.

Let $t(D, C)$ denote the number of non-degenerated triple points of D with respect to an X_S-coloring C.

Proposition 3.3. *Let* $[\theta] \in H^3_{\mathrm{Q}}(X, A)_S$ *be a third cohomology class. Assume that the quandle cocycle invariant* $\Phi_\theta(K)$ *of a* 2*-knot* K *contains a non-zero element. Then for any diagram* D *of* K*, it holds that* $t(D, C) \geq \ell([\theta])$.

Proof. By assumption, there is an X_S-coloring C for D such that $\langle [\gamma_{D,C}], [\theta] \rangle \neq 0$. Hence, it holds that $\ell([\gamma_{D,C}]) \geq \ell([\theta])$. On the other hand, it follows by definition that $t(D, C) \geq \ell([\gamma_{D,C}])$. Hence, we have $t(D, C) \geq \ell([\theta])$. □

We remark that the number $t(D, C)$ is originally introduced to give a lower bound of the triple point number as follows.

Theorem 3.4 ([20]). *If* $\Phi_\theta(K)$ *contains a non-zero element, then it holds that*

$$\mathrm{t}(K) \geq \ell([\theta]).$$

Proof. Any diagram D of K satisfies $t(D) \geq t(D, C) \geq \ell([\theta])$ by Proposition 3.3. □

4. Lower bound of sheet number

Recall that a diagram D of a 2-knot K is the projection image $\pi(K)$ equipped with crossing information. For a double point p, the preimage $(\pi|_K)^{-1}(p)$ consists of a pair of points, which are called the *lower* and *upper points* with respect to the height function of the projection. Let $\Lambda_- = \Lambda_-(D)$ denote the closure of the lower points in K. The set Λ_- is regarded as a disjoint union of a graph and a finite number of circles embedded in K. In particular, every vertex of Λ_- has degree 1 or 4. More precisely, a branch point b of D gives a 1-valent vertex $b^* = (\pi|_K)^{-1}(b)$ of Λ_-, and a triple point t gives a 4-valent vertex t^* on the bottom disk. We call t^* the *bottom point* of t. See Figure 4, where the solid and dotted lines mean Λ_- and the closure of the set of upper points in K, respectively.

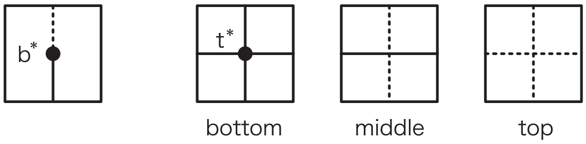


Figure 4

Remark 4.1. (i) The set of the connected regions of the complement $K \backslash \Lambda_-$ has a one-to-one correspondence to the set of the sheets of D.

(ii) If D is X_S-colored, then we give each region of $K \backslash \Lambda_-$ the color assigned to the corresponding sheet of D naturally.

Let C be an X_S-coloring for a diagram D. We define the subgraph $\Lambda_-(C)$ of Λ_- whose edges and circles satisfy the following condition: The regions of $K \backslash \Lambda_-(C)$ on both sides of an edge/circle have different colors with respect to C. In particular, any 1-valent vertex b^* of Λ_- and the edge incident to b^* do not belong to $\Lambda_-(C)$.

Lemma 4.2. *Let X be an active quandle, C an X_S-coloring for a diagram D, and t a triple point of D.*

(i) *If t is a degenerated triple point with respect to C, then the bottom point t^* has degree* 2 *or* 4 *in $\Lambda_-(C)$, or does not belong to $\Lambda_-(C)$.*

(ii) *If t is a non-degenerated triple point, then t^* has degree* 3 *or* 4 *in $\Lambda_-(C)$.*

Proof. Let a_i $(i = 1, 2, 3, 4)$, b_j $(j = 1, 2)$, and c be the colors of the bottom, middle, and top sheets, respectively, and e_k $(k = 1, 2, 3, 4)$ the edges incident to t^* in Λ_- as shown in Figure 5.

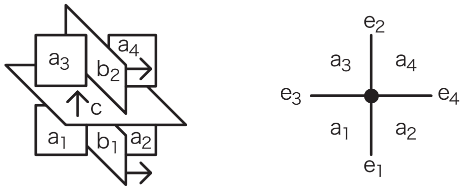


Figure 5

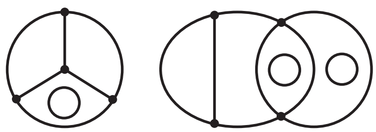
(i) If $a_1 = b_1 = c$, then we have $a_1 = a_2 = a_3 = a_4$. Hence, the four edges $e_1, \dots, e_4$, and the bottom point t^* do not belong to $\Lambda_-(C)$.

If $a_1 = b_1 \neq c$, then we have $a_1 = a_2 \neq a_3 = a_4$. Hence, the edges e_3 and e_4 belong to $\Lambda_-(C)$ and e_1 and e_2 do not belong to $\Lambda_-(C)$. In particular, t^* has degree 2 in $\Lambda_-(C)$.

If $a_1 \neq b_1 = c$, then $a_1 \neq a_2 = a_3 \neq a_4$. Hence, the four edges $e_1, \dots, e_4$ belong to $\Lambda_-(C)$, and t^* has degree 4 in $\Lambda_-(C)$.

(ii) Since $a_1 \neq b_1 \neq c$, we have $a_1 \neq a_2$ and $a_3 \neq a_4$. Hence, the edges e_1 and e_2 belong to $\Lambda_-(C)$. Assume that neither edge of e_3 nor e_4 belong to $\Lambda_-(C)$, that is, $a_1 = a_3$ and $a_2 = a_4$. Since $a_1 * c = a_3$, $a_2 * c = a_4$, and X is active, we have $c = a_1 = a_2$. This contradicts to $a_1 \neq a_2$. Hence, at least one of e_3 and e_4 belongs to $\Lambda_-(C)$, and t^* has degree 3 or 4 in $\Lambda_-(C)$. □

Assume that $\Lambda_-(C)$ is a disjoint union of m connected graphs and n circles. Let v_i $(i = 3, 4)$ denote the number of vertices of degree i, and r the number of the connected regions of $K \backslash \Lambda_-(C)$. The following is easily obtained by the calculation of the Euler characteristic of a 2-sphere. See Figure 6.



r=11, v_3=6, v_4=2, m=2, n=3

Figure 6

Lemma 4.3. $r = \frac{1}{2}v_3 + v_4 + m + n + 1$.

Proposition 4.4. *Let X be an active quandle, and C an X_S-coloring for a diagram D. If $t(D, C) > 0$, then it holds that $sh(D) \geq \frac{1}{2}t(D, C) + 2$.*

Proof. Recall that $sh(D)$ is coincident with the number of the connected regions of the complement $K \backslash \Lambda_-$. Since $\Lambda_-(C) \subset \Lambda_-$, it holds that $sh(D) \geq r$. On the other hand, it follows by Lemma 4.2 that $\frac{1}{2}v_3 + v_4 \geq \frac{1}{2}(v_3 + v_4) \geq \frac{1}{2}t(D, C)$. Furthermore, it holds that $m \geq 1$ by $t(D, C) > 0$. By $n \geq 0$ and Lemma 4.3, we have $sh(D) \geq \frac{1}{2}t(D, C) + 2$ immediately. □

Theorem 4.5. *Let X be an active quandle, and $[\theta] \in H^3_{\mathrm{Q}}(X, A)_S$ a third cohomology class. If the quandle cocycle invariant $\Phi_\theta(K)$ of a 2-knot K contains a non-zero element, then it holds that*

$$\mathrm{sh}(K) \geq \frac{1}{2}\ell([\theta]) + 2.$$

Proof. It follows by Propositions 3.3 and 4.4 that any diagram D of K satisfies $sh(D) \geq \frac{1}{2}t(D, C) + 2 \geq \frac{1}{2}\ell([\theta]) + 2$. $\square$

5. Upper bound of sheet number

Let k be a 1-knot, and r a non-negative integer. We take a tangle T in the upper-half space $\mathbf{R}^3_+ = \{(x, y, z, 0) \mid x, y \in \mathbf{R}, z \geq 0\}$ whose knotting represents the 1-knot k. By spinning $\mathbf{R}^3$ about the axis $\mathbf{R}^2 = \{(x, y, 0, 0) \mid x, y \in \mathbf{R}\}$, we recover the 4-space $\mathbf{R}^4 = \{(x, y, z\cos\theta, z\sin\theta) \mid x, y \in \mathbf{R}, z \geq 0, \theta \in S^1\}$.

We take a 3-ball B in $\mathbf{R}^3_+$ such that the knotting part of T is entirely contained in B. In the spinning process of $\mathbf{R}^3_+$, we simultaneously rotate B r full twists with keeping the points $T \cap \partial B$. The trace of T provides a 2-knot. We call it the *r-twist-spun knot*, and denote it by $\tau^r k$ [22]. See Figure 7.

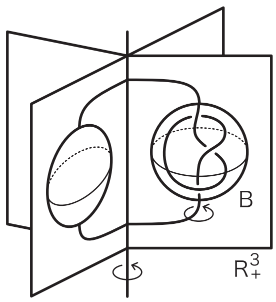


Figure 7

Remark 5.1. (i) The 0-spun knot $\tau^0 k$ is called the *spun knot* simply. The spun knot has a diagram which is obtained from a tangle diagram of k in the upper-half plane by spinning it about the axis. The diagram has neither triple point nor branch point. Hence, we have $\mathrm{t}(\tau^0 k) = 0$ and $\mathrm{sh}(\tau^0 k) \leq \mathrm{c}(k) + 1$, where $\mathrm{c}(k)$ is the crossing number of the 1-knot k.

(ii) Every 1-twist-spun knot is a trivial 2-knot; that is, it bounds a 3-ball embedded in $\mathbf{R}^4$ [22].

(iii) If $r \geq 2$ and k is a non-trivial 1-knot, then $\tau^r k$ is always non-ribbon [6]. Moreover, any diagram of $\tau^r k$ must have at least four triple points [15].

To construct a diagram of a twist-spun knot $\tau^r k$, we consider the sequence of Reidemeister moves for a tangle diagram T of k in the upper-half plane $\mathbf{R}^2_+$ as shown in the upper row of Figure 8. Assume that T has n crossings.

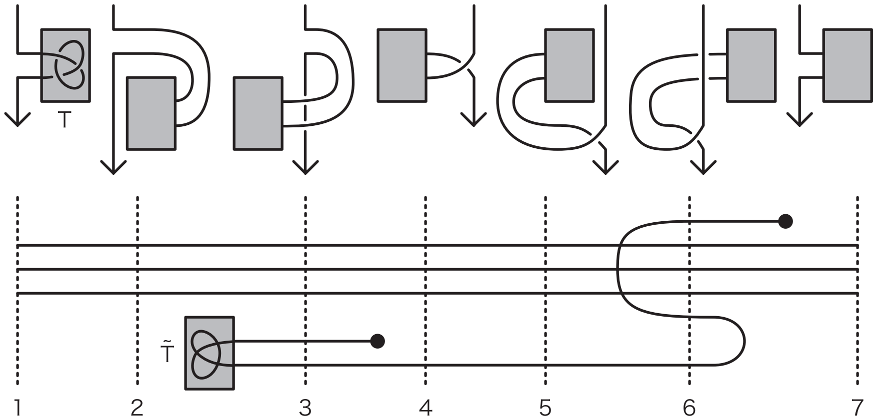


Figure 8

$1 \to 2$: The deformation is realized by an ambient isotopy of $\mathbf{R}^2_+$.
$2 \to 3$: The tangle goes over the terminal path with several Reidemeister moves II and n Reidemeister moves III.
$3 \to 4$: A single Reidemeister move I is performed.
$4 \to 5$: The deformation is realized by an ambient isotopy of $\mathbf{R}^2_+$.
$5 \to 6$: The tangle goes under the initial path with several Reidemeister moves II and n Reidemeister moves III.
$6 \to 7$: A Reidemeister move II and a Reidemeister move I are performed.

It is known that the sequence represents a full twist of the tangle (cf. [20]). We take r copies of the sequence in a pile to obtain a diagram D of $\tau^r k$ in $\mathbf{R}^3$ with open book structure. In particular, Reidemeister moves I and III in the sequence correspond to a branch point and a triple point of D, respectively.

To obtain the set Λ_- from D, we arrange the lower crossings in a line at each stage of the sequence. See the lower row of Figure 8. Here, $\tilde{T}$ indicates the immersed curve obtained from the diagram T by ignoring crossing information, and the number of the parallel curves is equal to n. We put r copies of the trace in a pile to obtain the set Λ_- on a 2-sphere. See Figure 9.

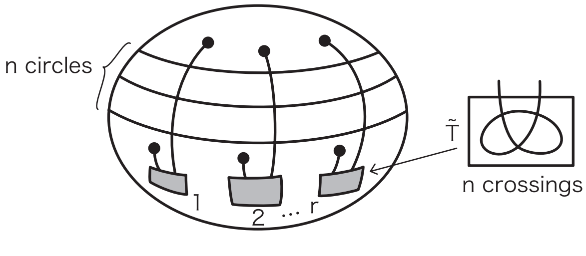


Figure 9

Lemma 5.2. *For a non-trivial* 1-*knot* k *and* $r \geq 2$, *it holds that*

$$\mathrm{sh}(\tau^r k) \leq \{2\mathrm{c}(k) - 1\}r + 2.$$

Proof. We take a tangle diagram of k which realizes the crossing number $n = \mathrm{c}(k)$. For the graph Λ_- constructed as above, it is not difficult to count the number of the connected regions of the complement $\tau^r k \backslash \Lambda_-$ as follows;

$$sh(D) = 1 + (n-1)r + nr + 1 = (2n-1)r + 2.$$

Since $\mathrm{sh}(\tau^r k) \leq sh(D)$, we have the conclusion. □

Assume that a tangle diagram T has a particular pair of crossings labeled a and b as shown in the top-left of Figure 10, where the boxed sub-tangle T'

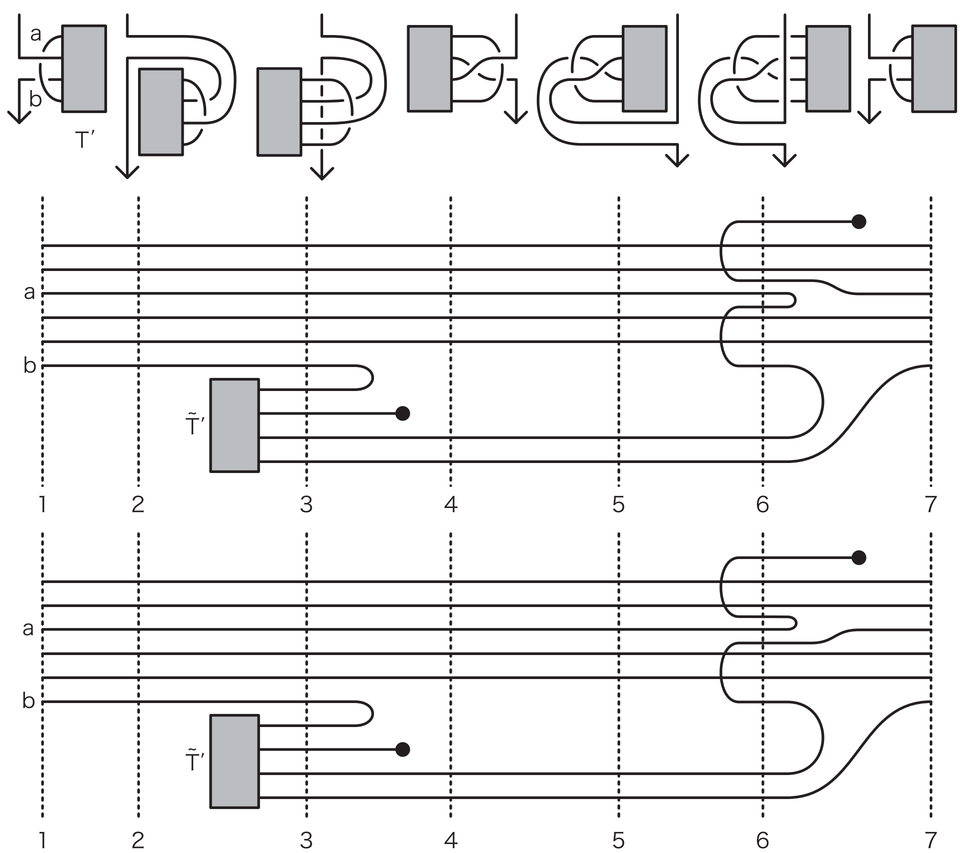


Figure 10

has $n-2$ crossings. We consider the sequence of Reidemeister moves for T as in the top row of the figure.

The sequence is similar to the previous one, and the differences are as follows:

$2 \to 3$: The sub-tangle T' goes over the terminal path with several Reidemeister moves II and $n-2$ Reidemeister moves III.

$3 \to 4$: A Reidemeister move II and a Reidemeister move I are performed.

$5 \to 6$: The sub-tangle T' goes under the initial path with several Reidemeister moves II and $n-2$ Reidemeister moves III.

$6 \to 7$: A pair of Reidemeister moves II and a Reidemeister move I are performed.

It is known that the sequence also represents a full twist of the tangle (cf. [20]). In the middle and bottom rows of Figure 10, we illustrate the trace of the lower crossings arranged in a line at each stage, where the middle row is the case that the under-crossing of a comes before the over-crossing of b with respect to the orientation of T, and the bottom row is the opposite case.

Theorem 5.3. *Suppose that a non-trivial* 1-*knot* k *has a minimal diagram which contains the portion*  *or* . *Then for* $r \geq 2$, *it holds that*

$$\mathrm{sh}(\tau^r k) \leq \{2\mathrm{c}(k) - 5\}r + 2.$$

Proof. We may assume that k has a tangle diagram which contains a sub-tangle T' with $n-2$ crossings as above, where $n = \mathrm{c}(k)$.

We consider the case that the set Λ_- is obtained by taking r copies of the traces in the middle row of Figure 10. The case in the bottom row can be similarly proved. By observing Λ_- as shown in Figure 11, we count the number of the connected regions of the complement $\tau^r k \backslash \Lambda_-$ as follows;

$$sh(D) = 1 + (n-2)r + (n-3)r + 1 = (2n-5)r + 2.$$

Since $\mathrm{sh}(\tau^r k) \leq sh(D)$, we have the conclusion. $\square$

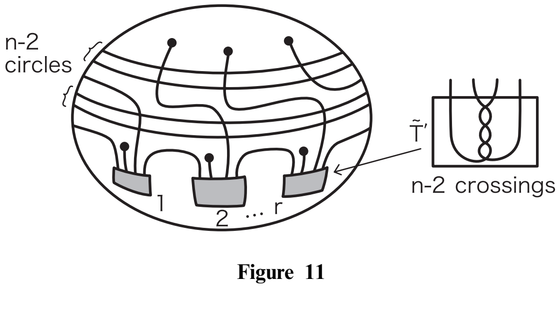


Figure 11

6. 2- and 3-twist-spun trefoils

Let k be the trefoil knot. It follows by Theorem 5.3 that $\mathrm{sh}(\tau^r k) \le r+2$. In particular, we have $\mathrm{sh}(\tau^2 k) \le 4$ and $\mathrm{sh}(\tau^3 k) \le 5$.

THEOREM 6.1. (i) *The* 2-*twist-spun trefoil has the sheet number four.*
(ii) *The* 3-*twist-spun trefoil has the sheet number five.*

PROOF. (i) In [16], we prove that if a 2-knot K admits a non-trivial X-coloring for some quandle X, then it holds that $\mathrm{sh}(K) \ge 4$. Since the 2-twist-spun trefoil $\tau^2 k$ admits a non-trivial R_3-coloring, it holds that $\mathrm{sh}(\tau^2 k) \ge 4$. Hence, we have $\mathrm{sh}(\tau^2 k) = 4$. (Recently, we prove that $\mathrm{sh}(K) \ge 4$ for any non-trivial 2-knot K [17, 18].)

(ii) It is known that $H^3_{\mathrm{Q}}(S_4, \mathbf{Z}_2) \cong (\mathbf{Z}_2)^3$ (cf. [1]). Let $[\theta]$ be a non-zero cohomology class of this group. Then the quandle cocycle invariant of the 3-twist-spun trefoil is given by

$$\Phi_\theta(\tau^3 k) = \{0 \ (4 \text{ times}),\ 1 \ (12 \text{ times})\},$$

which contains a non-zero element.

In [20], we prove that if a homology class $[\gamma] \in H_3^{\mathrm{Q}}(S_4)_{\mathbf{Z}_2}$ satisfies $\langle [\gamma], [\theta] \rangle \ne 0 \in \mathbf{Z}_2$, then it holds that $\ell([\gamma]) \ge 6$ and hence $\ell([\theta]) \ge 6$. Here, the product is taken by regarding $[\theta]$ as a cohomology class of $H^3_{\mathrm{Q}}(S_4, \mathbf{Z}_2)_{\mathbf{Z}_2}$. By Theorem 4.5, we have

$$\mathrm{sh}(\tau^3 k) \ge \frac{1}{2}\ell([\theta]) + 2 \ge 5.$$

Hence, it holds that $\mathrm{sh}(\tau^3 k) = 5$. $\square$

We have an alternative proof of Theorem 6.1(i) similarly to that of (ii). In fact, for a generator $[\theta]$ of $H^3_{\mathrm{Q}}(R_3, \mathbf{Z}_3) \cong \mathbf{Z}_3$, we have

$$\Phi_\theta(\tau^2 k) = \{0 \ (3 \text{ times}),\ 1 \ (6 \text{ times})\}$$

and $\ell([\theta]) = 4$. Hence, it holds that $\mathrm{sh}(\tau^2 k) \ge \frac{4}{2} + 2 = 4$.

QUESTION 6.2. Does the r-twist-spun trefoil have the sheet number $r+2$ for $r \ge 4$?

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