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Holomorphic functions taking values in a quotient of Fréchet-Schwartz spaces

Belmesnaoui AQZZOUZ and Hassan M. EL ALJ

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ABSTRACT. We define a space of holomorphic functions $O_1(U, E|F)$ on a domain of holomorphy U of $\mathbf{C}^n$, taking their values in quotient bornological spaces $E|F$ as the kernel of a sheaf-morphism. We show that if E is a Schwartz b-space and F a Fréchet-Schwartz b-space, then $O_1(U, E|F)$ and $O(U, E) \,|\, O(U, F)$ are naturally isomorphic.

1. Introduction and notation

In studying spectral theory of topological algebras, L. Waelbroeck introduced a class of spaces that he called b-spaces [16], i.e. complete and separate convex bornological vector spaces in the sense of Hogbe Nlend [9], and he succeeded in solving some problems related to the new class. To give the definition of a b-space, we need to recall some definitions.

Let E be a real or complex vector space, and let B be an absolutely convex set of E. Let E_B be the vector space generated by B i.e. $E_B = \bigcup_{\lambda>0} \lambda B$. The Minkowski functional of B is a semi-norm on E_B. It is a norm, if and only if B does not contain any nonzero subspace of E. The set B is said to be completant if its Minkowski functional is a Banach norm.

A bounded structure β on the vector space E is defined by a family of "bounded" subsets of E with the following properties:

(1) Every finite subset of E is bounded.
(2) Every union of two bounded subsets is bounded.
(3) Every subset of a bounded subset is bounded.
(4) A set homothetic to a bounded subset is bounded.
(5) Each bounded subset is contained in a completant bounded subset.

A b-space (E, β) is a vector space E with a boundedness β. A subspace F of a b-space E is bornologically closed if $F \cap E_B$ is closed in E_B for every completant bounded subset B of E.

On the other hand, if U is a domain of holomorphy of $\mathbf{C}^n$, we denote by $O(U)$ the space of holomorphic functions on U endowed with its von

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Neumann boundedness. If E is a b-space, then we define the space of E-valued holomorphic functions on U as the b-space $O(U,E) = \lim_B O(U,E_B)$ where $\lim_B$ is the bornological inductive limit, E_B is the Banach space generated by B, $O(U,E_B)$ is the space of holomorphic functions on U taking their values in E_B and B ranges over bounded completant subsets of E. It is well known that a function f is holomorphic if it is locally holomorphic, in other words, if E is a b-space, then $O(.,E)$ is a sheaf which takes its values in the category of b-spaces.

By using the projective tensor product $\otimes_q$ of G. Noël [12], L. Waelbroeck [17] defined a space of holomorphic functions on a domain of holomorphy U of $\mathbf{C}^n$, which takes values in a quotient Banach space $E|F$ as the space $O(U) \otimes_q (E|F) = (O(U) \hat{\otimes}_{\pi_b} E) \,|\, (O(U) \hat{\otimes}_{\pi_b} F)$ where $\otimes_{\pi_b}$ is the projective tensor product in the category of b-spaces [9]. His definition gave a presheaf and it did not give a sheaf. In 1986, F. H. Vasilescu [14] defined a space of holomorphic functions on a complex manifold U taking their values in a quotient of Fréchet space $E|F$ as $O(U) \hat{\otimes}_\pi (E|F) = (O(U) \hat{\otimes}_\pi E) \,|\, (O(U) \hat{\otimes}_\pi F)$ where $\otimes_\pi$ is the projective tensor product of Grothendieck [7]. In the general situation the definition of Vasilescu gives also a presheaf and not a sheaf.

In [6], we tried to define a space of holomorphic functions which must be a sheaf. For this reason we defined in [6] a new space of holomorphic functions $O(U,E)$ on a domain of holomorphy U of $\mathbf{C}^n$, valued in a b-space E as the kernel of the sheaf-morphism $\bar{\partial} : \mathscr{E}(.,E) \to \mathscr{E}(.,E) \otimes \mathbf{C}^{*n}$, where $\mathbf{C}^{*n}$ is the space of antilinear forms on $\mathbf{C}^n$.

In this paper, we will extend our results in [6] to the category of quotient bornological spaces in the sense of Waelbroeck [19]. In this direction, we will define two spaces of holomorphic functions on a domain of holomorphy U of $\mathbf{C}^n$, taking their values in a quotient bornological space $E|F$. The first one is the space $O(U,E|F) \simeq \lim_V (O(V)\varepsilon(E|F))$ where V ranges over relatively compact subsets of U and ε is the ε-product defined in the category $\mathbf{q}$ [1] and the second one $O_1(U,E|F)$ is the kernel of the sheaf-morphism $\bar{\partial} : \mathscr{E}(.,E|F) \to \mathscr{E}(.,E|F) \otimes \mathbf{C}^{*n}$. $O(.,E|F)$ is also a presheaf. But if $E|F$ is a quotient bornological space such that E is a Schwartz b-space and F is a Fréchet-Schwartz b-space, we will prove that $O(U,E|F) \simeq O(U,E) \,|\, O(U,F)$. Finally, we will prove that in general, the quotient bornological space $O(U,E|F)$ is naturally isomorphic to a subquotient of $O_1(U,E|F)$.

Let us fix some notation and recall some definitions that will be used in this paper. Let **E.V.** be the category of vector spaces and linear mappings over the scalar field **R** or **C**, and **Ban** the subcategory of Banach spaces and bounded linear mappings.

1- Let $(E, \|.\|_E)$ be a Banach space. A Banach subspace F of E is a vector subspace endowed with a Banach norm $\|.\|_F$ such that the inclusion

$(F, \|.\|_F) \to (E, \|.\|_E)$ is continuous. A quotient Banach space $E|F$ is a vector space E/F, where E is a Banach space and F a Banach subspace of E.

Given two quotient Banach spaces $E|F$ and $E_1|F_1$. A strict morphism $u : E|F \to E_1|F_1$ is a linear mapping $u : x + F \mapsto u_1(x) + F_1$ where $u_1 : E \to E_1$ is a bounded linear mapping such that $u_1(F) \subseteq F_1$. We say that u_1 induces u. Two bounded linear mappings $u_1, u_2 : E \to E_1$ which induce a strict morphism, induce the same strict morphism iff $u_1 - u_2$ is a bounded linear mapping $E \to F_1$. A pseudo-isomorphism $u : E|F \to E_1|F_1$ is a strict morphism induced by a surjective bounded linear mapping $u_1 : E \to E_1$ such that $u_1^{-1}(F_1) = F$.

Let $E|F$ be a quotient Banach space and E_0 be a Banach subspace of E such that F is a Banach subspace of E_0. Then the natural injection $E_0 \to E$ induces a strict morphism $E_0|F \to E|F$, and the identity mapping $Id_E : E \to E$ induces a strict morphism $E|F \to E|E_0$.

We call $\tilde{\mathbf{q}}\mathbf{Ban}$ the category of quotient Banach spaces and strict morphisms. It is a subcategory of **EV** and contains the category **Ban** (any Banach space E will be identified with the quotient Banach space $E|\{0\}$, and moreover if $u_1 : E \to E_1$ is a bounded linear mapping, then u_1 induces a strict morphism $E|\{0\} \to E_1|\{0\}$ and every strict morphism $E|\{0\} \to E_1|\{0\}$ is induced by a unique bounded linear mapping $u_1 : E \to E_1$).

The category $\tilde{\mathbf{q}}\mathbf{Ban}$ is not abelian, if E is a Banach space and F a closed subspace of E. It would be very nice if the quotient Banach space $E|F$ is isomorphic to the quotient $(E/F)\,|\,\{0\}$. This is not the case in $\tilde{\mathbf{q}}\mathbf{Ban}$ unless F is complemented in E.

L. Waelbroeck [18] introduced an abelian category **qBan** generated by $\tilde{\mathbf{q}}\mathbf{Ban}$ and inverses of pseudo-isomorphims. It has the same objects as $\tilde{\mathbf{q}}\mathbf{Ban}$. Every morphism u of **qBan** can be expressed as $u = v \circ s^{-1}$, where s is a pseudo-isomorphism and v is a strict morphism. For more information about quotient Banach spaces we refer the reader to [18].

2- In a similar way, we define the category of quotient bornological spaces. Given two b-spaces (E, β_E) and (F, β_F), a linear mapping $u : E \to F$ is bounded, if it maps bounded subsets of E into bounded subsets of F. The mapping $u : E \to F$ is said to be bornologically surjective if for every $B' \in \beta_F$, there exists $B \in \beta_E$ such that $u(B) = B'$.

We denote by $\mathbf{b}(E_1, E_2)$ the space of bounded linear mappings between the b-spaces E_1 and E_2. It is a b-space for the following equibounded boundedness: a subset B of $\mathbf{b}(E_1, E_2)$ is bounded if the set $\{u(x) : u \in B, x \in B'\}$ is bounded in E_2 for all B' bounded in E_1. And we denote by **b** the category of b-spaces and bounded linear mappings. For more information about this category we refer the reader to [9] and [16].

Let (E, β_E) be a b-space. A b-subspace of E is a subspace F with a boundedness β_F such that (F, β_F) is a b-space and $\beta_F \subseteq \beta_E$. A quotient bornological space $E|F$ is a vector space E/F, where E is a b-space and F a b-subspace of E.

Given two quotient bornological spaces $E|F$ and $E_1|F_1$, a strict morphism $u : E|F \to E_1|F_1$ is induced by a bounded linear mapping $u_1 : E \to E_1$ whose restriction to F is a bounded linear mapping $F \to F_1$. Two bounded linear mappings $u_1, v_1 : E \to E_1$, which induce a strict morphism, induce the same strict morphism $E|F \to E_1|F_1$ iff $u_1 - v_1$ is a bounded linear mapping $E \to F_1$. A strict morphism u is a class of equivalence, of bounded linear mappings, for the equivalence just defined.

The class of quotient bornological spaces and strict morphisms is a category, that we call $\tilde{\mathbf{q}}$. A pseudo-isomorphism $u : E|F \to E_1|F_1$ is a strict morphism induced by a bounded linear mapping $u_1 : E \to E_1$ which is bornologically surjective and such that $u_1^{-1}(F_1) = F$ as b-spaces i.e. $B \in \beta_F$ if $B \in \beta_E$ and $u_1(B) \in \beta_{F_1}$).

As in the category $\tilde{\mathbf{q}}\mathbf{Ban}$, there are pseudo-isomorphisms which do not have strict inverses. L. Waelbroeck [19] constructed an abelian category $\mathbf{q}$ that contains $\tilde{\mathbf{q}}$ such that all pseudo-isomorphisms of $\tilde{\mathbf{q}}$ are isomorphisms. For more informations about quotient bornological spaces, we refer the reader to [19].

3- A Banach space E is an $\mathscr{L}_{\infty,\lambda}$-space, $\lambda \geq 1$, if every finite-dimensional subspace F of E is contained in a finite-dimensional subspace F_1 of E such that $d(F_1, l_n^\infty) \leq \lambda$, where $n = \dim F_1, l_n^\infty$ is $\mathbf{K}^n$ ($\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$) with the norm $\sup_{1 \leq i \leq n} |x_i|$, and $d(X, Y) = \inf\{\|T\| \, \|T^{-1}\| : T : X \to Y \text{ isomorphism}\}$ is the Banach-Mazur distance of the Banach spaces X and Y. A Banach space E is an $\mathscr{L}_\infty$-space if it is an $\mathscr{L}_{\infty,\lambda}$-space for some $\lambda \geq 1$. For more information about $\mathscr{L}_\infty$-spaces we refer the reader to [11].

4- Let E and F be two Banach spaces. A bounded linear mapping $u : E \to F$ is nuclear if there exist bounded sequences $(x_n')_n \subset E'$, $(y_n)_n \subset F$ and the one $(\lambda_n) \subset l^1$ such that for all $x \in E$ we have $u(x) = \sum_{n=1}^{+\infty} \lambda_n x_n'(x) y_n$. A b-space G is nuclear if all bounded completant B of G is included in a bounded completant A of G such that the inclusion $i_{AB} : G_B \to G_A$ is a nuclear mapping. For more informations about nuclear b-spaces we refer the reader to [9].

2. Preliminaries

If $E|F$ and $E_1|F_1$ are two quotient bornological spaces, we denote by $\mathbf{q}(E|F, E_1|F_1)$ the quotient bornological space $\mathbf{q}^1(E|F, E_1|F_1) \,|\, \mathbf{q}^0(E|F, E_1|F_1)$, where $\mathbf{q}^1(E|F, E_1|F_1)$ is the space of $f \in \mathbf{b}(E, E_1)$ such that the restriction $f_{|_F} \in \mathbf{b}(F, F_1)$ **satisfies** the following boundedness: a subset B of $\mathbf{q}^1(E|F, E_1|F_1)$ is bounded if it is equibounded in $\mathbf{b}(E, E_1)$ and $B_{|_F} = \{f_{|_F} : f \in B\}$ is equibounded in $\mathbf{b}(F, F_1)$, and $\mathbf{q}^0(E|F, E_1|F_1) = \mathbf{b}(E, F_1)$.

If $E|F, \ldots, E_n|F_n$ are quotient bornological spaces, we define by induction:

$$\mathbf{q}_1(E|F, E_1|F_1) = \mathbf{q}(E|F, E_1|F_1)$$

and

$$\mathbf{q}_n(E|F, \ldots, E_{n-1}|F_{n-1}; E_n|F_n) = \mathbf{q}(E|F, \mathbf{q}_{n-1}(E_1|F_1, \ldots, E_{n-1}|F_{n-1}; E_n|F_n)).$$

The projective tensor product of two b-spaces E and F is the b-space $E \otimes_{\pi_b} F$ defined as $\lim_{B,C}(E_B \hat{\otimes}_\pi F_C)$, where B (resp. C) ranges over bounded completant subsets of E (resp. F). The inductive limit is taken in the category $\mathbf{b}$ and $E_B \hat{\otimes}_\pi F_C$ is the completion of the normed space $(E_B \otimes F_C, \| \ \|_\pi)$ where $\| \ \|_\pi$ is the projective tensor norm given by the formula

$$\|u\|_\pi = \inf\Big\{\sum\nolimits_{k=1}^n \|x_k\| \, \|y_k\| : u = \sum\nolimits_{k=1}^n x_k \otimes y_k\Big\}.$$

Recall the definition of the projective tensor product $\otimes_q$ of G. Noël [12] in the category $\mathbf{q}$. Let $E|F$ and $E_1|F_1$ be two quotient bornological spaces. These spaces have a projective tensor product $(E|F) \otimes_q (E_1|F_1)$ if a quotient bornological space $E_2|F_2$ exists and a functor isomorphism of $\sigma\mathbf{q}_2(E|F, E_1|F_1, .)$ with $\sigma\mathbf{q}(E_2|F_2, .)$. The projective tensor product of $E|F$ and $E_1|F_1$ is naturally isomorphic to $E_2|F_2$. By G. Noël [12], for all couples of quotient bornological spaces $E|F$ and $E_1|F_1$, the projective tensor product $(E|F) \otimes_q (E_1|F_1)$ is defined, and if $u : E|F \to E'|F'$ and $v : E_1|F_1 \to E_1'|F_1'$ are morphisms, then $u \otimes_q v : (E|F) \otimes_q (E_1|F_1) \to (E'|F') \otimes_q (E_1'|F_1')$ is a morphism. The projective tensor product $\otimes_q$ defines a right exact functor $\mathbf{q} \times \mathbf{q} \to \mathbf{q}$.

If X is a set and $E|F$ is a quotient bornological space, G. Noël showed in [12] that

$$l^1(X, E|F) \simeq l^1(X) \otimes_q (E|F) \simeq (l^1(X) \hat{\otimes}_{\pi_b} E) \,|\, (l^1(X) \hat{\otimes}_{\pi_b} F).$$

The ε-product of two Banach spaces E and F is the Banach space $E\varepsilon F$ of linear mappings $E' \to F$ whose restrictions to the closed unit ball $B_{E'}$ of E' are continuous for the topology $\sigma(E', E)$ where E' is the topological dual of E. It follows from the proposition 2 of [15] that the ε-product is symmetric i.e. the Banach spaces $E\varepsilon F$ and $F\varepsilon E$ are isometrically isomorphic. If E_i and F_i are Banach spaces and $u_i : E_i \to F_i$ are bounded linear mappings, $i = 1, 2$, the ε-product of u_1 and u_2 is the bounded linear mapping $u_1\varepsilon u_2 : E_1\varepsilon E_2 \to F_1\varepsilon F_2$, $f \mapsto u_2 \circ f \circ u_1'$, where u_1' is the dual mapping of u_1. It is clear that $u_1\varepsilon u_2$ is injective when u_1 and u_2 are injections. If G and E are Banach spaces and F is a Banach subspace of E, then $G\varepsilon F$ is a Banach subspace of $G\varepsilon E$. For more informations about the ε-product the reader is refered to [15].

Recall from [2] that the ε-product $G\varepsilon E$ of a b-space G by a Banach space E is defined as the b-space $\bigcup_B (G_B \varepsilon E)$ where B ranges over bounded completant subsets of the b-space G. If F is a b-subspace in G, the space $F\varepsilon E$ is a b-subspace in $G\varepsilon E$. Now, if G and E are two b-spaces, the ε-product of G and E is the b-space $G\varepsilon E = \bigcup_{B,C} (G_B \varepsilon E_C)$ where B (resp. C) ranges over bounded completant subsets of the b-spaces G (resp. E).

If U is an open subset of $\mathbf{C}^n$ and E is a b-space, the b-space of E-valued holomorphic functions on U is defined as the b-space $O(U,E) = \lim_B O(U,E_B)$ where $\lim_B$ is the bornological inductive limit and B ranges over bounded completant subsets of the b-spaces E. Since $O(U,E_B) \simeq O(U)\varepsilon E_B$, we obtain $O(U,E) \simeq O(U)\varepsilon E$.

Also, we recall that for each Banach space E, we have $c_0 \varepsilon E \simeq c_0(E)$. Since the inductive limit is an exact functor, it follows that if E is a b-space, we have $c_0 \varepsilon E \simeq c_0(E)$ where c_0 is the Banach space of all sequences which converge to 0.

In [1], we defined the ε-product of an $\mathscr{L}_\infty$-space G by a quotient Banach space $E|F$ as the quotient Banach space $G\varepsilon(E|F) = (G\varepsilon E)\,|\,(G\varepsilon F)$. By Proposition 6.2 of [1], the functor $G\varepsilon. : \mathbf{b} \to \mathbf{b}$ is exact, and it follows from Theorem 4.1 of [19], that this functor admits an exact extension $G\varepsilon. : \mathbf{q} \to \mathbf{q}$. This shows that if $E|F$ is a quotient bornological space, then $G\varepsilon(E|F) = (G\varepsilon E)\,|\,(G\varepsilon F)$.

Recall that a Banach space H has the approximation property if the identity mapping $Id_H : H \to H$ belongs to the closure of $(H)' \otimes H$ in the topology of the uniform convergence on the compact subsets of the Banach space H.

The following result shows that for nuclear b-spaces, our ε-product defined in [1] is isomorphic to the projective tensor product $\otimes_q$ of G. Noël [12].

THEOREM 2.1. *Let N be a nuclear b-space and $E|F$ be a quotient bornological space. Then $G \otimes_q (E|F) \simeq G\varepsilon(E|F)$.*

PROOF. If N is a nuclear b-space, then by [9], we have $N = \lim_B N_B$ where each Banach space N_B is isometrically isomorphic to the $\mathscr{L}_\infty$-space c_0. Since each functor $N_B\varepsilon. : \mathbf{b} \to \mathbf{b}$ is exact and the inductive limit $\lim_B$ is an exact functor on the category $\mathbf{b}$, the functor $N\varepsilon. = \lim_B (N_B \varepsilon.) : \mathbf{b} \to \mathbf{b}$ is exact. Now, it follows from Theorem 4.1 of [19], that this functor has an exact extension $N\varepsilon. : \mathbf{q} \to \mathbf{q}$. Then for every quotient bornological space $E|F$, we have $N\varepsilon(E|F) = (N\varepsilon E)\,|\,(N\varepsilon F)$.

On the other hand, since N is a nuclear b-space, it follows from [9] that $N\varepsilon E = N \hat{\otimes}_{\pi_b} E$. Hence $N\varepsilon(E|F) = (N \hat{\otimes}_{\pi_b} E)\,|\,(N \hat{\otimes}_{\pi_b} F)$. Now, by [12], we have $G \otimes_q (E|F) = (N \hat{\otimes}_{\pi_b} E)\,|\,(N \hat{\otimes}_{\pi_b} F)$. This establishes the result.

As a consequence, if U is an open subset of $\mathbf{C}^n$, the b-space $O(U)$ is nuclear for its von Neumann boundedness, and then we obtain

$$O(U) \otimes_q (E|F) \simeq O(U)\varepsilon(E|F) \simeq (O(U)\varepsilon E) \,|\, (O(U)\varepsilon F) \simeq O(U,E) \,|\, O(U,F).$$

3. Definition of the presheaf $O(.,E|F)$

Let U be an open subset of $\mathbf{R}^n$ and let $\mathscr{C}_U$ be the set of all open relatively compact subsets of U. If $V \in \mathscr{C}_U$, the space $O(V)$ with its von Neumann boundedness is a nuclear b-space, and then defines an exact functor $O(V)\varepsilon. = O(V,.)$ on the category **b**. If E is a b-space and F is a bornologically closed subspace of E, the b-space

$$O(V,E/F) = O(V)\varepsilon(E/F)$$

is defined as

$$(O(V)\varepsilon E)/(O(V)\varepsilon F) = O(V,E)/O(V,F).$$

If $W, V \in \mathscr{C}_U$ such that $W \subset V$, we have a bounded linear mapping

$$\Psi : O(V) \to O(W), \qquad f \mapsto f_{|_W}$$

where $f_{|_W}$ is the restriction of f to W. We can show that $(O(V))_{V \in \mathscr{C}_U}$ is a projective system in the category **b**. If E is a b-space the family $(O(V)\varepsilon E)_{V \in \mathscr{C}_U}$ is also a projective system in **b**, and then has a projective limit in the category **b**.

We define

$$O(U,E) = \lim_{V \in \mathscr{C}_U}(O(V)\varepsilon E).$$

Also we define the presheaf $O(.,E|F)$.

Definition 3.1. *Let U be an open subset of $\mathbf{C}^n$ and $E|F$ be a quotient bornological space. Then we define the space of holomorphic functions $O(U,E|F)$ as the quotient bornological space $\lim_V(O(V)\varepsilon(E|F))$ where V ranges over open relatively compact subsets of U.*

It is clear that $O(U,E|F) = \lim_V((O(V)\varepsilon E) \,|\, (O(V)\varepsilon F))$. To prove that $\lim_V((O(V)\varepsilon E) \,|\, (O(V)\varepsilon F)) = \lim_V(O(V)\varepsilon E) \,|\, \lim_V(O(V)\varepsilon F)$, we need to recall from [3] some definitions.

The boundedness of a Fréchet space has a property that a general bornology does not have. b-Spaces whose boundedness have this property were called Fréchet b-spaces in [3].

Definition 3.2. *A b-space E is a Fréchet b-space if for all sequences of bounded subsets $(B_n)_n$ of E, there exists a sequence of positive real numbers $(\lambda_n)_n$ such that $\bigcup_n \lambda_n B_n$ is bounded in E.*

If $U' \subset U$ is open, the morphism $O_1(U, E|F) \to O_1(U', E|F)$ is the projective limit of the restrictions $O(V)\varepsilon(E|F) \to O(V')\varepsilon(E|F)$ with V open, relatively compact in U, V' open, relatively compact in U' and $V' \subset V$. It follows that $O_1(., E|F)$ is a presheaf.

Recall that both Borel and Mittag-Leffler [10] considered a class of mappings with a dense range. In [3], we studied this class that we called "approximatively surjective mappings".

DEFINITION 3.3. *Let (E, β_E) and (F, β_F) be b-spaces. A bounded linear mapping $u : E \to F$ is approximatively surjective if for each completant bounded subset $B \in \beta_F$, there exist bounded completant bounded subsets $B_1 \in \beta_F$ and $C \in \beta_E$ such that $B \subset B_1$, $u(C) \subset B_1$ and for every $\varepsilon > 0$, we have $B_1 \subset \varepsilon B_1 + \bigcup_M Mu(C)$.*

It is clear that in the Banach case, a mapping is approximatively surjective if and only if it has a dense range.

For such a class of mappings, we proved in (cf. [3]) a version of Bartle-Graves theorem.

THEOREM 3.4 (cf. [3]). *Let $u : E \to F$ be an approximatively surjective bounded linear mapping between b-spaces and X a compact space. The bounded linear mapping $C(X, u) : C(X, E) \to C(X, F)$, $f \mapsto u \circ f$ is approximatively surjective.*

Theorem 3.4 is useful to establish the exactness of the projective limit functor on the category of b-spaces as the following Theorem shows:

THEOREM 3.5 [3]. *Let (E_n) and (F_n) be projective systems in the category of b-spaces such that for each $n \in \mathbf{N}$, F_n is a Fréchet b-space which is a bornologically closed subspace of E_n. For each $n \in \mathbf{N}$, let $u_{n+1} : E_{n+1} \to E_n$ be a bounded linear mapping whose restriction $v_{n+1} = u_{n+1|_{F_{n+1}}} : F_{n+1} \to F_n$ is an approximatively surjective bounded linear mapping. Then* $\lim_n(E_n/F_n) \simeq (\lim_n E_n)/(\lim_n F_n)$.

As an immediate consequence, we obtain an analogue in the category of quotients bornological spaces.

COROLLARY 3.6. *For each $n \in \mathbf{N}$, let E_n be a b-space and F_n be a Fréchet b-space which is a b-subspace of E_n and let $u_{n+1} : E_{n+1} \to E_n$ be a bounded linear mapping whose restriction $v_{n+1} = u_{n+1|_{F_{n+1}}} : F_{n+1} \to F_n$ is an approximatively surjective bounded linear mapping. Then* $\lim_n(E_n|F_n) \simeq (\lim_n E_n) \,|\, (\lim_n F_n)$.

PROOF. In fact, the projective limit functor $\lim_n$ is exact on the category of b-spaces **b**, hence by Theorem 4.1 of [19], the functor $\lim_n$ admit an exact extension to the category **q**. This proves the result.

If G is a b-space, we denote by G_c the space G that we endow with its Schwartz boundedness (i.e. a subset A of G is bounded if there exists a completant bounded subset B of G such that A is compact in the Banach space G_B). The space G_c is a Schwartz b-space. If G is a Schwartz b-space, then $G = G_c$.

Our first principal result is the following:

THEOREM 3.7. *Let U be an open subset of $\mathbf{C}^n$ and $E|F$ be a quotient bornological space such that E is a Schwartz b-space and F is a Fréchet-Schwartz b-space. Then the quotient bornological spaces $O(U, E|F)$ and $O(U,E)\,|\,O(U,F)$ are naturally isomorphic.*

PROOF. The set U is not assumed to be a domain of holomorphy of $\mathbf{C}^n$. Let $\tilde{U} = V$ be its associated domain of holomorphy. In V, each compact subset L is contained in a compact and holomorphically convex subset of V, then V is the union of a sequence of compact subset K_n such that, for each n, we have $K_n \subset \dot{K}_{n+1}$ and K_n is a holomorphically convex subset of V where $\dot{K}_{n+1}$ is the interior of K_{n+1}.

It is well known that the Runge Theorem implies that the restriction $O(U,G) \to O(V,G)$ has a dense range whenever G is a Banach space and V is holomorphically convex.

On the other hand, let $E|F$ be a quotient bornological space. Since F is a b-space, the restriction $O(K_{n+1})\varepsilon F \to O(K_n)\varepsilon F$ is an approximatively surjective mapping. Now, $E|F$ defines the following exact sequence in $\mathbf{q}$:

$$0 \to F \to E \to E|F \to 0.$$

Its image by each exact functor $O(K_n)\varepsilon. : \mathbf{q} \to \mathbf{q}$ is the following exact sequence:

$$0 \to O(K_n)\varepsilon F \to O(K_n)\varepsilon E \to O(K_n)\varepsilon(E|F) \to 0.$$

We obtain then the following infinite commutative diagram:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & O(K_{n+1})\varepsilon F \arrow[r] \arrow[d] & O(K_{n+1})\varepsilon E \arrow[r] \arrow[d] & (O(K_{n+1})\varepsilon E \,|\, O(K_{n+1})\varepsilon F) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & O(K_n)\varepsilon F \arrow[r] \arrow[d] & O(K_n)\varepsilon E \arrow[r] \arrow[d] & (O(K_n)\varepsilon E \,|\, O(K_n)\varepsilon F) \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

where the rows are exact and the vertical arrows $O(K_{n+1})\varepsilon F \to O(K_n)\varepsilon F$ are approximatively surjective for each n. Since each b-space F_n is a Fréchet b-space, it follows from Theorem 2.8 that

$$\lim_n(E_n|F_n) \simeq (\lim_n E_n) \,|\, (\lim_n F_n).$$

By Theorem 2.7, the bounded linear mapping

$$C(K, u_{n+1}) : C(K, E_{n+1}) \to C(K, E_n)$$

is approximatively surjective if K is compact and $u_{n+1} : E_{n+1} \to E_n$ is an approximatively surjective mapping. It follows that

$$C(K, \lim_n(E_n|F_n)) \simeq C(K, (\lim_n E_n)) \,|\, C(K, (\lim_n F_n)).$$

Because we assume that each E_n and F_n has a Schwartz boundedness, it follows that $E = \bigcup_A E_A = \bigcup_A (E_A)_c$ (resp. $F = \bigcup_B F_B = \bigcup_B (F_B)_c$) where A (resp. B) ranges over bounded completant subsets of E (resp. F) and $(E_A)_c$ (resp. $(F_B)_c$) is the space E_A (resp. F_B) with its Schwartz boundedness. The bounded linear mapping

$$O(K_{n+1}, F) \to O(K_n, F)$$

is then approximatively surjective and therefore

$$\lim_n(O(K_n, E) \,|\, O(K_n, F)) = O(U, E|F) \simeq O(U, E) \,|\, O(U, F)$$

and the Theorem 3.7 is proved.

4. The sheaf $O_1(., E|F)$

To give the definition of the sheaf of holomorphic functions $O_1(., E|F)$, we recall that in [4], we defined several sheaves of functions which take the values in a quotient bornological space $E|F$, such as, $C(., E|F)$, $C^r(., E|F)$ if $r \in \mathbf{R}^+ \backslash \mathbf{N}$, $C_b(., E|F)$, $C_e(., E|F)$ and $\theta(., E|F)$. By [1], for every quotient bornological space $E|F$, we have $\mathscr{F}(X, E|F) = \mathscr{F}(X, E) \,|\, \mathscr{F}(X, F)$ where $\mathscr{F}(X) = C(X), C_b(X), C_e(X)$ and $\theta(\mathbf{R}, w_o)$.

In this paper, we need to use the sheaf $\mathscr{E}(., E|F)$. Recall that the space of holomorphic functions that L. Waelbroeck [17] defined as $O(.) \otimes_q (E|F)$ is a presheaf but not a sheaf. In view of this we defined in [6] another space of holomorphic functions $O_1(U, E)$ which define a sheaf on the category **b**. To extend it to the category of quotient bornological spaces **q**, we need to recall first the space $\mathscr{E}(U, E)$ when E is a b-space [6].

The elements of $\mathscr{E}(U, E)$ are functions $f : U \to E$ such that for all $x \in U$, there exist a coordinate neighbourhood U_x of x and a completant bounded subset B_x of E such that $f_{|U_x} \in C^\infty(U_x, E_{B_x})$. A subset C of $\mathscr{E}(U,E)$ is bounded if for every $x \in U$, there exist a neighbourhood U_x of x and a bounded completant subset B_x of E such that $C_{|_{U_x}} = \{f_{|_{U_x}} : f \in C\}$ is bounded in the Fréchet space $C^\infty(U_x, E_{B_x})$. For $E = \mathbf{C}$, one writes $\mathscr{E}(U)$ instead of $\mathscr{E}(U, \mathbf{C})$.

By Proposition 2.1 of [6], if $u : E \to F$ is a bornologically surjective bounded linear mapping between b-spaces, then the bounded linear mapping $\mathscr{E}(U,u) : \mathscr{E}(U,E) \to \mathscr{E}(U,F)$, $f \mapsto u \circ f$ is bornologically surjective. Hence, the functor $\mathscr{E}(U,.) : \mathbf{b} \to \mathbf{b}$ is exact. Now, Proposition 4.1 of [19] implies that this functor has an exact extension $\mathscr{E}(U,.) : \mathbf{q} \to \mathbf{q}$. As a consequence, we obtain

$$\mathscr{E}(U, E|F) \simeq \mathscr{E}(U)\varepsilon(E|F) \simeq \mathscr{E}(U,E) \,|\, \mathscr{E}(U,F).$$

Let X be a topological space. We define a category $\mathbf{Open}_X$ whose objects are the open subsets of X such that if Y and Z are open subsets of X such that $Z \subset Y$, then a unique morphism $i_{YZ} : Z \to Y$ exists. If $K \subset Z \subset Y$, then the composition of the two morphisms $i_{ZK} : K \to Z$ and $i_{YZ} : Z \to Y$ is the unique morphism $i_{YK} : K \to Y$. The category $\mathbf{Open}_X^{op}$ is the opposite category to $\mathbf{Open}_X$.

To give the definition of the sheaf of holomorphic functions $O_1(., E|F)$, we need the following lemma:

Lemma 4.1. *Let X be a topological space. If $\mathscr{F}_1$ and $\mathscr{F}_2$ are sheaves $\mathbf{Open}_X^{op} \to \mathbf{q}$ and $u : \mathscr{F}_1 \to \mathscr{F}_2$ is a morphism of sheaves, then* $\mathrm{Ker}(u)(.)$ *is a sheaf.*

Proof. Let U be an open subset of a topological space X. If $\mathscr{F}_1$ and $\mathscr{F}_2$ are presheaves and $u : \mathscr{F}_1 \to \mathscr{F}_2$ is a morphism of presheaves then $\ker(u(U))$ is the kernel of $u(U) : \mathscr{F}_1(U) \to \mathscr{F}_2(U)$. If $V \subset U$, we have a morphism $\mathrm{Ker}(u(U)) \to \mathrm{Ker}(u(V))$ i.e. $\mathrm{Ker}(u(.))$ is a presheaf.

Let $\mathscr{G}(U) \simeq \mathrm{Ker}(u(U))$ and let $(U_i)_{i \in I}$ be an open covering of U. For all $i \in I$, we have a morphism $\mathscr{G}(U) \to \mathscr{G}(U_i)$ which is the "restriction morphism" given by the structure of the presheaf, and hence we define a morphism

$$\delta_{0(U_i)} :\ \mathscr{G}(U) \to \prod\nolimits_i \mathscr{G}(U_i)$$

as the direct product of the restriction morphisms $\mathscr{G}(U) \to \mathscr{G}(U_i)$. We shall need a second morphism

$$\delta_{1(U_i)} : \prod\nolimits_i \mathscr{G}(U_i) \to \prod\nolimits_{i,j} \mathscr{G}(U \cap U_j).$$

To define it, we first observe that $U_i = \bigcup_j (U_i \cap U_j)$. Hence we have a morphism $\mathscr{G}(U_i) \to \prod_j \mathscr{G}(U_i \cap U_j)$, and then the morphism $\prod_i \mathscr{G}(U_i) \to \prod_{i,j} \mathscr{G}(U_i \cap U_j)$.

Instead of looking at U_i, we could consider U_j, $U_j = \bigcup_i (U_i \cap U_j)$. We consider a morphism

$$G(U_j) \to \prod_i G(U_i \cap U_j)$$

and therefore a morphism

$$\prod_j \mathscr{G}(U_j) \to \prod_{i,j} \mathscr{G}(U_i \cap U_j).$$

Note that $\prod_i \mathscr{G}(U_i) = \prod_j \mathscr{G}(U_j)$. In this way, we obtain a second morphism $\prod_i \mathscr{G}(U_i) \to \prod_{i,j} \mathscr{G}(U_i \cap U_j)$. The morphism

$$\delta_{1(U_j)} : \prod_i \mathscr{G}(U_i) \to \prod_{i,j} \mathscr{G}(U_i \cap U_j)$$

is the difference between the two morphisms described above. It is clear that $\delta_{1(U_i)} \circ \delta_{0(U_j)} = 0$.

To prove that $\mathrm{Ker}(u)(.)$ is a sheaf whenever $\mathscr{F}_1$ and $\mathscr{F}_2$ are sheaves, we use a 3×3 Lemma in [13]. The following diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathscr{G}(U) & \longrightarrow & \Pi_i \mathscr{G}(U_i) & \longrightarrow & \Pi_{i,j} \mathscr{G}(U_i \cap U_j) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathscr{F}_1(U) & \longrightarrow & \Pi_i \mathscr{F}_1(U_i) & \longrightarrow & \Pi_{i,j} \mathscr{F}_1(U_i \cap U_j) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathscr{F}_2^{0,1}(U) & \longrightarrow & \Pi_i \mathscr{F}_2^{0,1}(U_i) & \longrightarrow & \Pi_{i,j} \mathscr{F}_2^{0,1}(U_i \cap U_j)
\end{array}$$

is commutative. Since its three columns and its second and third rows are left exact, the first row is left exact, and then $\mathscr{G}(.)$ is a sheaf.

In our definition we shall use the quotient bornological space $\mathscr{E}(U, E|F)$. For this purpose we first prove that $\mathscr{E}(., E|F)$ is a sheaf.

In fact, if $U' \subset U$, we have a natural morphism $\mathscr{E}(U, E|F) \to \mathscr{E}(U', E|F)$. It is clear that $\mathscr{E}(., E|F)$ is a presheaf.

THEOREM 4.2. *Let U be an open subset of $\mathbf{C}^n$ and $E|F$ a quotient bornological space. Then the presheaf $\mathscr{E}(., E|F)$ is a sheaf.*

Proof. We must show that the preshef $\mathscr{E}(.,E|F)$ is a sheaf. We consider an open covering (U_i) of U. We assume that (f_i) is a system with $f_i \in \mathscr{E}(U_i, E)$ such that

$$f_{i|U_i \cap U_j} - f_{j|U_i \cap U_j} \in \mathscr{E}(U_i \cap U_j, F).$$

Using a partition of unity (φ_i) subordinate to the open cover (U_i) of U, we let $f = \sum \varphi_i f_i$. We have $f = \sum \varphi_i f_i \in \mathscr{E}(U, F)$. In a similar way B is bounded in $\mathscr{E}(U,F)$ if it is bounded in $\mathscr{E}(U,E)$ and for every i, the set $B_{|U_i} = \{f_{|U_i} = f_i : f \in B\}$ is bounded in $\mathscr{E}(U_i, F)$. In this way the morphism $\delta_{0(U_i)}$ is monic.

Let $x \in U$, there exists a neighbourhood W of x which meets only a finite number of supports of the functions (φ_i). Consider

$$f_{|W} - f_{i|W} = \sum\nolimits_j \varphi_{j|W}(f_{j|W} - f_{i|W}).$$

We know that on W, we have

$$f_{|W} - f_{i|W} \in \mathscr{E}(W, F).$$

We wish also to prove that the kernel of $\delta_{1(U_i)}$ is naturally isomorphic to the coimage of δ_0. Again the b-space in the definition of the kernel has as bounded subsets the ranges of mappings $W \to \prod_i \mathscr{E}(U_i, E)$ such that the differences of the restrictions to $U_i \cap U_j$ are bounded in $\mathscr{E}(U_i \cap U_j, F)$. The same construction gives a bounded mapping of W into $\mathscr{E}(U,E)$ such that $\forall i : g_{|W} - g_{i|W}$ is a bounded mapping from W into $\mathscr{E}(W,F)$. Therefore the sequence $(0, \delta_{0(U_i)}, \delta_{1(U_i)})$ is left exact, and the presheaf $\mathscr{E}(.,E|F)$ is a sheaf.

Now, we are in position to give the definition of the sheaf $O_1(.,E|F)$.

Definition 4.3. *Let $E|F$ be a quotient bornological space. The sheaf $O_1(.,E|F)$ is the kernel of the sheaf-morphism $\bar{\partial} : \mathscr{E}(.,E|F) \to \mathscr{E}(.,E|F) \otimes_q \mathbf{C}^{*n}$, where $\mathbf{C}^{*n}$ is the space of antilinear forms on $\mathbf{C}^n$ and $\otimes_q$ is the projective tensor product in* $\mathbf{q}$.

Theorem 4.4. *Let $E|F$ be a quotient bornological space such that E is a Schwartz b-space and F is a Fréchet-Schwartz b-space and let U be an open subset of $\mathbf{C}^n$. Then the quotient bornological space $O(U,E|F)$ is naturally isomorphic to a subquotient of $O_1(U,E|F)$.*

Proof. Let V be an open relatively compact subset of U. Since the b-spaces $O(V)$ and $\mathscr{E}(V)$ are nuclear, then $O(V)\varepsilon(E|F) = (O(V)\varepsilon E) \,|\, (O(V)\varepsilon F)$ and $\mathscr{E}(V)\varepsilon(E|F) = (\mathscr{E}(V)\varepsilon E) \,|\, (\mathscr{E}(V)\varepsilon F)$. On the other hand, we have an injection $i : O(V) \to \mathscr{E}(V)$, and then the morphisms $i_E : O(V,E) \to \mathscr{E}(V,E)$

and $i_F : O(V,F) \to \mathscr{E}(V,F)$ are injectives such that the restriction of i_E to $O(V,F)$ coincides with i_F. Hence the bounded linear mapping i_E induces a strict morphism $(O(V)\varepsilon E) \mid (O(V)\varepsilon F) \to (\mathscr{E}(V)\varepsilon E) \mid (\mathscr{E}(V)\varepsilon F)$. We have to prove that it is monic. This is equivalent to showing that $O(V,F) = O(V,E) \cap \mathscr{E}(V,F)$ where the equality is bornological.

In fact, in one dimension, we use Morera's Theorem. Let $V \subset \mathbf{C}$ be open and simply connected and $f \in O(V,E) \cap \mathscr{E}(V,F)$. Let $z_o \in V$. Then f has a primitive

$$F(z) = \int_\gamma f(t)dt.$$

It is continuous, F-valued and of class C^1 as an F-valued function.

It satisfies the Cauchy-Riemann relations. It is holomorphic, F-valued. Its derivative f is also holomorphic, F-valued. A bounded subset of $O(V,E)$ which is bounded in $\mathscr{E}(V,F)$ is in a similar way bounded in $O(V,F)$. If V is not simply connected, it is locally simply connected, and its holomorphy is local.

Consider $f \in O(V,E) \cap \mathscr{E}(V,F)$. Then there exists a completant bounded subset B of E such that $f \in O(V,E_B)$. By Hartog's Theorem, applied to holomorphic functions taking their values in the Banach space E_B, the function f is continuous and separately analytic. Hence $f \in O(V,F)$. The same proof shows that bounded subsets of $O(V,E) \cap \mathscr{E}(V,F)$ are bounded in $O(V,F)$.

Now, as $O_1(V,E|F)$ is the kernel of the sheaf-morphism $\bar{\partial} : \mathscr{E}(V,E|F) \to \mathscr{E}(V,E|F) \otimes_q \mathbf{C}^{*n}$, it follows that $O(V)\varepsilon(E|F)$ is a subquotient of the quotient bornological space $O_1(V,E|F)$.

Finally, since $O(U,E|F)$ is the quotient bornological space $\lim_V(O(V)\varepsilon(E|F))$ where V ranges over open relatively compact subsets of U (definition 3.1) and the quotient bornological spaces $O(U,E|F)$ and $O(U,E) \mid O(U,F)$ are naturally isomorphic whenever E is a Schwartz b-space and F is a Fréchet-Schwartz b-space (Theorem 3.7), it follows that the morphism

$$O_1(U,E|F) \to O(U,E|F)$$

is monic because it is the projective limit of the monic morphisms $O(V)\varepsilon(E|F) \to O_1(V,E|F))$.

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References

[1] B. Aqzzouz, The ε_c-product of a Schwartz b-space by a quotient Banach space and applications. Appl. Categ. Structures 10 (6) (2002), 603–616.

[2] B. Aqzzouz, On some isomorphism on the category of b-spaces. Siberian Mathematical Journal, volume 44, (2003), 749–756.

[3] B. Aqzzouz and R. Nouira, Bartle and Graves Theorem for approximatively surjective mappings between b-spaces. Methods Funct. Anal. Topology, vol. 11, no. 4, (2005), 320–326.

[4] B. Aqzzouz, H. EL Alj and R. Nouira, Sur le Théorème de Bartle-Graves dans la catégorie des quotients bornologiques. Mathematica Bohemica, vol. 130, No. 4, (2005), 371–385.

[5] B. Aqzzouz, Spaces of continuous functions taking their values in the ε-product. Rev. R. Acad. Cien. Serie A. Mat. Volumen 99(2), (2005), 143–148.

[6] B. Aqzzouz, M. T. Belghiti, H. EL Alj and R. Nouira, Some results on the space of holomorphic functions taking their values in b-spaces. Methods Funct. Anal. Topology, vol. 12, no. 2, (2006), 113–123.

[7] A. Grothendieck, Produits tensoriels topologiques et espaces nucléaires. Mem. Amer. Math. Soc. 1966.

[8] R. Gunning and H. Rossi, Analytic functions of several complex variables, Prentice Hall, 1965.

[9] H. Hogbe Nlend, Théorie des bornologies et applications. Lecture Notes in Math., vol. 213. Springer-Verlag, Berlin-New York, 1971.

[10] L. Hörmander, The analysis of Linear partial differential operators I, Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin-New York 1983.

[11] J. Lindenstrauss and L. Tzafriri, Classical Banach spaces, Lecture Notes in Math., vol. 338. Springer-Verlag, Berlin-New York, 1973.

[12] G. Noël, Produit tensoriel et platitude des q-espaces, Bull. Soc. Math. Belgique, 22 (1970), 119–142.

[13] N. Popescu and L. Popescu, Theory of category. Editura Academiei Bucuresti Roumania, 1979.

[14] F. H. Vasilescu, Spectral theory in quotient Fréchet spaces, Revue Roumaine de Math. Pures et Appl. 32 (1986), 561–571.

[15] L. Waelbroeck, Duality and the injective tensor product. Math. Ann. 163 (1966), 122–126.

[16] L. Waelbroeck, Topological vector spaces and algebras, Lectures Notes in Math., vol. 230. Springer-Verlag, Berlin-New York, 1971.

[17] L. Waelbroeck, Fonctions à valeurs dans des quotients banachiques, Bull. Cl. Sc. Acad. Roy. Belg. LXVII (1981), 319–327.

[18] L. Waelbroeck, Quotient Banach spaces, Banach Center Publ. (1982), 553–562, Warsaw.

[19] L. Waelbroeck, The category of quotient bornological spaces, J. A. Barosso (ed.), Aspects of Mathematics and its Applications, Elsevier Sciences Publishers B.V. (1986), 873–894.

[20] L. Waelbroeck, Holomorphic functions taking their values in a quotient bornological space. Operator Theory: Advances and Applications, vol. 43 (1990), 323–335.

Belmesnaoui Aqzzouz
Université Mohammed V-Souissi
Faculté des Sciences Economiques
Juridiques et Sociales
Département d'Economie
B.P. 5295, *Sala Eljadida, Morocco*
E-mail: baqzzouz@hotmail.com

Hassan M. El Alj
Université Ibn Tofail
Faculté des Sciences
Département de Mathématiques
B.P. 133, *Kénitra, Morocco*
E-mail: hassanelalj@yahoo.fr