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Lebesgue spaces with variable exponent on a probability space

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Abstract. We show that the Lebesgue space with a variable exponent $L_{p(\cdot)}$ is a rearrangement–invariant space if and only if p is constant. In addition, we give a necessary and sufficient condition on a variable exponent for a martingale inequality to hold.

1. Introduction

Let p be a variable exponent, i.e., let $p : \mathbf{R}^n \to [1, \infty)$ be a measurable function. The Lebesgue space with a variable exponent $L_{p(\cdot)}$ is defined to be the set of measurable functions f on $\mathbf{R}^n$ such that for some $\lambda > 0$,

$$\int_{\mathbf{R}^n} |\lambda f(x)|^{p(x)} dx < \infty.$$

Such Lebesgue spaces were studied by O. Kováčik and J. Rákosník [6], X. Fan and D. Zhao [3] and others.

In this paper, we consider such Lebesgue spaces $L_{p(\cdot)}$ on a probability space Ω: one of our purposes is to prove that $L_{p(\cdot)}$ is a rearrangement–invariant space (see Definition 5) if and only if p is constant.

Another purpose is to prove the weak type Doob inequality with a variable exponent. Let $(\Omega, \Sigma, \mathbf{P})$ be a probability space with a *filtration* $\mathscr{F} = (\mathscr{F}_n)_{n \in \mathbf{Z}_+}$, where we mean by filtration an increasing sequence of sub-σ-algebras of Σ. We define $Mf = \sup_{n \in \mathbf{Z}_+} |f_n|$ and $f_\infty = \lim_{n \to \infty} f_n$ almost surely (a.s.). Let $f = (f_n)_{n \in \mathbf{Z}_+}$ be a uniformly integrable $\mathscr{F}$-martingale, that is, $f = (f_n)$ is a $\mathscr{F}$-martingale such that $f_n = \mathbf{E}[f_\infty | \mathscr{F}_n]$ a.s. for $n \in \mathbf{Z}_+$. Let $p \geq 1$ be a constant. Then

$$\lambda^p \mathbf{P}(Mf > \lambda) \leq \mathbf{E}[|f_\infty|^p] \qquad (\lambda > 0). \tag{1}$$

This inequality was proved by J. L. Doob.

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It is well known that inequality (1) is a probabilistic analogue of the Hardy-Littlewood weak type inequality

$$|\{x \in G : Mf(x) > t\}| \le \frac{C}{t^p}\int_G |f(y)|^p dy, \qquad 1 \le p < \infty,$$

where $G \subset \mathbf{R}^n$ is an open set and M denotes the Hardy–Littlewood maximal operator.

Recently, this inequality was generalized as follows (see [2]):

If there exists a constant c such that for every ball B,

$$\frac{1}{p(x)} \le \frac{c}{|B|}\int_B \frac{1}{p(y)}\, dy, \qquad x \in B, \tag{2}$$

then there exists a constant C such that for any $t > 0$,

$$|\{x \in G : Mf(x) > t\}| \le C\int_G \left(\frac{|f(y)|}{t}\right)^{p(y)} dy.$$

We show that an analogous result holds in our setting; under the condition that

$$\frac{1}{p} \le C\mathbf{E}\left[\frac{1}{p}\middle|\mathscr{F}_\tau\right] \qquad \text{for all stopping times } \tau,$$

we prove the inequality

$$\mathbf{P}(Mf > \lambda) \le \mathbf{E}\left[\left|\frac{f_\infty}{\lambda}\right|^p\right]. \tag{3}$$

We also consider the inequality

$$\mathbf{E}[\lambda^p 1_{\{Mf>\lambda\}}] \le \mathbf{E}[|f_\infty|^p]. \tag{4}$$

2. Preliminaries

Let $(\Omega, \Sigma, \mathbf{P})$ be a complete probability space. We denote by $\mathbf{F}$ the set of all filtrations of $(\Omega, \Sigma, \mathbf{P})$, by $\mathscr{M}(\mathscr{F})$ the set of all uniformly integrable martingales with respect to $\mathscr{F} \in \mathbf{F}$, and by $\mathscr{S} \equiv \mathscr{S}(\mathscr{F})$ the set of all stopping times with respect to $\mathscr{F} \in \mathbf{F}$. For $f = (f_n)_{n\in\mathbf{Z}_+} \in \mathscr{M}(\mathscr{F})$, we let

$$Mf = \sup_{n\in\mathbf{Z}_+} |f_n| \qquad \text{and} \qquad f_\infty = \lim_{n\to\infty} f_n \ \ a.s.$$

Next we fix some notation concerning generalized Lebesgue spaces.

Let p be a variable exponent, i.e., let $p : \Omega \to [1, \infty)$ be a random variable. We put $p^+ = \operatorname{ess\,sup}_\Omega p$ and $p^- = \operatorname{ess\,inf}_\Omega p$. For a random variable x we define the functional $\rho_{p(\cdot)}$ by

$$\rho_{p(\cdot)}(x) = \mathbf{E}[|x|^p] = \int_\Omega |x(\omega)|^{p(\omega)} d\mathbf{P}(\omega).$$

The Lebesgue space $L_{p(\cdot)}$ with the variable exponent p on Ω is defined to be the set of all random variables such that $\rho_{p(\cdot)}(\lambda x) < \infty$ for some $\lambda > 0$. It is easy to see that the functional $\|\cdot\|_{p(\cdot)}$ defined by

$$\|x\|_{p(\cdot)} = \inf\{\lambda > 0 : \rho_{p(\cdot)}(x/\lambda) \le 1\}$$

is a norm on $L_{p(\cdot)}$.

PROPOSITION 1. *Let p be a variable exponent and let $x \in L_{p(\cdot)}$. Then $\|x\|_{p(\cdot)} \le 1$ if and only if $\rho_{p(\cdot)}(x) \le 1$.*

See [6] for a proof.

Let X and Y be normed linear spaces. We write $X \hookrightarrow Y$ if X is continuously embedded in Y, i.e., if $X \subset Y$ and the inclusion map is continuous.

PROPOSITION 2. *Let p and q be variable exponents. If $p \le q$ a.s. on Ω, then $L_{q(\cdot)} \hookrightarrow L_{p(\cdot)}$.*

See [6, Theorem 2.8] for a proof.

DEFINITION 1. A Banach function space over a probability space is a real Banach space $(X, \|\cdot\|_X)$ of random variables such that:

(B1) $L_\infty \hookrightarrow X \hookrightarrow L_1$.

(B2) If $x \in X$ and $|y| \le |x|$ a.s., then $y \in X$ and $\|y\|_X \le \|x\|_X$.

(B3) If $x_n \in X$ for all n, $0 \le x_n \uparrow x$ a.s., and $\sup_n \|x_n\|_X < \infty$, then $x \in X$ and $\|x\|_X = \sup_n \|x_n\|_X$.

PROPOSITION 3. *The space $(L_{p(\cdot)}, \|\cdot\|_{p(\cdot)})$ is a Banach function space over a probability space Ω.*

PROOF. The completeness can be proved as in [6] or [5]. It is easy to prove (B1), (B2) and (B3).

If x and y are random variables, we write $x \simeq_d y$ when they have the same distribution.

DEFINITION 2. A rearrangement-invariant (r.i.) space is a Banach function space $(X, \|\cdot\|_X)$ such that:

(R) If $x \simeq_d y$ and $x \in X$, then $y \in X$ and $\|x\|_X = \|y\|_X$.

It is known (cf. [1, p. 43]) that if p is constant on Ω, then $(L_{p(\cdot)}, \|\cdot\|_{p(\cdot)})$ is an r.i. space.

3. Results

Throughout this section, we will consider martingales $f = (f_n)$ such that $f_\infty \in L_{p(\cdot)}$ and filtrations $\mathscr{F} = (\mathscr{F}_n)$ such that $\mathscr{F}_0$ includes the family of $\mathbf{P}$-negligible subsets of Ω. We denote by 1_A the indicator function of $A \in \Sigma$.

Our main result is as follows:

THEOREM 1. *Suppose that $(\Omega, \Sigma, \mathbf{P})$ is a nonatomic space, i.e., that $(\Omega, \Sigma, \mathbf{P})$ contains no atom. If there exists a norm on $L_{p(\cdot)}$ which is equivalent to $\|\cdot\|_{p(\cdot)}$ and with respect to which $L_{p(\cdot)}$ is an r.i. space, then p is constant.*

PROOF. We assume that p is not constant. It suffices to show that there are random variables x and y such that $x \in L_{p(\cdot)}$, $x \simeq_d y$ and $\|y\|_{p(\cdot)} = \infty$. Then there exist numbers m_1 and m_2 such that $1 \leq m_1 < m_2 < \infty$ and both $A = \{p \leq m_1\}$ and $B = \{m_2 \leq p\}$ have positive measure. Since Ω is non-atomic, there exist measurable sets A' and B' such that $A' \subset A$, $B' \subset B$ and $\mathbf{P}(A') = \mathbf{P}(B') > 0$. Again since Ω is a nonatomic, there exist sequences $\{A_n\}_{n \in \mathbf{N}}$ and $\{B_n\}_{n \in \mathbf{N}}$ of pairwise disjoint measurable subsets of Ω such that

$$A_n \subset A', \quad B_n \subset B' \qquad (n \in \mathbf{N}),$$

and

$$\mathbf{P}(A_n) = \frac{1}{2^n}\mathbf{P}(A'), \quad \mathbf{P}(B_n) = \frac{1}{2^n}\mathbf{P}(B') \qquad (n \in \mathbf{N}).$$

Define

$$x = \sum_{n=1}^{\infty} 2^{n/m_2} 1_{A_n} \qquad \text{and} \qquad y = \sum_{n=1}^{\infty} 2^{n/m_2} 1_{B_n}.$$

Then we have $x \simeq_d y$ and

$$\mathbf{E}[x^p] = \sum_{n=1}^{\infty} \mathbf{E}[2^{(n/m_2)p} 1_{A_n}] \leq \sum_{n=1}^{\infty} \mathbf{E}[2^{(m_1/m_2)n} 1_{A_n}] = \sum_{n=1}^{\infty} \left\{ \left(\frac{1}{2}\right)^{1-m_1/m_2} \right\}^n \mathbf{P}(A') < \infty.$$

Therefore $x \in L_{p(\cdot)}$. Moreover, for each $\lambda > 0$, there is a number $N_\lambda \in \mathbf{N}$ such that $\lambda \leq 2^{N_\lambda/m_2}$. Then we have

$$\mathbf{E}\left[\left|\frac{y}{\lambda}\right|^p\right] \geq \sum_{n=N_\lambda}^{\infty} \mathbf{E}\left[\left(\frac{2^{n/m_2}}{\lambda}\right)^p 1_{B_n}\right] \geq \sum_{n=N_\lambda}^{\infty} \frac{1}{\lambda^{m_2}} \mathbf{P}(B') = \infty.$$

Hence $\|y\|_{p(\cdot)} = \inf\{\lambda > 0 : \mathbf{E}[|y/\lambda|^p] \leq 1\} = \infty$.

If $(\Omega, \Sigma, \mathbf{P})$ has an atom, then the conclusion of Theorem 1 may not hold.

Example. Let A_1, A_2 and A_3 be measurable subsets of Ω such that

$$\mathbf{P}(A_1) = \mathbf{P}(A_2) = \frac{2}{5}, \qquad \mathbf{P}(A_3) = \frac{1}{5} \qquad \text{and} \qquad A_i \cap A_j = \varnothing \quad (i \neq j).$$

We let $\Sigma = \sigma(A_1, A_2, A_3)$. Now we define

$$p = \begin{cases} 1 & \text{on } A_1 \cup A_2, \\ 2 & \text{on } A_3. \end{cases}$$

For random variables x and y, we can write

$$x = \sum_{i=1}^{3} a_i 1_{A_i}, \qquad y = \sum_{i=1}^{3} b_i 1_{A_i},$$

where $a_i, b_i \in \mathbf{R}$.

We observe that if $x \neq y$ and $x \simeq_d y$, then $a_1 \neq a_2$, $a_1 = b_2$, $a_2 = b_1$ and $a_3 = b_3$. As a result, for $\lambda > 0$,

$$\begin{aligned} \mathbf{E}\left[\left|\frac{x}{\lambda}\right|^p\right] &= \left|\frac{a_1}{\lambda}\right| \mathbf{P}(A_1) + \left|\frac{a_2}{\lambda}\right| \mathbf{P}(A_2) + \left|\frac{a_3}{\lambda}\right|^2 \mathbf{P}(A_3) \\ &= \left|\frac{b_2}{\lambda}\right| \mathbf{P}(A_1) + \left|\frac{b_1}{\lambda}\right| \mathbf{P}(A_2) + \left|\frac{b_3}{\lambda}\right|^2 \mathbf{P}(A_3) = \mathbf{E}\left[\left|\frac{y}{\lambda}\right|^p\right], \end{aligned}$$

that is, $\|x\|_{p(\cdot)} = \|y\|_{p(\cdot)}$. Thus $(L_{p(\cdot)}, \|\cdot\|_{p(\cdot)})$ is the r.i. space, but p is not constant.

Now we study some martingale inequalities.

The next proposition is an analogue of the result of [2, Theorem 1.8]. The method of [2] is available for the proof of the proposition.

Proposition 4. *Let $\mathscr{F} \in \mathbf{F}$. If there is a constant $C \geq 1$ such that*

$$\frac{1}{p} \leq C\mathbf{E}\left[\frac{1}{p}\middle|\mathscr{F}_\tau\right] \quad a.s.$$

for all $\tau \in \mathscr{S}$, then for any $f \in \mathscr{M}(\mathscr{F})$ such that $f_\infty \in L_{p(\cdot)}$ and for any $\lambda > 0$,

$$\mathbf{P}(Mf > \lambda) \leq C\mathbf{E}\left[\left|\frac{f_\infty}{\lambda}\right|^p\right].$$

Proof. For $\lambda > 0$, we define

$$\tau = \inf\{n \in \mathbf{Z}_+ : |f_n| > \lambda\} \in \mathscr{S},$$

with the convention that $\inf \emptyset = \infty$. It is easy to prove that $\{Mf > \lambda\} = \{\tau < \infty\}$ and $\{\tau < \infty\} \subset \{|f_\tau| > \lambda\}$. Note that

$$\mathbf{E}\left[\left|\frac{f_\infty}{\lambda}\right| \middle| \mathscr{F}_\tau\right] > 1 \ a.s. \text{ on } \{\tau < \infty\}. \tag{5}$$

By Young's inequality,

$$\mathbf{E}\left[\left|\frac{f_\infty}{\lambda}\right| \middle| \mathscr{F}_\tau\right] \leq \mathbf{E}\left[\frac{1}{p}\left|\frac{f_\infty}{\lambda}\right|^p \middle| \mathscr{F}_\tau\right] + \mathbf{E}\left[\frac{1}{q}\middle| \mathscr{F}_\tau\right] \ a.s., \tag{6}$$

where q is the conjugate function of p. By (5) and (6), we have

$$\begin{aligned}
\mathbf{E}\left[\frac{1}{p}\middle| \mathscr{F}_\tau\right] &= 1 - \mathbf{E}\left[\frac{1}{q}\middle| \mathscr{F}_\tau\right] \\
&< \mathbf{E}\left[\left|\frac{f_\infty}{\lambda}\right| \middle| \mathscr{F}_\tau\right] - \mathbf{E}\left[\frac{1}{q}\middle| \mathscr{F}_\tau\right] \\
&\leq \mathbf{E}\left[\frac{1}{p}\left|\frac{f_\infty}{\lambda}\right|^p \middle| \mathscr{F}_\tau\right] \ a.s. \text{ on } \{\tau < \infty\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbf{P}(Mf > \lambda) &\leq \mathbf{E}\left[\frac{1}{\mathbf{E}[1/p \mid \mathscr{F}_\tau]}\mathbf{E}[1/p \mid \mathscr{F}_\tau]1_{\{\tau<\infty\}}\right] \\
&\leq \mathbf{E}\left[\frac{1}{\mathbf{E}[1/p \mid \mathscr{F}_\tau]}\mathbf{E}\left[\frac{1}{p}\left|\frac{f_\infty}{\lambda}\right|^p \middle| \mathscr{F}_\tau\right]1_{\{\tau<\infty\}}\right] \\
&\leq \mathbf{E}\left[\frac{1}{\mathbf{E}[1/p \mid \mathscr{F}_\tau]} \cdot \frac{1}{p} \cdot \left|\frac{f_\infty}{\lambda}\right|^p 1_{\{\tau<\infty\}}\right] \\
&\leq C\mathbf{E}\left[\left|\frac{f_\infty}{\lambda}\right|^p 1_{\{\tau<\infty\}}\right] \leq C\mathbf{E}\left[\left|\frac{f_\infty}{\lambda}\right|^p\right].
\end{aligned}$$

This completes the proof of Proposition 4.

For a specific filtration, we prove the inequality (4). In order to prove it, we need the following lemmas.

Lemma 1. *Let $\mathscr{A}$ be a sub-σ-algebra of Σ and let p be an $\mathscr{A}$-measurable variable exponent. Then*

$$\mathbf{E}[|x| \,|\mathscr{A}]^p \leq \mathbf{E}[|x|^p|\mathscr{A}] \ a.s.$$

for all $x \in L_{p(\cdot)}$.

Proof. Suppose that p is an $\mathscr{A}$-measurable simple function and $p \geq 1$ on Ω. Then we can write

$$p = \sum_{k=1}^{n} \alpha_k 1_{A_k} \ \ a.s.,$$

where the sets A_k are pairwise disjoint subsets of Ω and $\alpha_k \geq 1$ $(k = 1, 2, \dots)$. We find that

$$\begin{aligned}\mathbf{E}[|x|\,|\mathscr{A}]^p &= \sum_{k=1}^{n} \mathbf{E}[|x|\,|\mathscr{A}]^{\alpha_k} 1_{A_k} \leq \sum_{k=1}^{n} \mathbf{E}[|x|^{\alpha_k}|\mathscr{A}] 1_{A_k} \\ &= \mathbf{E}\left[\sum_{k=1}^{n} |x|^{\alpha_k} 1_{A_k} \middle| \mathscr{A}\right] = \mathbf{E}[|x|^p|\mathscr{A}] \ \ a.s.\end{aligned}$$

Suppose now that p is an arbitrary $\mathscr{A}$-measurable variable exponent. Then there exists a sequence $(p_n)_{n \in \mathbf{N}}$ of $\mathscr{A}$-measurable simple functions such that

$$p_n \uparrow p \quad (n \to \infty) \qquad \text{and} \qquad p_n \geq 1 \quad (n \in \mathbf{N}).$$

Since $|x|^{p_n} \leq 1 + |x|^p \in L_1$ $(n \in \mathbf{N})$ when $x \in L_{p(\cdot)}$, the dominated convergence theorem gives

$$\mathbf{E}[|x|\,|\mathscr{A}]^p \leq \mathbf{E}[|x|^p|\mathscr{A}] \ \ a.s.$$

This completes the proof of Lemma 1.

Lemma 2. *Let $\{A_k\}_{k \in \mathbf{N}}$ be a sequence of pairwise disjoint measurable subsets of Ω such that $\Omega = \bigcup_{k \in \mathbf{N}} A_k$, and let $\mathscr{F}_0 = \sigma(\{A_k; k \in \mathbf{N}\})$. If there exists a constant $C \geq 1$, independent of $x \in L^+_{p(\cdot)}$, such that*

$$\mathbf{E}[\mathbf{E}[x|\mathscr{F}_0]^p \,|\, \mathscr{F}_0] \leq C\mathbf{E}[x^p|\mathscr{F}_0] \ \ a.s., \tag{7}$$

then p is $\mathscr{F}_0$-measurable. Here $L^+_{p(\cdot)} = \{x \in L_{p(\cdot)} : x \geq 0\}$.

Proof. Let $\mathscr{A} = \mathscr{F}_0$. Assume that p is not $\mathscr{A}$-measurable, i.e., that there exists an index $N \in \mathbf{N}$ such that p is not constant on A_N. Then there exists a number m such that $1 \leq m < \infty$, and both $\Lambda_1 = \{p \leq m\} \cap A_N$ and $\Lambda_2 = \{m < p\} \cap A_N$ have positive measure. For $a > 1$, we define a random variable x_a by $x_a = a1_{\Lambda_1} \in L^+_{p(\cdot)}$. Then we have

$$\mathbf{E}[x_a^p|\mathscr{A}] = \frac{1_{A_N}}{\mathbf{P}(A_N)} \mathbf{E}[a^p 1_{\Lambda_1}] \leq a^m \frac{\mathbf{P}(\Lambda_1)}{\mathbf{P}(A_N)} 1_{A_N} \ \ a.s.,$$

and

$$\mathbf{E}[\mathbf{E}[x_a|\mathscr{A}]^p \mid \mathscr{A}]1_{A_N} = \mathbf{E}\left[\frac{1_{A_N}}{\mathbf{P}(A_N)^p}\mathbf{E}[a1_{A_1}]^p \mid \mathscr{A}\right]$$
$$\geq \frac{1_{A_N}}{\mathbf{P}(A_N)}\mathbf{E}\left[a^p\left(\frac{\mathbf{P}(A_1)}{\mathbf{P}(A_N)}\right)^p 1_{A_2}\right] \quad a.s.$$

Hence

$$\frac{\mathbf{E}[\mathbf{E}[x_a|\mathscr{A}]^p \mid \mathscr{A}]}{\mathbf{E}[x_a^p|\mathscr{A}]}1_{A_N} \geq \frac{1_{A_N}}{\mathbf{P}(A_1)}\left[a^{p-m}\left(\frac{\mathbf{P}(A_1)}{\mathbf{P}(A_N)}\right)^p 1_{A_2}\right] \quad a.s.$$

Since $p - m > 0$ on A_2,

$$\lim_{a\to\infty}\frac{\mathbf{E}[\mathbf{E}[x_a|\mathscr{A}]^p \mid \mathscr{A}]}{\mathbf{E}[x_a^p|\mathscr{A}]}1_{A_N} = \infty \quad a.s.$$

Therefore there is no constant $C \geq 1$ satisfying (7). This completes the proof of Lemma 2.

Theorem 2. *Let $\{A_k\}_{k\in\mathbf{N}}$ be as in Lemma 2 and let $\mathscr{F} = (\mathscr{F}_n)_{n\in\mathbf{Z}_+} \in \mathbf{F}$ be such that $\mathscr{F}_0 = \sigma(\{A_k; k \in \mathbf{N}\})$. Then the following are equivalent:*

(i) *There exists a constant $C \geq 1$, independent of $f \in \mathscr{M}(\mathscr{F})$, such that for any $\lambda > 0$,*

$$\mathbf{E}[\lambda^p 1_{\{Mf>\lambda\}}] \leq C\mathbf{E}[|f_\infty|^p]. \tag{8}$$

(ii) *p is $\mathscr{F}_0$-measurable.*

Proof. (i) $\Rightarrow$ (ii): By (8), we have for any $A \in \mathscr{F}_0$,

$$\mathbf{E}[\lambda^p 1_{\{\mathbf{E}[|f_\infty|\,|\mathscr{F}_0]>\lambda\}}1_A] = \mathbf{E}[\lambda^p 1_{\{\mathbf{E}[|f_\infty 1_A|\mathscr{F}_0]>\lambda\}}] \leq C\mathbf{E}[|f_\infty|^p 1_A].$$

Hence

$$\sup_{\lambda>0}\ \mathbf{E}[\lambda^p|\mathscr{F}_0]1_{\{\mathbf{E}[|f_\infty|\,|\mathscr{F}_0]>\lambda\}} \leq C\mathbf{E}[|f_\infty|^p|\mathscr{F}_0] \quad a.s.$$

This implies that for any $f \in \mathscr{M}(\mathscr{F})$,

$$\mathbf{E}[\mathbf{E}[|f_\infty|\,|\mathscr{F}_0]^p \mid \mathscr{F}_0] \leq C\mathbf{E}[|f_\infty|^p|\mathscr{F}_0] \quad a.s.$$

Thus, by Lemma 2, p is $\mathscr{F}_0$-measurable.

(ii) $\Rightarrow$ (i): For $\lambda > 0$, we define $\tau \in \mathscr{S}$ as in Proposition 4. Then, by Lemma 1, we have

$$\mathbf{E}[\lambda^p 1_{\{Mf>\lambda\}}] = \mathbf{E}[\lambda^p 1_{\{\tau<\infty\}}] \leq \mathbf{E}[|f_\tau|^p 1_{\{\tau<\infty\}}]$$
$$\leq \mathbf{E}[|f_\infty|^p 1_{\{\tau<\infty\}}] \leq \mathbf{E}[|f_\infty|^p].$$

This completes the proof.

Corollary 1. *Let $\mathscr{F} = (\mathscr{F}_n)$ be as in Theorem 2, and $1 < p^- \le p^+ < \infty$. Then the following conditions are equivalent:*

(i) *There exists a constant $C \ge 1$, independent of $f \in \mathscr{M}(\mathscr{F})$, such that*

$$\|Mf\|_{p(\cdot)} \le C\|f_\infty\|_{p(\cdot)}.$$

(ii) *p is $\mathscr{F}_0$-measurable.*

Proof. We may assume $\|f_\infty\|_{p(\cdot)} \le 1$.

(i) $\Rightarrow$ (ii): Since $p^+ < \infty$ and $\mathbf{E}[|f_\infty|^p] \le 1$, we have $\mathbf{E}[(Mf)^p] \le C^{p^+}$. Therefore, we have

$$\mathbf{E}[\lambda^p 1_{\{Mf>\lambda\}}] \le C^{p^+}$$

for any $f_\infty \in L_{p(\cdot)}$. Thus, by Theorem 2, p is $\mathscr{F}_0$-measurable.

(ii) $\Rightarrow$ (i): We set $q = p/p^-$. By Lemma 1,

$$(Mf)^q = \sup_n |f_n|^q \le \sup_n \mathbf{E}[|f_\infty|^q|\mathscr{F}_n].$$

We have

$$\begin{aligned}\rho_{p(\cdot)}(Mf) &= \mathbf{E}[(Mf)^{q\cdot p^-}] = \|(Mf)^q\|_{p^-}^{p^-} \\ &\le \left\|\sup_n \mathbf{E}[|f_\infty|^q|\mathscr{F}_n]\right\|_{p^-}^{p^-}.\end{aligned}$$

Here, by the strong type Doob inequality, we obtain

$$\left\|\sup_n \mathbf{E}[|f_\infty|^q|\mathscr{F}_n]\right\|_{p^-}^{p^-} \le q^+\|f_\infty^q\|_{p^-}^{p^-},$$

where $q^+ = p^+/p^-$. Therefore

$$\rho_{p(\cdot)}(Mf) \le q^+\|f_\infty^q\|_{p^-}^{p^-} \le q^+.$$

Hence, $\|Mf\|_{p(\cdot)} \le q^+$.

Remark. Let $p^- > 1$ and $p^+ < \infty$. According to [7] (cf. [4, Remark 4.5]), the following are equivalent:

(i) There exists a constant $c \ge 1$ such that for any $f \in L_{p(\cdot)}(\mathbf{R}^n)$,

$$\int_{\mathbf{R}^n} (Mf(x))^{p(x)}dx \le c\int_{\mathbf{R}^n} |f(x)|^{p(x)}dx$$

holds, where M denotes the Hardy–Littlewood maximal operator.

(ii) p is constant.

A probabilistic analogue of this result is as follows:

Let $(\Omega, \Sigma, \mathbf{P})$ be a nonatomic probability space, and let p be a variable exponent. The following are equivalent:

(i) If there exists a norm on $L_{p(\cdot)}$ which is equivalent to $\|\cdot\|_{p(\cdot)}$ and with respect to which $L_{p(\cdot)}$ is an r.i. space.

(ii) There exists a constant $C \geq 1$, independent of $f \in \mathcal{M}(\mathcal{F})$, such that

$$\mathbf{E}[\lambda^p 1_{\{Mf>\lambda\}}] \leq C\mathbf{E}[|f_\infty|^p]$$

for any $\mathcal{F} \in \mathbf{F}$ and $\lambda > 0$.

(iii) p is constant.

The equivalence of (i) and (iii) follows from Theorem 1. The equivalence of (ii) and (iii) follows from Theorem 2.

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