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Compact Toeplitz operators on parabolic Bergman spaces

Dedicated to Professor Yoshihiro Mizuta on the occasion of his sixtieth birthday

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Abstract. Parabolic Bergman space $\boldsymbol{b}_\alpha^p$ is a Banach space of all p-th integrable solutions of a parabolic equation $(\partial/\partial t + (-\Delta)^\alpha)u = 0$ on the upper half space, where $0 < \alpha \le 1$ and $1 \le p < \infty$. In this note, we consider the Toeplitz operator from $\boldsymbol{b}_\alpha^p$ to $\boldsymbol{b}_\alpha^q$ where $p \le q$, and discuss the condition that it be compact.

1. Introduction

Let $\boldsymbol{R}_+^{n+1}$ be the upper half space of the $(n+1)$-dimensional Euclidean space $(n \ge 1)$. We denote by $X = (x, t)$ a point in $\boldsymbol{R}_+^{n+1} = \boldsymbol{R}^n \times (0, \infty)$, and by $L^{(\alpha)}$ the α-parabolic operator on $\boldsymbol{R}_+^{n+1}$:

$$L^{(\alpha)} := \frac{\partial}{\partial t} + (-\Delta_x)^\alpha,$$

where $\Delta_x := \partial_{x_1}^2 + \cdots + \partial_{x_n}^2$ is the Laplacian on the x-space $\boldsymbol{R}^n$ and $0 < \alpha \le 1$. We consider the parabolic Bergman space on the upper half space

$$\boldsymbol{b}_\alpha^p := \{u \in L^p(V); u \text{ is } L^{(\alpha)}\text{-harmonic on } \boldsymbol{R}_+^{n+1}\},$$

where $1 \le p \le \infty$ and V is the Lebesgue measure on $\boldsymbol{R}_+^{n+1}$. We give the definition of $L^{(\alpha)}$-harmonic functions in §2 (see also [3]). The orthogonal projection from $L^2(V)$ to $\boldsymbol{b}_\alpha^2$ is an integral operator with kernel R_α, called the α-parabolic Bergman kernel (see [2]). Then for a positive Borel measure μ on the upper half space $\boldsymbol{R}_+^{n+1}$, we can consider the Toeplitz operator with symbol μ, defined by

$$(T_\mu u)(X) := \int R_\alpha(X, Y) u(Y) d\mu(Y).$$

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In this paper, we only consider Borel measures μ such that $0 \le \mu(K) < \infty$ for all compact sets K. Then we call such a measure a positive Borel measure, simply.

B. R. Choe, H. Koo and H. Yi [1] studied the Toeplitz operators on the harmonic Bergman spaces on $\boldsymbol{R}_+^{n+1}$. It was shown in [2] that when $\alpha = 1/2$, our $1/2$-parabolic Bergman spaces coincide with their harmonic Bergman spaces. Our investigation generalizes some results in [1].

In our previous paper [4], we treated the boundedness of the Toeplitz operator $T_\mu \equiv T_{\mu,p,q} : \boldsymbol{b}_\alpha^p \to \boldsymbol{b}_\alpha^q$, where $p \le q$, related to that of the Carleson inclusion $\imath_\mu \equiv \imath_{\mu,p,q} : \boldsymbol{b}_\alpha^p \to L^q(\mu)$. In this paper, we shall discuss their compactness. We also treat the parabolic Bloch space

$$\mathscr{B}_\alpha := \{u \in C^1(\boldsymbol{R}_+^{n+1});$$
$$\|u\|_{\mathscr{B}_\alpha} := |u(X_0)| + \sup_{(x,t) \in \boldsymbol{R}_+^{n+1}} (t^{1/2\alpha}|\nabla_x u(x,t)| + t|\partial_t u(x,t)|) < \infty\},$$

where $X_0 = (0,1)$ and ∇_x denotes the gradient operator on the x-space $\boldsymbol{R}^n$. It is natural to consider $\mathscr{B}_\alpha/\boldsymbol{R}$ rather than $\boldsymbol{b}_\alpha^\infty$ when we treat with $q = \infty$, where $\boldsymbol{R}$ is considered as the set of constant functions.

First, we shall state the results obtained in [4] with some definitions. We introduce some auxiliary functions. Let μ be a positive Borel measure on $\boldsymbol{R}_+^{n+1}$, $\tau \in \boldsymbol{R}$ and m be a nonnegative integer. For $Y = (y,s) \in \boldsymbol{R}_+^{n+1}$, we put

$$\hat{\mu}_\tau^{(\alpha)}(Y) := s^{-\tau(n/2\alpha+1)}\mu(Q^{(\alpha)}(Y)),$$
$$\tilde{\mu}_{\tau,m}^{(\alpha)}(Y) := s^{(2-\tau)(n/2\alpha+1)} \int R_\alpha^m(X,Y)^2 d\mu(X),$$

where $Q^{(\alpha)}(Y)$ is an α-parabolic Carleson box, defined by

$$Q^{(\alpha)}(Y) := \{(x_1,\dots,x_n,t); s \le t \le 2s, |x_j - y_j| \le 2^{-1}s^{1/2\alpha}, j = 1,\dots,n\}, \tag{1}$$

and where R_α^m is a modified reproducing kernel, defined by

$$R_\alpha^m(X,Y) = R_\alpha^m(x,t;y,s) := \frac{(-2)^m}{m!} s^m \partial_s^m R_\alpha(x,t;y,s).$$

We note that $R_\alpha^0 = R_\alpha$ and write simply $\tilde{\mu}_\tau^{(\alpha)} := \tilde{\mu}_{\tau,0}^{(\alpha)}$. A relation between the above two functions is stated in Lemma 3 below.

Definition 1. Let $\tau \in \boldsymbol{R}$ and let $\mu \ge 0$ be a Borel measure on $\boldsymbol{R}_+^{n+1}$.

(i) μ is called a τ-Carleson measure (in the α-parabolic sense) if $\|\hat{\mu}_\tau^{(\alpha)}\|_\infty < \infty$, where $\|\cdot\|_\infty$ stands for the usual supremum norm.

(ii) μ is called a vanishing τ-Carleson measure (in the α-parabolic sense) if $\lim_{Y \to \mathscr{A}} \hat{\mu}_\tau^{(\alpha)}(Y) = 0$, where $\mathscr{A}$ denotes the Alexandroff point (infinity of the one point compactification) of the upper half space $\boldsymbol{R}_+^{n+1}$.

We denote by $\mathscr{E}_m$ the vector space generated by $\{R_\alpha^m(\cdot, Y)\}_{Y \in \boldsymbol{R}_+^{n+1}}$. Remark that $\mathscr{E}_m$ is dense in $\boldsymbol{b}_\alpha^p$ for $1 \le p < \infty$ when $m \ge 1$. If $1 < p < \infty$, then $\mathscr{E}_0$ is also dense in $\boldsymbol{b}_\alpha^p$. Theorems obtained in [4] are the following.

Theorem A. *Let $1 \le p \le q \le \infty$ with $p \ne \infty$, $q \ne 1$ and put $\tau = 1 + \frac{1}{p} - \frac{1}{q}$. Let μ be a positive Borel measure on $\boldsymbol{R}_+^{n+1}$ and $m \ge 1$ be an integer. Then we have the following inequalities:*

$$\|T_{\mu,p,q}\| \le C_1 \|\hat{\mu}_\tau^{(\alpha)}\|_\infty \le C_2 \|\tilde{\mu}_{\tau,m}^{(\alpha)}\|_\infty,$$

where $T_{\mu,p,q}$ is the Toeplitz operator $\boldsymbol{b}_\alpha^p \to \boldsymbol{b}_\alpha^q$ or $\boldsymbol{b}_\alpha^p \to \mathscr{B}_\alpha/\boldsymbol{R}$ according as $q \ne \infty$ or $q = \infty$, and $\|T_{\mu,p,q}\|$ denotes the operator norm. Here we remark that the above positive constants C_1, C_2 can be taken independently of μ.

Under some additional conditions, the opposite inequalities also hold.

Theorem B. *In the same situation as above, we assume, in addition,*

$$\int |R_\alpha^m(X, Y)| d\mu(X) < \infty \qquad \text{for every } Y \in \boldsymbol{R}_+^{n+1} \tag{2}$$

for some integer $m \ge 1$. Then we have

$$\|\tilde{\mu}_{\tau,m}^{(\alpha)}\|_\infty \le C_3 \|T_{\mu,p,q}\|,$$

where the above positive constant C_3 can be chosen independently of μ.

Concerning the theorem, we give a remark.

Remark 1. *In* [4], *we showed Theorem B under the condition*

$$\int |R_\alpha^m(X, Y)| d\mu(X) < \infty \qquad \text{for } V\text{-a.e. } Y \in \boldsymbol{R}_+^{n+1}. \tag{3}$$

Remark that if $T_{\mu,p,q}$ is bounded, then (3) *is equvalent to* (2) *(see* [4, Theorem 2]*).*

The above theorems are closely related to the boundedness of the Carleson inclusion.

Theorem C. *For $1 \le p \le q < \infty$, put $\tau = q/p$. Let $\mu \ge 0$ be a Borel measure on $\boldsymbol{R}_+^{n+1}$. Then there exists a constant $C_4 \ge 1$ independent of μ such that the inequalities*

$$C_4^{-1} \|\hat{\mu}_\tau^{(\alpha)}\|_\infty^{1/q} \le \|\imath_{\mu,p,q}\| \le C_4 \|\hat{\mu}_\tau^{(\alpha)}\|_\infty^{1/q}$$

hold when μ *is a* τ*-Carleson measure, where* $\iota_\mu = \iota_{\mu,p,q}$ *denotes the inclusion map* $\boldsymbol{b}_\alpha^p \to L^q(\mu) : \iota_\mu u = u$ *and* $\|\iota_{\mu,p,q}\|$ *denotes the operator norm.*

REMARK 2. *In the above theorem, even when* μ *is a* τ*-Carleson measure, the inclusion map* ι_μ*, which we call the Carleson inclusion, is not necessarily injective.*

Now, we shall state our main results.

THEOREM 1. *Let* $1 < p \le q \le \infty$ *with* $p \ne \infty$ *and put* $\tau = 1 + \frac{1}{p} - \frac{1}{q}$*, and let* μ *be a positive Borel measure on* $\boldsymbol{R}_+^{n+1}$ *satisfying* (2). *Then the following statements are equivalent:*
(i) *The Toeplitz operator* $T_{\mu,p,q}$ *is compact;*
(ii) μ *is a vanishing* τ*-Carleson measure, i.e.,* $\lim_{Y \to \mathscr{A}} \hat{\mu}_\tau^{(\alpha)}(Y) = 0$*;*
(iii) $\lim_{Y \to \mathscr{A}} \tilde{\mu}_\tau^{(\alpha)}(Y) = 0$.

REMARK 3. *In the above theorem, we can also handle the case where* $p = 1$. *In this case, we use the notion of "$*$-compact operator" instead of "compact operator" (see* §2 *later, cf.* [6]*) and when* $p = 1$*,* $q = \infty$*, we have to replace* $\tilde{\mu}_\tau^{(\alpha)}$ *by* $\tilde{\mu}_{\tau,m}^{(\alpha)}$ *with* $m \ge 1$ *in* (iii). *We can state the above assertions in a unified form if we use the notion of "$*$-compact operator" (see Theorem* 3 *below).*

We shall also give a characterization of the compactness of the Carleson inclusion.

THEOREM 2. *For* $1 \le p \le q < \infty$*, we put* $\tau := q/p$. *Then* $\iota_{\mu,p,q}$ *is* $*$*-compact if and only if* μ *is a vanishing* τ*-Carleson measure.*

Throughout this paper, C will denote a positive constant whose value is not important, not depending on measures μ or functions u, and not necessarily the same at each occurence; it may vary even within a line.

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2. Preliminaries

In this section, we recall fundamental properties of $L^{(\alpha)}$-harmonic functions and compact operators.

In order to define $L^{(\alpha)}$-harmonic functions on $\boldsymbol{R}_+^{n+1}$, we shall recall how the adjoint operator $\tilde{L}^{(\alpha)} = -\partial/\partial t + (-\Delta)^\alpha$ acts on $C_c^\infty(\boldsymbol{R}_+^{n+1})$, the space of all infinitely differentiable functions with compact supports on $\boldsymbol{R}_+^{n+1}$. Since the case $\alpha = 1$ is trivial, we only consider the case $0 < \alpha < 1$ here. Then $(-\Delta)^\alpha$ is the convolution operator defined by $-c_{n,\alpha}\text{p.f.}|x|^{-n-2\alpha}$, where p.f. stands for the finite part,

$$c_{n,\alpha} = -4^{\alpha}\pi^{-n/2}\Gamma((n+2\alpha)/2)/\Gamma(-\alpha) > 0$$

and $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$. Hence for $\varphi \in C_c^{\infty}(\boldsymbol{R}_+^{n+1})$,

$$\tilde{L}^{(\alpha)}\varphi(x,t) = -\frac{\partial}{\partial t}\varphi(x,t) - c_{n,\alpha}\lim_{\delta\downarrow 0}\int_{|y|>\delta}(\varphi(x+y,t)-\varphi(x,t))|y|^{-n-2\alpha}dy.$$

It is easily seen that if $\mathrm{supp}(\varphi)$, the support of φ, is contained in $\{|x|<r, t_1<t<t_2\}$, then

$$|\tilde{L}^{(\alpha)}\varphi(x,t)| \le 2^{n+2\alpha}c_{n,\alpha}\left(\sup_{t_1<s<t_2}\int_{\boldsymbol{R}^n}|\varphi(y,s)|dy\right)\cdot|x|^{-n-2\alpha}$$

for (x,t) with $|x| \ge 2r$.

Definition 2. Let $0<\alpha\le 1$. A continuous funcion u on $\boldsymbol{R}_+^{n+1}$ is said to be $L^{(\alpha)}$-harmonic, if $L^{(\alpha)}u = 0$ in the sense of distribution, i.e., $\int u\tilde{L}^{(\alpha)}\varphi\, dV = 0$ for every $\varphi \in C_c^{\infty}(\boldsymbol{R}_+^{n+1})$.

Next, we introduce the fundamental solution $W^{(\alpha)}$ of $L^{(\alpha)}$, defined by

$$W^{(\alpha)}(x,t) = \begin{cases} (2\pi)^{-n}\int_{\boldsymbol{R}^n}\exp(-t|\xi|^{2\alpha}+\sqrt{-1}x\cdot\xi)d\xi & t>0 \\ 0 & t\le 0. \end{cases}$$

When $\alpha=1$ or $\alpha=1/2$, we know the explicit form. In fact, for $t>0$,

$$W^{(1)}(x,t) = (4\pi t)^{-n/2}e^{-|x|^2/4t} \quad \text{and} \quad W^{(1/2)}(x,t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}}\frac{t}{(t^2+|x|^2)^{(n+1)/2}}.$$

The following homogeneity of $W^{(\alpha)}$ is useful:

$$\partial_x^{\beta}\partial_t^k W^{(\alpha)}(x,t) = t^{-((n+|\beta|)/2\alpha+k)}(\partial_x^{\beta}\partial_t^k W^{(\alpha)})(t^{-1/2\alpha}x,1),$$

where $\beta = (\beta_1,\dots,\beta_n)$ is a multi-index and $k\ge 0$ is an integer.

The following estimate plays an important role in our argument.

Lemma 1 ([4, Lemma 1]). *Let $\beta=(\beta_1,\dots,\beta_n)$ be a multi-index of non-negative integers and $k\ge 0$ be an integer. Then there exists a constant $C>0$ such that*

$$|\partial_x^{\beta}\partial_t^k W^{(\alpha)}(x,t)| \le C(t+|x|^{2\alpha})^{-(n+|\beta|)/2\alpha-k}$$

for all $(x,t)\in\boldsymbol{R}_+^{n+1}$.

We list some properties of α-parabolic Bergman kernels R_α and R_α^m. Recall that

$$R_\alpha(x,t;y,s) := -2\partial_t W^{(\alpha)}(x-y,t+s),$$

$$R_\alpha^m(x,t;y,s) := \frac{(-2)^{m+1}}{m!}s^m\partial_t^{m+1}W^{(\alpha)}(x-y,t+s).$$

These kernels have the following reproducing property: For $m \geq 0$, $1 \leq p < \infty$ and for every $u \in \boldsymbol{b}_\alpha^p$, $R_\alpha^m u = u$, i.e.,

$$R_\alpha^m u(x,t) := \int R_\alpha^m(x,t;y,s)u(y,s)dV(y,s) = u(x,t). \tag{4}$$

Lemma 1 gives the following estimate for R_α^m. For an integer $m \geq 0$, there exists a constant $C > 0$ such that

$$|R_\alpha^m(x,t;y,s)| \leq Cs^m(t+s+|x-y|^{2\alpha})^{-(n/2\alpha+1)-m}. \tag{5}$$

We also need an estimate from below. Then there exist constants $C > 0$ and $\rho > 0$ such that

$$|R_\alpha^m(x,t;y,s)| \geq Cs^{-(n/2\alpha+1)} \tag{6}$$

for all $(y,s) \in \boldsymbol{R}_+^{n+1}$ and $(x,t) \in Q^{(\alpha)}(y,\rho s)$ ([5, Corollary 1]).

If $m > \left(\frac{n}{2\alpha}+1\right)\left(\frac{1}{p}-1\right)$, then we have

$$\|R_\alpha^m(\cdot,Y)\|_{L^p(V)} = Cs^{(n/2\alpha+1)(1/p-1)} \tag{7}$$

with some constant $C > 0$ independent of $Y = (y,s) \in \boldsymbol{R}_+^{n+1}$. Indeed, (5) and next lemma ensure $\|R_\alpha^m(\cdot,Y)\|_{L^p(V)} < \infty$, so that the homogeneity of $W^{(\alpha)}$ gives the equality (7).

LEMMA 2. *Let $\gamma, \eta \in \boldsymbol{R}$. If $-1 < \gamma < \eta - \left(\frac{n}{2\alpha}+1\right)$, then*

$$\int t^\gamma(t+s+|x-y|^{2\alpha})^{-\eta}dV(x,t) = Cs^{\gamma-\eta+n/2\alpha+1}$$

with some constant $C > 0$ independent of $(y,s) \in \boldsymbol{R}_+^{n+1}$.

LEMMA 3. *Let $\mu \geq 0$ be a Borel measure on $\boldsymbol{R}_+^{n+1}$. For $\tau > 1 - \left(\frac{n}{2\alpha}+1\right)^{-1}$ and an integer $m > \left(\frac{\tau-2}{2}\right)\left(\frac{n}{2\alpha}+1\right)$, we have the following relations:*

(i) *μ is a τ-Carleson measure if and only if $\tilde{\mu}_{\tau,m}^{(\alpha)}$ is bounded.*

(ii) *μ is a vanishing τ-Carleson measure if and only if $\lim_{Y\to\mathscr{A}} \tilde{\mu}_{\tau,m}^{(\alpha)}(Y) = 0$.*

PROOF. (i) is shown in [4, Lemma 6], and the "if" part of (ii) also follows from [4, Lemma 6]. Hence we will show the "only if" part of (ii). We assume that μ is a vanishing τ-Carleson measure. We use the following Whitney type decomposition of $\boldsymbol{R}_+^{n+1}$. For $\nu = (\beta_1,\dots,\beta_n,k) \in \boldsymbol{Z}^{n+1}$, we put $t_\nu := 2^k$, $x_\nu := 2^{k/2\alpha}(\beta_1,\dots,\beta_n)$ and $Q_\nu := Q^{(\alpha)}(X_\nu)$, where $Q^{(\alpha)}(X_\nu)$ is the Carleson box defined by (1) and $X_\nu = (x_\nu,t_\nu)$. Then in a similar manner to the proof of [4, Proposition 2], we have

$$\begin{aligned}
\tilde{\mu}^{(\alpha)}_{\tau,m}(Y) &= s^{(2-\tau)(n/2\alpha+1)} \int R^m_\alpha(X,Y)^2 d\mu(X) \\
&\le C s^{2m+(2-\tau)(n/2\alpha+1)} \sum_{\nu \in \boldsymbol{Z}^{n+1}} \int_{Q_\nu} (t+s+|x-y|^{2\alpha})^{-2(n/2\alpha+1+m)} d\mu(x,t) \\
&\le C s^{2m+(2-\tau)(n/2\alpha+1)} \sum_{\nu \in \boldsymbol{Z}^{n+1}} (t_\nu+s+|x_\nu-y|^{2\alpha})^{-2(n/2\alpha+1+m)} \mu(Q_\nu) \\
&= C s^{2m+(2-\tau)(n/2\alpha+1)} \\
&\quad \times \sum_{\nu \in \boldsymbol{Z}^{n+1}} (t_\nu+s+|x_\nu-y|^{2\alpha})^{-2(n/2\alpha+1+m)} t_\nu^{(\tau-1)(n/2\alpha+1)} \hat{\mu}^{(\alpha)}_\tau(X_\nu) V(Q_\nu) \\
&\le C s^{2m+(2-\tau)(n/2\alpha+1)} \\
&\quad \times \int (t+s+|x-y|^{2\alpha})^{-2(n/2\alpha+1+m)} t^{(\tau-1)(n/2\alpha+1)} \hat{\mu}^{(\alpha)}_\tau(X) dV(X).
\end{aligned}$$

Now let $\delta > 0$ be arbitrary given and let us take a compact set K in $\boldsymbol{R}^{n+1}_+$ such that $\hat{\mu}^{(\alpha)}_\tau(X) < \delta$ for every $X \in \boldsymbol{R}^{n+1}_+ \setminus K$. Then we have

$$\begin{aligned}
\tilde{\mu}^{(\alpha)}_{\tau,m}(Y) \le C\delta + C\|\hat{\mu}^{(\alpha)}_\tau\|_\infty s^{2m+(2-\tau)(n/2\alpha+1)} \\
\times \int_K (t+s+|x-y|^{2\alpha})^{-2(n/2\alpha+1+m)} t^{(\tau-1)(n/2\alpha+1)} dV(X),
\end{aligned}$$

which implies

$$\varlimsup_{Y \to \mathscr{A}} \tilde{\mu}^{(\alpha)}_{\tau,m}(Y) \le C\delta.$$

This completes the proof.

Next, we recall some general properties on compact operators.

Definition 3 (cf. [6]). Let $\mathscr{X}$, $\mathscr{Y}$ be Banach spaces and $T : \mathscr{X} \to \mathscr{Y}$ be a bounded linear operator. Assume that $\mathscr{X}$ has a predual Banach space.

(i) $T : \mathscr{X} \to \mathscr{Y}$ is said to be weakly compact if for every sequence $(u_j)_j$ in $\mathscr{X}$ such that w-$\lim_{j\to\infty} u_j = 0$, Tu_j converges to 0 in $\mathscr{Y}$.

(ii) $T : \mathscr{X} \to \mathscr{Y}$ is said to be $*$-compact if for every sequence $(u_j)_j$ in $\mathscr{X}$ such that w*-$\lim_{j\to\infty} u_j = 0$, Tu_j converges to 0 in $\mathscr{Y}$.

(iii) $T : \mathscr{X} \to \mathscr{Y}$ is said to be compact if for every bounded sequence $(u_j)_j$ in $\mathscr{X}$, there exists a subsequence $(u_{j_k})_k$ such that $(Tu_{j_k})_k$ converges in $\mathscr{Y}$.

The relations of these notions are given by the following lemma.

LEMMA 4. *Let $\mathscr{X}$, $\mathscr{Y}$ be Banach spaces with $\mathscr{X} = \mathscr{Z}^*$ for some Banach space $\mathscr{Z}$. Then we have the following:*
(i) *If $T : \mathscr{X} \to \mathscr{Y}$ is $*$-compact, then T is compact.*
(ii) *If $T : \mathscr{X} \to \mathscr{Y}$ is compact, then T is weakly compact.*
(iii) *If a Banach space $\mathscr{X}$ is reflexive, i.e., $\mathscr{Z} = \mathscr{X}^*$, then the notions of "weakly compact", "compact" and "$*$-compact" for bounded linear operators from $\mathscr{X}$ to $\mathscr{Y}$ are equivalent to each other.*

LEMMA 5. *Let $\mathscr{X}$, $\mathscr{Y}$ be Banach spaces with $\mathscr{X} = \mathscr{Z}^*$ for some Banach space $\mathscr{Z}$. The space of all $*$-compact operators $T : \mathscr{X} \to \mathscr{Y}$ is a closed subspace in the Banach space of all bounded linear operators.*

PROOF. Let $(T_k)_k$ be a sequence of $*$-compact operators which converges to a bounded operator T in the norm sense. Take any sequence $(u_j)_j$ in $\mathscr{X}$ such that $\text{w}^*\text{-}\lim_{j\to\infty} u_j = 0$. First, we remark that $\sup_j \|u_j\| < \infty$ by the uniform boundedness principle. Then we have

$$\|Tu_j\| \le \|Tu_j - T_k u_j\| + \|T_k u_j\| \le \|T - T_k\|\,\|u_j\| + \|T_k u_j\|.$$

Since T_k is $*$-compact, letting $j \to \infty$, we have

$$\limsup_{j\to\infty} \|Tu_j\| \le \|T - T_k\| \limsup_{j\to\infty} \|u_j\|,$$

which shows our desired result $\lim_{j\to\infty} \|Tu_j\| = 0$.

Let $\mathscr{B}_{\alpha,0}$ denote the α-parabolic little Bloch space,

$$\mathscr{B}_{\alpha,0} := \left\{ u \in \mathscr{B}_\alpha;\ \lim_{(x,t)\to\mathscr{A}} (t^{1/2\alpha} |\nabla_x u(x,t)| + t|\partial_t u(x,t)|) = 0 \right\}$$

(see [2] for detail). Note that $\mathscr{B}_{\alpha,0}$ is separable. In this paper, we always consider the predual of $\boldsymbol{b}_\alpha^1$ as $\mathscr{B}_{\alpha,0}/\boldsymbol{R}$.

We close this section by remarking the following facts.

LEMMA 6. *Let $1 \le p < \infty$. For $m > \left(\frac{n}{2\alpha} + 1\right)\left(\frac{1}{p} - 1\right)$, we have*

$$\text{w}^*\text{-}\lim_{Y\to\mathscr{A}} \left(\frac{R_\alpha^m(\cdot, Y)}{\|R_\alpha^m(\cdot, Y)\|_{L^p(V)}} \right) = 0$$

in $\boldsymbol{b}_\alpha^p$, where $\boldsymbol{b}_\alpha^p \simeq (\boldsymbol{b}_\alpha^{p'})^*$ if $1 < p < \infty$ and $\boldsymbol{b}_\alpha^1 \simeq (\mathscr{B}_{\alpha,0}/\boldsymbol{R})^*$. Here p' is the exponent conjugate to p.

PROOF. Take an arbitrary sequence $(Y_j)_j = ((y_j, s_j))_j$ in $\boldsymbol{R}_+^{n+1}$ which converges to $\mathscr{A}$ and put

$$v_j(X) := \frac{R_\alpha^m(X, Y_j)}{\|R_\alpha^m(\cdot, Y_j)\|_{L^p(V)}}.$$

We may assume w*-$\lim_{j\to\infty} v_j = v$ for some $v \in \boldsymbol{b}_\alpha^p$, because the sequence is bounded in $\boldsymbol{b}_\alpha^p$.

Let $1 < p < \infty$. For every $X \in \boldsymbol{R}_+^{n+1}$, since $R_\alpha(X,\cdot) \in \boldsymbol{b}_\alpha^{p'}$,

$$\begin{aligned} v(X) &= \langle v, R_\alpha(X,\cdot)\rangle = \lim_{j\to\infty} \langle v_j, R_\alpha(X,\cdot)\rangle \\ &= \lim_{j\to\infty} v_j(X) = \lim_{j\to\infty} R_\alpha^m(X, Y_j) s_j^{(n/2\alpha+1)(1/p')} \\ &= 0, \end{aligned}$$

by (4), and (5), (7), where $\langle\cdot,\cdot\rangle$ denotes the pairing of the duality.

Let $p = 1$. Since $R_\alpha(X,\cdot) \in \mathscr{B}_{\alpha,0}$, we have

$$\lim_{j\to\infty} \langle v_j, R_\alpha(X,\cdot)\rangle = \langle v, R_\alpha(X,\cdot)\rangle.$$

By the definition of the pairing on $\boldsymbol{b}_\alpha^1 \times (\mathscr{B}_{\alpha,0}/\boldsymbol{R})$ ([2, Theorem 9.3]),

$$\begin{aligned} \langle v_j, R_\alpha(X,\cdot)\rangle &= -2 \int v_j(Y) s\partial_s R_\alpha(X, Y) dV(Y) \\ &= \int v_j(Y) R_\alpha^1(X, Y) dV(Y) \\ &= v_j(X). \end{aligned}$$

The last equality follows from (4). Hence $v(X) = \lim_{j\to\infty} v_j(X)$ for every $X \in \boldsymbol{R}_+^{n+1}$. On the other hand,

$$v_j(X) = \frac{R_\alpha^m(X, Y_j)}{\|R_\alpha^m(\cdot, Y_j)\|_{L^1(V)}} \to 0$$

as $j \to \infty$ by (5) and (7), which implies $v = 0$. This completes the proof.

Lemma 7. *Let $1 \le p < \infty$. A sequence $(u_j)_j$ in $\boldsymbol{b}_\alpha^p$ converges to $u \in \boldsymbol{b}_\alpha^p$ in the* w**-topology, if and only if the sequence $(u_j)_j$ is bounded in $\boldsymbol{b}_\alpha^p$ and converges to u uniformly on every compact set in $\boldsymbol{R}_+^{n+1}$.*

Proof. First we shall show the "only if" part. Assume w*-$\lim_{j\to\infty} u_j = u$. By the uniform boundedness principle, $(u_j)_j$ is bounded in $\boldsymbol{b}_\alpha^p$. Then [2, Proposition 5.2 and Theorem 5.4] shows the local uniform boundedness and the equicontinuity. Taking any subsequence $(u_{j_k})_k$ which converges to some $v \in \boldsymbol{b}_\alpha^p$ uniformly on every compact set in $\boldsymbol{R}_+^{n+1}$, we have

$$\lim_{k\to\infty} u_{j_k}(X) = \lim_{k\to\infty} \langle u_{j_k}, R_\alpha(X,\cdot)\rangle = u(X)$$

for every $X \in \boldsymbol{R}_+^{n+1}$, because $R_\alpha(X,\cdot)$ is in the predual of $\boldsymbol{b}_\alpha^p$ and it has the reproducing property (4). Next we show the "if" part. By the w*-

compactness of bounded sets, we may assume the sequence $(u_j)_j$ converges to some $v \in \boldsymbol{b}_\alpha^p$ in the w*-topology. By the "only if" part, which we have already shown, we find that $(u_j)_j$ converges to v uniformly on every compact set, which implies $v = u$ and this completes the proof.

REMARK 4. *The above assertion also holds for* $\tilde{\mathscr{B}}_\alpha = \{u \in \mathscr{B}_\alpha; u(X_0) = 0\}$ *where* $X_0 = (0, 1)$. *Here we consider* $\tilde{\mathscr{B}}_\alpha \simeq \mathscr{B}_\alpha/\boldsymbol{R} \simeq (\boldsymbol{b}_\alpha^1)^*$. *In fact, by using* [2, Proposition 7.2 and Theorem 7.3] *instead of* [2, Proposition 5.2 and Theorem 5.4] *and by taking* $\tilde{R}_\alpha(X, \cdot) := R_\alpha(X, \cdot) - R_\alpha(X_0, \cdot) \in \boldsymbol{b}_\alpha^1$ *instead of* $R_\alpha(X, \cdot)$, *we can carry out the above arguments.*

3. Measures with compact support

From now on, we start to prove our theorems. First, in this section, we treat measures whose supports are compact. In this case, we need not assume $p \leq q$.

PROPOSITION 1. *Let* $1 \leq p < \infty$, $1 \leq q \leq \infty$ *and* $\mu \geq 0$ *be a Borel measure on* $\boldsymbol{R}_+^{n+1}$ *with compact support. Then*
(i) *the operator* $T_{\mu,p,q}$ *is* $*$-*compact if* $q > 1$, *and*
(ii) *the operator* $\iota_{\mu,p,q}$ *is* $*$-*compact if* $q < \infty$.

PROOF. We first show the boundedness. For $u \in \boldsymbol{b}_\alpha^p$, by (7) and [2, Proposition 5.2], we have

$$\begin{aligned}\|T_\mu u\|_{L^q(V)} &\leq \int \|R_\alpha(\cdot, Y)\|_{L^q(V)} |u(Y)| d\mu(Y) \\ &\leq C\|u\|_{L^p(V)} \int s^{-\tau(n/2\alpha+1)}\, d\mu(y, s),\end{aligned}$$

where $\tau = \frac{1}{p} + 1 - \frac{1}{q}$. Remarking the boundedness of the inclusion $\boldsymbol{b}_\alpha^\infty \subset \mathscr{B}_\alpha$, which follows from [2, Theorem 5.4], we also have the boundedness of $T_{\mu,p,\infty} : \boldsymbol{b}_\alpha^p \to \mathscr{B}_\alpha/\boldsymbol{R}$. Since

$$\|u\|_{L^q(\mu)} \leq \sup_{Y \in \operatorname{supp}(\mu)} |u(Y)| \cdot \left(\int d\mu\right)^{1/q} \leq C\|u\|_{L^p(V)},$$

the boundedness of $\iota_{\mu,p,q}$ can be easily verified, where the last inequality above follows from the boundedness of the point evaluation [2, Proposition 5.2].

Next, to show the compactness, we take an arbitrary sequence $(u_j)_j$ from $\boldsymbol{b}_\alpha^p$ which converges to 0 in the w*-topology. We may assume that $(u_j)_j$ be in $\mathscr{E}_m$ for $m \geq 1$, because $\mathscr{E}_m$ is dense in $\boldsymbol{b}_\alpha^p$. Since $T_\mu u_j(X) = \int R_\alpha(X, Y) u_j(Y) d\mu(Y)$, Lemma 7 implies that

$$\|T_\mu u_j\|_{L^q(V)} \leq C \int \|R_\alpha(\cdot, Y)\|_{L^q(V)} |u_j(Y)| d\mu(Y)$$
$$\leq C \int s^{(n/2\alpha+1)(1/q-1)} |u_j(y,s)| d\mu(y,s) \to 0$$

as $j \to \infty$ for $1 < q \leq \infty$. We also have

$$\|u_j\|_{L^q(\mu)} \leq \sup_{\mathrm{supp}(\mu)} |u_j| \cdot \left(\int d\mu \right)^{1/q} \to 0$$

as $j \to \infty$. These complete the proof.

Remark 5. *In the above proposition, when $q = \infty$, we have $\|T_\mu u\|_{\mathscr{B}_\alpha} \leq C\|T_\mu u\|_\infty \leq C\|u\|_p$ for $u \in \boldsymbol{b}_\alpha^p$. Hence $T_\mu \boldsymbol{b}_\alpha^p \subset \boldsymbol{b}_\alpha^\infty \cap \mathscr{B}_{\alpha,0}$ holds. In fact, for any $u \in \mathscr{E}_m$,*

$$|t\partial_t(T_\mu u)(x,t)| \leq \int |t\partial_t R_\alpha(x,t,y,s) u(y,s)| d\mu(y,s)$$
$$\leq \left(\sup_{Y \in \mathrm{supp}(\mu)} |t\partial_t R_\alpha(X,Y)| \right) \left(\sup_{\mathrm{supp}(\mu)} |u| \right) \int d\mu$$
$$\to 0 \qquad \text{as } X \to \mathscr{A}.$$

By [2, Lemma 9.2], *we see $T_\mu u \in \mathscr{B}_{\alpha,0}$.*

4. Proof of Theorem 1

We begin with the following proposition.

Proposition 2. *For $1 \leq p \leq q \leq \infty$ with $p \neq \infty$ and $q \neq 1$, we put $\tau := \frac{1}{p} + 1 - \frac{1}{q}$. If $\lim_{X \to \mathscr{A}} \hat{\mu}_\tau^{(\alpha)}(X) = 0$, then $T_{\mu,p,q}$ is $*$-compact.*

Proof. Take an exhaustion $(\omega_j)_j$ of $\boldsymbol{R}_+^{n+1}$ and put

$$\mu_j := \mu|_{\omega_j} \qquad \text{and} \qquad \nu_j := \mu - \mu_j.$$

Then by the assumption that $\lim_{Y \to \mathscr{A}} \hat{\mu}_\tau^{(\alpha)}(Y) = 0$, $((\hat{\nu}_j)_\tau^{(\alpha)})_j$ converges to 0 uniformly on $\boldsymbol{R}_+^{n+1}$. Theorem A shows

$$\|T_\mu - T_{\mu_j}\| = \|T_{\nu_j}\| \leq C_1 \|(\hat{\nu}_j)_\tau^{(\alpha)}\|_\infty \to 0$$

as $j \to \infty$. Hence T_μ is $*$-compact, because each T_{μ_j} is $*$-compact by Proposition 1. □

Next, we consider the converse assertion.

PROPOSITION 3. *Let $\mu \geq 0$ be a Borel measure on $\boldsymbol{R}_+^{n+1}$ satisfying* (2) *for some $m \geq 1$. For $1 \leq p \leq q \leq \infty$ with $p \neq \infty$ and $q \neq 1$, we put $\tau := \frac{1}{p} + 1 - \frac{1}{q}$. If $T_{\mu,p,q}$ is $*$-compact, then $\lim_{X \to \mathscr{A}} \tilde{\mu}_{\tau,m}^{(\alpha)}(X) = 0$.*

PROOF. Since μ is a τ-Carleson measure, we have

$$\int R_\alpha^m(X, Y)^2 d\mu(X) = \int T_\mu R_\alpha^m(\cdot, Y) \cdot R_\alpha^m(\cdot, Y) dV$$

for $Y \in \boldsymbol{R}_+^{n+1}$ by [4, Proposition 3]. Hence it follows from (7) that

$$\begin{aligned}
\tilde{\mu}_{\tau,m}^{(\alpha)}(Y) &= \int T_\mu R_\alpha^m(\cdot, Y) \cdot R_\alpha^m(\cdot, Y) dV \cdot s^{(n/2\alpha+1)(2-\tau)} \\
&\leq \|T_\mu R_\alpha^m(\cdot, Y)\|_{L^q(V)} \cdot \|R_\alpha^m(\cdot, Y)\|_{L^{q'}(V)} \cdot s^{-(n/2\alpha+1)(1/p-1/q-1)} \\
&= C\|T_\mu R_\alpha^m(\cdot, Y)\|_{L^q(V)} \cdot \|R_\alpha^m(\cdot, Y)\|_{L^p(V)}^{-1} \\
&= C\left\| T_\mu\left(\frac{R_\alpha^m(\cdot, Y)}{\|R_\alpha^m(\cdot, Y)\|_{L^p(V)}}\right)\right\|_{L^q(V)}
\end{aligned}$$

if $1 < q < \infty$. When $q = \infty$, we similarly have the estimate

$$\tilde{\mu}_{\tau,m}^{(\alpha)}(Y) \leq C\left\| T_\mu\left(\frac{R_\alpha^m(\cdot, Y)}{\|R_\alpha^m(\cdot, Y)\|_{L^p(V)}}\right)\right\|_{\mathscr{B}_\alpha/\boldsymbol{R}}.$$

Therefore

$$\lim_{Y \to \mathscr{A}} \tilde{\mu}_{\tau,m}^{(\alpha)}(Y) = 0,$$

because of the $*$-compactness of T_μ and Lemma 6.

We can now prove our main theorem.

THEOREM 3. *Let $1 \leq p \leq q \leq \infty$ with $p \neq \infty$, $q \neq 1$ and put $\tau = 1 + \frac{1}{p} - \frac{1}{q}$, and let μ be a positive Borel measure on $\boldsymbol{R}_+^{n+1}$ satisfying* (2) *with some integer $m \geq 1$. Then the following statements are equivalent:*

(i) *The Toeplitz operator $T_{\mu,p,q}$ is $*$-compact;*
(ii) *μ is a vanishing τ-Carleson measure, i.e., $\lim_{Y \to \mathscr{A}} \hat{\mu}_\tau^{(\alpha)}(Y) = 0$;*
(iii) *$\lim_{Y \to \mathscr{A}} \tilde{\mu}_{\tau,m}^{(\alpha)}(Y) = 0$;*
(iv) *$\lim_{Y \to \mathscr{A}} \tilde{\mu}_{\tau,k}^{(\alpha)}(Y) = 0$ for every integer $k > \left(\frac{n}{2\alpha} + 1\right)\left(\frac{\tau-2}{2}\right)$.*

PROOF. In Propositions 2 and 3, we have shown the implications "(ii) $\Rightarrow$ (i)" and "(i) $\Rightarrow$ (iii)". Lemma 3 (ii) shows the implication "(iii) $\Rightarrow$ (ii) $\Leftrightarrow$ (iv)" and we have the theorem.

Since $\boldsymbol{b}_\alpha^p$ is reflexive for $1 < p < \infty$, Theorem 1 follows from Theorem 3. Finally, we give a remark.

Remark 6. *When $q = \infty$ and $T_{\mu,p,\infty}$ is $*$-compact, the image of $T_{\mu,p,\infty}$ is in the little Bloch space $\mathscr{B}_{\alpha,0}/\boldsymbol{R}$, which follows from Remark* 5 *and the proof of Proposition* 2.

5. Proof of Theorem 2

Finally, we consider the Carleson inclusion. Combining the following propositions, we have Theorem 2.

Proposition 4. *For $1 \leq p \leq q < \infty$, we put $\tau := q/p$. If $\lim_{Y \to \mathscr{A}} \hat{\mu}_\tau^{(\alpha)}(Y) = 0$, then $\iota_{\mu,p,q}$ is $*$-compact.*

Proof. Let $(\omega_j)_j$ be an exhaustion of $\boldsymbol{R}_+^{n+1}$ and define $\iota_j : \boldsymbol{b}_\alpha^p \to L^q(\mu)$ by $\iota_j u = u \cdot 1_{\omega_j} \in L^q(\mu)$ for $u \in \boldsymbol{b}_\alpha^p$. Putting $\mu_j := \mu|_{\omega_j}$ and $\nu_j := \mu - \mu_j$, we have

$$\lim_{j \to \infty} \|(\hat{\nu}_j)_\tau^{(\alpha)}\|_\infty = 0$$

from assumption. Here we remark that for $u \in \boldsymbol{b}_\alpha^p$, by Theorem C,

$$\|(\iota_{\mu,p,q} - \iota_j)u\|_{L^q(\mu)} = \|u\|_{L^q(\nu_j)} \leq C_4 \|(\hat{\nu}_j)_\tau^{(\alpha)}\|_\infty^{1/q} \|u\|_{L^p(V)},$$

which shows

$$\|\iota_{\mu,p,q} - \iota_j\| \leq C_4 \|(\hat{\nu}_j)_\tau^{(\alpha)}\|_\infty^{1/q} \to 0$$

as $j \to \infty$ from assumption. On the other hand, ι_j is $*$-compact from the $*$-compactness of $\iota_{\mu_j,p,q}$ by Proposition 1. Thus we see that $\iota_{\mu,p,q}$ is $*$-compact, because $\|u\|_{L^q(\mu_j)} = \|\iota_j u\|_{L^q(\mu)}$.

Proposition 5. *For $1 \leq p \leq q < \infty$, we put $\tau := q/p$. If $\iota_{\mu,p,q}$ is $*$-compact, then $\lim_{Y \to \mathscr{A}} \hat{\mu}_\tau^{(\alpha)}(Y) = 0$.*

Proof. Let $m > \left(\frac{n}{2\alpha} + 1\right)\left(\frac{1}{p} - 1\right)$. For $Y \in \boldsymbol{R}_+^{n+1}$, restricting the domain of the integral to $Q^{(\alpha)}(y, \rho s)$, we have the estimate

$$\left\| \left(\frac{R_\alpha^m(\cdot, Y)}{\|R_\alpha^m(\cdot, Y)\|_{L^p(V)}} \right) \right\|_{L^q(\mu)} \geq C \hat{\mu}_\tau^{(\alpha)}(y, \rho s)$$

by (6). Since $\iota_{\mu,p,q}$ is $*$-compact, the left hand side tends to 0 as $Y \to \mathscr{A}$ by Lemma 6. Then we have

$$\lim_{Y \to \mathscr{A}} \hat{\mu}_\tau^{(\alpha)}(Y) = 0.$$

6. A relation between Toeplitz operators and Carleson inclusions

In the definition of the Toeplitz operator, we may use a modified kernel R_α^m. Then the treatment is a little simpler, especially for the case $p = 1$ or $q = \infty$. Nevertheless, in this paper, we only consider the Toeplitz operator defined by the original Bergman kernel R_α. Hence the Toeplitz operator T_μ is formally self-adjoint. Moreover the formal adjoint of the Carleson inclusion ι_μ is closely related to T_μ, i.e., $T_\mu = \iota_\mu^* \iota_\mu$ holds. In this section, we explain this relation more exactly.

We consider a positive Borel measure μ satisfying (2) with $m \geq 1$. In this case, $\iota_\mu \equiv \iota_{\mu,p,q}$ is defined densely on $\boldsymbol{b}_\alpha^p$ and we can define the adjoint operator.

REMARK 7. *Let μ be a positive Borel measure on $\boldsymbol{R}_+^{n+1}$ satisfying* (2) *for some $m \geq 1$. Then, for every $1 \leq p, q < \infty$, the inclusion $\iota_{\mu,p,q} : \boldsymbol{b}_\alpha^p \to L^q(\mu)$ defined on $\mathscr{E}_m$ is closable. In fact, let $(u_j)_j$ be a sequence in $\mathscr{E}_m$ such that there exist $u \in \boldsymbol{b}_\alpha^p$ and $v \in L^q(\mu)$ with $\lim_{j\to\infty} u_j = u$ in $\boldsymbol{b}_\alpha^p$ and $\lim_{j\to\infty} u_j = v$ in $L^q(\mu)$. Then $u = v$ μ-a.e..*

Next, we remark that $T_\mu u(X)$ is defined pointwise for each $u \in \mathscr{E}_m$, i.e.,

$$T_\mu u(X) = \int R_\alpha(X, Y) u(Y) d\mu(Y)$$

is well-defined for all $X \in \boldsymbol{R}_+^{n+1}$ and $T_\mu u$ is $L^{(\alpha)}$-harmonic on $\boldsymbol{R}_+^{n+1}$. Indeed, since the estimate (5) shows $|R_\alpha(X, \cdot)| \leq Ct^{-(n/2\alpha+1)}$ for each fixed $X = (x, t) \in \boldsymbol{R}_+^{n+1}$, the integrability

$$\int |R_\alpha(X, Y) u(Y)| d\mu(Y) \leq Ct^{-(n/2\alpha+1)} \int |u| d\mu < \infty$$

follows from (2). This estimate gives the Huygens property of $T_\mu u$ ([2, (4.1)]), which shows that $T_\mu u$ is $L^{(\alpha)}$-harmonic ([2, Proposition 2.5]).

PROPOSITION 6. *Let $1 \leq p, q \leq \infty$ with $p \neq \infty$, $q \neq 1$ and put $\tau := \frac{1}{p} + \frac{1}{q'}$, where q' denotes the exponent conjugate to q. Let μ be a positive Borel measure on $\boldsymbol{R}_+^{n+1}$ satisfying* (2) *for an integer $m \geq 1$. For $u \in \mathscr{E}_m$, $T_\mu u \in \boldsymbol{b}_\alpha^q$ ($\mathscr{B}_\alpha$ when $q = \infty$) if and only if $(\iota_{\mu,p,\tau p})u$ is in the domain of $(\iota_{\mu,q',\tau q'})^*$ and $T_\mu u = (\iota_{\mu,q',\tau q'})^* (\iota_{\mu,p,\tau p}) u$ holds.*

PROOF. First we remark that τp is the exponent conjugate to $\tau q'$, since $\frac{1}{\tau p} + \frac{1}{\tau q'} = 1$. We assume that $u \in \mathscr{E}_m$ satisfy $T_\mu u \in \boldsymbol{b}_\alpha^q$ ($\mathscr{B}_\alpha$ when $q = \infty$). Let $v \in \mathscr{E}_m \subset \boldsymbol{b}_\alpha^{q'}$ be arbitrary, take $\delta > 0$ and put $v_\delta(x, t) = v(x, t + \delta)$. Then by the Schwarz inequality and (7), we have

$$\int |u(y, s)| \int |R_\alpha(x, t, y, s + \delta) v_\delta(x, t)| dV(x, t) d\mu(y, s) < \infty.$$

Hence the Fubini theorem yields

$$\int v_\delta(y, s+\delta)u(y,s)d\mu(y,s) = \int v_\delta(x,t)T_\mu u(x, t+\delta)dV(x,t)$$
$$= \int_{\{t>\delta\}} v(x,t)T_\mu u(x,t)dV(x,t).$$

Letting $\delta \downarrow 0$, we have $\int vu\,d\mu = \int vT_\mu u\,dV$, because v is bounded and $vT_\mu u \in L^1(V)$. Then

$$\langle (\iota_{\mu,p,\tau p})u, (\iota_{\mu,q',\tau q'})v\rangle = \langle T_\mu u, v\rangle,$$

which implies $(\iota_{\mu,p,\tau p})u$ is in the domain of $(\iota_{\mu,q',\tau q'})^*$ and $(\iota_{\mu,q',\tau q'})^*(\iota_{\mu,p,\tau p})u = T_\mu u$. The opposite direction is trivial, which completes the proof.

Corollary 1. *Let $1 \le p \le q \le \infty$ with $p \ne \infty$, $q \ne 1$ and put $\tau := \frac{1}{p} + \frac{1}{q'} \in [1,2]$. Let μ be a τ-Carleson measure on $\boldsymbol{R}_+^{n+1}$. Then, both operators $\iota_{\mu,p,\tau p} : \boldsymbol{b}_\alpha^p \to L^{\tau p}(\mu)$ and $(\iota_{\mu,q',\tau q'})^* : L^{\tau p}(\mu) \to (\boldsymbol{b}_\alpha^{q'})^* = \boldsymbol{b}_\alpha^q$ ($\mathscr{B}_\alpha/\boldsymbol{R}$ when $q = \infty$) are bounded, and*

$$T_{\mu,p,q} = (\iota_{\mu,q',\tau q'})^*(\iota_{\mu,p,\tau p})$$

holds.

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