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Oscillatory criteria for differential equations with deviating argument

Dedicated to Professor O. Boruvka on the occasion of his 90th birthday

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The aim of this paper is to give a new approach for considering the question concerning the oscillatory criteria for differential equations with deviating argument.

We will deal with the differential equation

$$L_n y(t) + h(t, y(\varphi(t)), y'(\varphi(t)), \dots, y^{(n-1)}(\varphi(t))) = 0\,, \qquad n > 1 \tag{E}$$

where $h: J \times R^n \to R$, $\varphi: J \to R$, $a_i: J \to (0, \infty)$, $i = 0, 1, \dots, n$, are continuous functions, $J = [t_0, \infty)$, and

$$L_0 y(t) = a_0(t) y(t)\,, \qquad L_i y(t) = a_i(t)(L_{i-1} y(t))'\,, \qquad i = 1, 2, \dots, n\,.$$

Under a solution $y(t)$ of (E) we will understand a solution existing on some ray $[T_y, \infty)$ and such that

$$\sup\{|y(t)| : t_1 \leqq t < \infty\} > 0 \qquad \text{for any} \quad t_1 \geqq T_y\,.$$

The following basic assumptions will be used:

1. $\int^\infty a_i^{-1}(t) dt = \infty$, $i = 1, 2, \dots, n-1$;
2. $y_0 h(t, y_0, y_1, \dots, y_{n-1}) > 0$ for all $t \in J$ and any $y_i \in R$, $i = 0, 1, \dots, n-1$, $y_0 \neq 0$;
3. $y_0 h(t, y_0, y_1, \dots, y_{n-1}) < 0$ for all $t \in J$ and any $y_i \in R$, $i = 0, 1, \dots, n-1$, $y_0 \neq 0$;
4. $\lim \varphi(t) = \infty$ as $t \to \infty$.

DEFINITION 1. A solution $y(t)$ of (E) will be called oscillatory if there exists an increasing sequence $\{t_i\}_{i=1}^\infty$ such that $\lim_{i\to\infty} t_i = \infty$ and $y(t_i) = 0$, $i = 1, 2, \dots$. A solution $y(t)$ of (E) will be called nonoscillatory if it is not oscillatory, i.e. there exists $T'_y \geqq T_y$ such that $y(t) > 0$ or $y(t) < 0$ on the interval $[T'_y, \infty)$.

It follows from the assumptions 1.–4. and from the equation (E) that to each nonoscillatory solution $y(t)$ of (E) there exists such a number $T_y'' \geqq T_y$ that on the interval $[T_y'', \infty)$ each quasiderivative $L_i y(t)$, $i = 0, 1, \dots, n$ has a constant sign and therefore, $L_i y(t)$, $i = 0, 1, \dots, n-1$ are monotone functions on $[T_y'', \infty)$, so that $\lim_{t\to\infty} L_i y(t)$, $i = 0, 1, \dots, n-1$ exist in the extended sense, i.e. $\lim_{t\to\infty} |L_i y(t)|$ is finite or ∞, $i = 0, 1, \dots, n-1$. Then for the nonoscillatory solutions the following two cases are possible:

a) $\lim_{t\to\infty} |L_i y(t)| = \infty$ for all $i = 0, 1, \dots, n-1$.

We note that in this case, if $\lim_{t\to\infty} |L_{n-1} y(t)| = \infty$, then $\lim_{t\to\infty} |L_i y(t)| = \infty$ for $i = 0, 1, \dots, n-2$ and $\operatorname{sgn} L_{n-1} y(t) = \operatorname{sgn} L_i y(t)$, $i = 0, 1, \dots, n-2$.

b) There exists $k \in \{0, 1, \dots, n-1\}$ such that $\lim_{t\to\infty} L_k y(t)$ is finite, $\lim_{t\to\infty} L_i y(t) = \infty \cdot \operatorname{sgn} y(t)$, $i = 0, 1, \dots, k-1$, and $\lim_{t\to\infty} L_i y(t) = 0$, $i = k+1, \dots, n-1$.

REMARK 1. The case a) cannot occur if the assumptions 1., 2., 4., are satisfied. Indeed, in such a case for a nonoscillatory solution $y(t)$, $y(t) \neq 0$ on $[T_y, \infty)$, we have $y(t) L_n y(t) < 0$ which implies that $|L_{n-1} y(t)|$ is nonincreasing and therefore $\lim_{t\to\infty} L_{n-1} y(t)$ is finite. Thus, $k \leqq n-1$.

REMARK 2. The number k in the case b) is uniquely determined and is such that (see [1, Lemma 2 and Lemma 5])

i) if the assumption 2. holds true, then
$(-1)^{i+1} y(t) L_i y(t) > 0$, $i = k+1, \dots, n-1$, for $t > T_y$ and n even,
$(-1)^i y(t) L_i y(t) > 0$, $i = k+1, \dots, n-1$, for $t > T_y$ and n odd;

ii) if the assumption 3. holds true, then
$(-1)^i y(t) L_i y(t) > 0$, $i = k+1, \dots, n-1$, for $t > T_y$ and n even,
$(-1)^{i+1} y(t) L_i y(t) > 0$, $i = k+1, \dots, n-1$, for $t > T_y$ and n odd.

DEFINITION 2. We will say that a nonoscillatory solution $y(t)$ of (E) belongs to the class V_n if the case a) occurs, i.e. $\lim_{t\to\infty} L_i y(t) = \infty \cdot \operatorname{sgn} y(t)$, $i = 0, 1, \dots, n-1$. We will say that a nonoscillatory solution $y(t)$ of (E) belongs to the class V_k, $k \in \{0, 1, \dots, n-1\}$, if the case b) occurs.

Evidently the classes V_k, $k = 0, 1, \dots, n$, are disjoint and each nonoscillatory solution of (E) belongs to one and only one class V_k.

Our aims are to state the conditions which guarantee that $\lim_{t\to\infty} L_k y(t) = 0$ for each solution $y(t) \in V_k$, $k \in \{0, 1, \dots, n-1\}$ and to state the conditions which guarantee that the class V_k, $k \in \{0, 1, \dots, n-1\}$, is empty. For the case $\varphi(t) = t$ these problems were discussed in [1] and for the case $\varphi(t) \neq t$ in [2], [3], [4] and others.

Let $t_0 \leqq c < t < \infty$. Denote

$$P_0(t, c) = 1\,, \qquad P_i(t, c) = \int_c^t a_1^{-1}(s_1)ds_1 \int_c^{s_1} a_2^{-1}(s_2)ds_2 \cdots \int_c^{s_{i-1}} a_i^{-1}(s_i)ds_i\,,$$
$$i = 1, 2, \ldots, n-1\,, \tag{1}$$

$$Q_n(t, c) = 1\,, \qquad Q_j(t, c) = \int_c^t a_{n-1}^{-1}(s_{n-1})ds_{n-1} \int_c^{s_{n-1}} a_{n-2}^{-1}(s_{n-2})ds_{n-2}$$
$$\cdots \int_c^{s_{j+1}} a_j^{-1}(s_j)ds_j\,, \qquad j = 1, 2, \ldots, n-1\,. \tag{2}$$

It is easy to see ([1, Lemma 3]) that

$$\lim_{t\to\infty} P_i(t, c) = \infty\,, \qquad \lim_{t\to\infty} Q_i(t, c) = \infty\,, \qquad i = 1, 2, \ldots, n-1$$

and

$$\lim_{t\to\infty} \frac{Q_j(t, c)}{Q_i(t, c)} = 0\,, \qquad 0 < i < j \leqq n\,,$$

$$\lim_{t\to\infty} \frac{P_i(t, c)}{P_j(t, c)} = 0\,, \qquad 0 \leqq i < j \leqq n-1\,.$$

LEMMA 1 ([1, Lemma 4]). *Let $z(t)$ be such that $z(t) \neq 0$ on $[t_1, \infty)$ and $L_n z(t)$ exists on $[t_1, \infty)$ and suppose that $z(t)L_n z(t) \leqq 0$ on $[t_1, \infty)$, where the equality may hold at isolated points eventually. Let the assumption* 1. *be valid. Let $k \in \{0, 1, \ldots, n-1\}$ from* b). *Then there exists a $T_1 \geqq t_1$ such that*

$$\operatorname{sgn} z(t) = \operatorname{sgn} L_k z(t) \qquad \textit{for} \quad t \geqq T_1\,.$$

If $n + k$ is even, then $|L_k z(t)|$ increases on $[T_1, \infty)$ and there exist two constants $0 < c_1 < c_2$ such that for $t > T_1$

$$0 < c_1 < |L_k z(t)| < c_2$$

and

$$0 < c_1 < \left|\lim_{t\to\infty} \frac{L_0 z(t)}{P_k(t, c)}\right| < c_2\,, \qquad \lim_{t\to\infty} \frac{L_0 z(t)}{P_{k+1}(t, c)} = 0\,.$$

If $n + k$ is odd, then $|L_k z(t)|$ decreases on $[T_1, \infty)$ and there exists a constant $c > 0$ such that, for $t > T_1$, $0 < |L_k z(t)| < c$ and

$$0 \leqq \left|\lim_{t\to\infty} \frac{L_0 z(t)}{P_k(t, c)}\right| < c\,, \qquad \lim_{t\to\infty} \frac{L_0 z(t)}{P_{k+1}(t, c)} = 0\,.$$

LEMMA 2 ([1, Lemma 6]). *Let $z(t)$ be such that $z(t) \neq 0$ on $[t_1, \infty)$ and $L_n z(t)$ exists on $[t_1, \infty)$ and suppose that $z(t)L_n z(t) \geqq 0$ for $t \geqq t_1$ where the equality may hold at isolated points eventually. Let* 1. *be valid. Then there exists $T_1 \geqq t_1$ such that the following is true:*

If $k \in \{0, 1, \dots, n-1\}$ *is the number from* b), *then* $\operatorname{sgn} z(t) = \operatorname{sgn} L_k z(t)$ *for* $t > T_1$. *If* $n + k$ *is odd, then* $|L_k z(t)|$ *increases and there exist two positive constants* c_1, c_2 *such that*

$$0 < c_1 < |L_k z(t)| < c_2 \qquad \textit{for} \quad t > T_1$$

and

$$0 < c_1 < \left| \lim_{t\to\infty} \frac{a_0(t) z(t)}{P_k(t, c)} \right| < c_2 , \qquad \lim_{t\to\infty} \frac{a_0(t) z(t)}{P_{k+1}(t, c)} = 0 .$$

If $n + k$ *is even, then* $|L_k z(t)|$ *decreases and there exists a positive constant* c_3 *such that*

$$0 < |L_k z(t)| < c_3 \qquad \textit{for} \quad t > T_1 ,$$

$$0 \leqq \left| \lim_{t\to\infty} \frac{a_0(t) z(t)}{P_k(t, c)} \right| < c_3 , \qquad \lim_{t\to\infty} \frac{a_0(t) z(t)}{P_{k+1}(t, c)} = 0 .$$

Lemma 3. *Let* $y(t) \in V_k$, $k \in \{0, 1, \dots, n-1\}$. *Then*

$$\lim_{t\to\infty} \frac{a_0(t) y(t)}{P_k(t, c)} = \lim_{t\to\infty} L_k y(t) = c_k . \tag{3}$$

If $c_k \neq 0$, *then there exist constants* $\alpha_k > 0$, $\beta_k > 0$ *and* $T_k > t_0$ *such that*

$$\frac{\alpha_k P_k(t, c)}{a_0(t)} \leqq |y(t)| \leqq \frac{\beta_k P_k(t, c)}{a_0(t)} , \qquad t > T_k . \tag{4}$$

Proof. This follows from l'Hospital's rule, Lemma 1 and Lemma 2.

Theorem 1. *Let the conditions* 1.–4. *be satisfied. Let* $G(t, u)$: $[t_0, \infty) \times [0, \infty) \to R_+$ *be continuous and nondecreasing in* u *for each fixed* t *and such that*

$$|h(t, y_0, y_1, \dots, y_{n-1})| \geqq G(t, |y_0|) \tag{5}$$

for all $(y_0, y_1, \dots, y_{n-1}) \in R^n$. *Moreover, let* $k \in \{0, 1, \dots, n-1\}$ *and suppose that*

$$\int_t^\infty \frac{1}{a_n(s)} Q_{k+1}(s, t) G\left(s, \frac{\alpha}{a_0(\varphi(s))} p_k(\varphi(s), c)\right) ds = \infty \tag{6}$$

for all $t \geqq T_k$ *such that* $\varphi(s) > c$ *for* $s > T_k$, $c \geqq t_0$ *and for each* $\alpha > 0$, *or*

$$\limsup_{t\to\infty} \int_t^\infty \frac{1}{a_n(s)} Q_{k+1}(s, t) G\left(\frac{\alpha}{a_0(\varphi(s))} p_k(\varphi(s), c)\right) ds > 0 \tag{7}$$

for each $\alpha > 0$. *Then for each* $y(t) \in V_k$ *we have* $\lim_{t\to\infty} L_k y(t) = 0$.

PROOF. Let $y(t) \in V_k$, $k \in \{0, 1, \ldots, n-1\}$ and let $\lim_{t\to\infty} L_k y(t) = c_k \neq 0$. Then, respecting the fact that $\lim_{t\to\infty} L_i y(t) = 0$, $i = k+1, \ldots, n-1$, integration of the equation (E) gives

$$(8) \qquad L_k y(t) = c_k + (-1)^{n-k+1} \int_t^\infty \frac{h(s, \tilde{y}(\varphi(s)))}{a_n(s)} Q_{k+1}(s, t)\, ds, \qquad t \geqq T_y,$$

where $\tilde{y}(t) = (y(t), y'(t), \ldots, y^{(n-1)}(t))$. Let $T_y > t_0$ be such that $y(t)$ has a constant sign for $t \geqq T_y$ and such that $\operatorname{sgn} y(t) = \operatorname{sgn} L_k y(t)$ for $t \geqq T_y$. Let $u \geqq T_y$ be such that $\varphi(t) \geqq T_y$ for $t \geqq u$. Then for $s \geqq t \geqq u \geqq T_y$ we have $\operatorname{sgn} y(\varphi(s)) = \operatorname{sgn} y(T_y) = \operatorname{sgn} L_k y(t)$. Multiplying the preceding equality by $\operatorname{sgn} y(T_y)$ we get

$$\operatorname{sgn} y(T_y)(L_k y(t) - c_k) = (-1)^{n-k+1} \int_t^\infty \frac{|h(s, \tilde{y}(\varphi(s)))|}{a_n(s)} Q_{k+1}(s, t)\, ds$$

for $t \geqq u$ or

$$|L_k y(t) - c_k| = \int_t^\infty \frac{|h(s, \tilde{y}(\varphi(s)))|}{a_n(s)} Q_{k+1}(s, t)\, ds.$$

Using (5) and (4) and the monotonicity of G we have

$$(9) \qquad |L_k y(t) - c_k| \geqq \int_t^\infty \frac{1}{a_n(s)} Q_{k+1}(s, t) G\left(s, \frac{\alpha_k}{a_0(\varphi(s))} P_k(\varphi(s), c)\right) ds$$

for $t \geqq u$. The expression on the left hand side is bounded, but this contradicts the assumption (6). If the assumption (7) is satisfied, then we get once more a contradiction because $\lim_{t\to\infty} |L_k y(t) - c_k| = 0$.

THEOREM 2. *Let all assumptions of Theorem 1 be satisfied. Then in the case that the condition 2. holds true the sets V_k are empty for $n + k$ even. In the case that the assumption 3. holds true the sets V_k are empty for $n + k$ odd.*

PROOF. From Theorem 1 we see that for $y(t) \in V_k$, $k \in \{0, 1, \ldots, n-1\}$, $\lim_{t\to\infty} |L_k y(t)| = 0$. But from Lemma 1 it follows that $|L_k y(t)|$ increases if $n + k$ is even and from Lemma 2 it follows that $|L_k y(t)|$ increases if $n + k$ is odd. This leads to a contradiction.

Let us denote

$$\gamma(t) = \sup\{s \geqq t_0 : \varphi(s) \leqq t\} \qquad \text{for all} \quad t \geqq t_0$$

and

$$m(t) = \max\{\gamma(t), t\}, \qquad t \geqq t_0.$$

Thus $m(t) \geqq t$. From the continuity of $\varphi(t)$ we have $\varphi(s) > t$ for $s > \gamma(t)$, and $\varphi(s) \geqq t$ for $s \geqq m(t)$, $t \geqq t_0$. Evidently $\lim_{t\to\infty} m(t) = \infty$.

REMARK 3. It follows from the definition of the classes V_k, $k \in \{0, 1, \dots, n-1\}$, that for any $y(t) \in V_k$, $\lim_{t\to\infty} L_{n-1} y(t)$ is finite. Thus

$$\lim_{t\to\infty} \int_t^\infty a_n^{-1}(s)|h(s, \tilde{y}(\varphi(s)))|\,ds = 0\,. \tag{10}$$

Taking the assumption (5) into the consideration we have

$$\lim_{t\to\infty} \int_t^\infty a_n^{-1}(s)G(s, |y(\varphi(s))|)ds = 0\,. \tag{11}$$

Our following considerations are now based on this fact.

Let the assumptions of Theorem 1 be satisfied. Then $\lim_{t\to\infty} L_k y(t) = 0$ for $y(t) \in V_k$, $k \in \{0, 1, \dots, n-1\}$, and therefore from (8) we have

$$L_k y(t) = (-1)^{n-k+1} \int_t^\infty a_n^{-1}(s) Q_{k+1}(s, t) h(s, \tilde{y}(\varphi(s)))ds\,, \qquad t \geqq T_y\,, \tag{12}$$

where T_y is such that

$$\operatorname{sgn} L_k y(t) = \operatorname{sgn} y(t) = \operatorname{sgn} y(\varphi(s))\,, \qquad s \geqq t \geqq T_y\,. \tag{13}$$

If the condition 2. is satisfied, then

$$\operatorname{sgn} y(t) = \operatorname{sgn} L_k y(t) = \operatorname{sgn} h(s, \tilde{y}(\varphi(s)))\,, \qquad s \geqq t \geqq T_y\,.$$

Therefore in this case $(-1)^{n-k+1} = +1$.

If the condition 3. is satisfied, then

$$\operatorname{sgn} y(t) = \operatorname{sgn} L_k y(t) = -\operatorname{sgn} h(s, \tilde{y}(\varphi(s)))\,, \qquad s \geqq t \geqq T_y\,.$$

In this case $(-1)^{n-k+1} = -1$.

a) Consider the case that $y(t) > 0$ for $t \geqq T_y$ and let $k > 0$. Then from (12) we get

$$L_k y(t) = \int_t^\infty a_n^{-1}(s) Q_{k+1}(s, t)|h(s, y(\varphi(s)))|ds\,, \qquad t \geqq T_y \tag{14}$$

in both cases 2. and 3. An integration of (14) between u and v, $T_y \leqq u \leqq v$ and the application of Fubini's theorem give

$$\begin{aligned} L_{k-1} y(v) - L_{k-1} y(u) = &\int_u^v a_n^{-1}(s)|h(s, \tilde{y}(\varphi(s)))| \int_u^s a_k^{-1}(t) Q_{k+1}(s, t)dtds \\ &+ \int_v^\infty a_n^{-1}(s)|h(s, \tilde{y}(\varphi(s)))| \int_u^v a_k^{-1}(t) Q_{k+1}(s, t)dtds\,. \end{aligned} \tag{15}$$

Taking into consideration that $L_{k-1} y(t) > 0$ and that both terms on the right hand side are nonnegative, we get

$$L_{k-1}y(v) \geqq \int_v^\infty a_n^{-1}(s)|h(s, \tilde{y}(\varphi(s)))| \int_u^v a_k^{-1}(t)Q_{k+1}(s, t)dtds \tag{16}$$

for $v > u \geqq T_y$. From the definition of $Q_{k+1}(s, t)$ it follows that for $t \leqq v \leqq s$

$$Q_{k+1}(s, t) \geqq Q_{k+1}(v, t) . \tag{17}$$

Using this fact we see from (16) that

$$L_{k-1}y(v) \geqq \int_u^v a_k^{-1}(t)Q_{k+1}(v, t)dt \int_v^\infty a_n^{-1}(s)|h(s, \tilde{y}(\varphi(s)))|ds . \tag{18}$$

Repeating this procedure $(k-1)$-times, we get

$$\begin{aligned} L_0 y(v) \geqq & \int_u^v a_1^{-1}(t_1) \int_u^{t_1} a_2^{-1}(t_2) \cdots \int_u^{t_{k-1}} a_k^{-1}(t)Q_{k+1}(t_{k-1}, t)dtdt_{k-1} \cdots dt_1 \\ & \cdot \int_v^\infty a_n^{-1}(s)|h(s, \tilde{y}(\varphi(s)))|ds , \qquad T_y \leqq u < v . \end{aligned} \tag{19}$$

Denote

$$R_k(v, u) = \int_u^v a_1^{-1}(t_1) \int_u^{t_1} a_2^{-1}(t_2) \cdots \int_u^{t_{k-1}} a_k^{-1}(t)Q_{k+1}(t_{k-1}, t)dw_k , \tag{20}$$

where $dw_k = dtdt_{k-1} \cdots dt_1$. Then we have

$$L_0 y(v) \geqq R_k(v, u) \int_v^\infty a_n^{-1}(s)|h(s, \tilde{y}(\varphi(s)))|ds , \qquad T_y \leqq u < v . \tag{21}$$

Taking into consideration (5), monotonicity of G and the properties of $m(t)$, we have

$$\begin{aligned} L_0 y(v) &\geqq R_k(v, u) \int_v^\infty a_n^{-1}(s)G(s, |y(\varphi(s))|)ds \\ &\geqq R_k(v, u) \int_{m(v)}^\infty a_n^{-1}(s)G(s, |y(\varphi(s))|)ds \\ &= R_k(v, u) \int_{m(v)}^\infty a_n^{-1}(s)G(s, a_0^{-1}(\varphi(s))|L_0 y(\varphi(s))|)ds . \end{aligned} \tag{22}$$

Note that $\varphi(s) \geqq v$ for $s \geqq m(v)$ so that $|L_0 y(\varphi(s))| \geqq |L_0 y(v)|$ because $|L_0 y(t)|$ is nondecreasing. Then since $G(t, z)$ is nondecreasing in z,

$$G(s, a_0^{-1}(\varphi(s))|L_0 y(\varphi(s))|) \geqq G(s, a_0^{-1}(\varphi(s))|L_0 y(v)|) , \tag{23}$$

and (22) implies that

$$(24) \quad L_0 y(v) \geqq R_k(v, u) \int_{m(v)}^{\infty} a_n^{-1}(s) G(s, a_0^{-1}(\varphi(s)) |L_0 y(v)|) ds \,, \qquad T_y \leqq u < v \,.$$

Respecting once more the monotonicity of $G(t, z)$ in z, we have

$$a_n^{-1}(s) G(s, a_0^{-1}(\varphi(s)) |L_0 y(v)|) \geqq a_n^{-1}(s) G(s, a_0^{-1}(\varphi(s)) R_k(v, u) \times \int_{m(v)}^{\infty} a_n^{-1}(\tau) G(\tau, a_0^{-1}(\varphi(\tau)) |L_0 y(v)|) d\tau)$$

or

$$\int_{m(v)}^{\infty} a_n^{-1}(s) G(s, a^{-1}(\varphi(s)) |L_0 y(v)|) ds \geqq \int_{m(v)}^{\infty} a_n^{-1}(s) G(s, a_0^{-1}(\varphi(s)) R_k(v, u) \times \int_{m(v)}^{\infty} a_n^{-1}(\tau) G(\tau, a_0^{-1}(\varphi(\tau)) |L_0 y(v)|) d\tau) ds \,.$$

Let us denote

$$(25) \qquad p(v) = \int_{m(v)}^{\infty} a_n^{-1}(s) G(s, a_0^{-1}(\varphi(s)) |L_0 y(v)|) ds \,.$$

Then we get

$$(26) \qquad p(v) \geqq \int_{m(v)}^{\infty} a_n^{-1}(s) G(s, a_0^{-1}(\varphi(s)) R_k(v, u) p(v)) ds \,, \qquad T_y \leqq u < v \,.$$

Taking (5), (23) and (26) into consideration, we obtain

$$L_{n-1} y(m(v)) = \int_{m(v)}^{\infty} a_n^{-1}(s) |h(s, \tilde{y}(\varphi(s)))| ds \geqq \int_{m(v)}^{\infty} a_n^{-1}(s) G(s, |y(\varphi(s))|) ds$$
$$\geqq \int_{m(v)}^{\infty} a_n^{-1}(s) G(s, a_0^{-1}(\varphi(s)) |L_0 y(v)|) ds = p(v)$$

and $0 = \lim_{v \to \infty} L_{n-1} y(m(v)) \geqq \lim_{v \to \infty} p(v) \geqq 0$. Thus

$$(27) \qquad \lim_{v \to \infty} p(v) = 0$$

b) Consider the case that $y(t) < 0$ for $t > T_y$ and $k > 0$. Then from (12) we get

$$(28) \qquad -L_k y(t) = \int_t^{\infty} a_n^{-1}(s) Q_{k+1}(s, t) |h(s, \tilde{y}(\varphi(s))| ds \,, \qquad t \geqq T_y$$

in both cases 2. and 3. An integration between u and v, $T_y \leqq u < v$, and the application of Fubini's theorem give

$$-L_{k-1}y(v)+L_{k-1}y(u)=\int_u^v a_n^{-1}(s)|h(s,\tilde{y}(\varphi(s)))|\int_u^s a_k^{-1}(t)Q_{k+1}(s,t)dtds$$
$$+\int_v^\infty a_n^{-1}(s)|h(s,\tilde{y}(\varphi(s)))|\int_u^v a_k^{-1}(t)Q_{k+1}(s,t)dtds\,.$$

Because $L_{k-1}y(u)<0$ and both terms on the right hand side are nonnegative, we have

$$-L_{k-1}y(v)\geqq\int_v^\infty a_n^{-1}(s)|h(s,\tilde{y}(\varphi(s)))|\int_u^v a_k^{-1}(t)Q_{k+1}(s,t)dtds\,.$$

Repeating the similar consideration as was done in the case $y(t)>0$, we get

$$-L_0y(v)\geqq R_k(v,u)\int_v^\infty a_n^{-1}(s)|h(s,\tilde{y}(\varphi(s)))|ds\,,\qquad T_y\leqq u<v\,,$$

and

$$-L_0y(v)\geqq R_k(v,u)\int_{m(v)}^\infty a_n^{-1}(s)G(s,a_0^{-1}(\varphi(s))|L_0y(\varphi(s))|)ds\,,$$

and finally

$$|L_0y(v)|\geqq R_k(v,u)\int_{m(v)}^\infty a_n^{-1}(s)G(s,a_0^{-1}(\varphi(s))|L_0y(v)|)ds\,,\qquad T_y\leqq u<v\,,\tag{29}$$

and

$$\int_{m(v)}^\infty a_n^{-1}(s)G(s,a_0^{-1}(\varphi(s))|L_0y(v)|)ds\geqq\int_{m(v)}^\infty a_n^{-1}(s)G(s,a_0^{-1}(\varphi(s))R_k(v,u)$$
$$\times\int_{m(v)}^\infty a_n^{-1}(\tau)G(\tau,a_0^{-1}(\varphi(\tau))|L_0y(v)|)d\tau)ds\,.$$

Denoting

$$q(v)=\int_{m(v)}^\infty a_n^{-1}(s)G(s,a_0^{-1}(\varphi(s))|L_0y(v)|)ds\tag{30}$$

we get

$$q(v)\geqq\int_{m(v)}^\infty a_n^{-1}(s)G(s,a_0^{-1}(\varphi(s))R_k(v,u)q(v))ds\,,\qquad T_y\leqq u<v\,.\tag{31}$$

Similar considerations as in the case $p(v)$ give us that

$$\lim_{v\to\infty}q(v)=0\,.\tag{32}$$

Thus for $-L_0y(v)=|L_0y(v)|$ and $q(v)$ in the case that $y(t)<0$ for $t\geqq T_y$ we have the same inequalities as for $|L_0 y(v)|$ and $p(v)$ in the case that $y(t) > 0$ for $t \geqq T_y$.

THEOREM 3. *Let all assumptions of Theorem 1 be satisfied. Moreover, suppose that for all fixed* $t \geqq t_0$

$$(33) \qquad y^{-1}G(t, y) \quad \text{nondecreasing for} \quad y > 0$$

and that for $k \in \{1, 2, \dots, n-1\}$

$$(34) \qquad \limsup_{v\to\infty} R_k(v, u) \int_{m(v)}^{\infty} a_n^{-1}(s)c^{-1}G(s, a_0^{-1}(\varphi(s))c)ds > 1$$

for some $c > 0$. *Then the set* V_k *is empty.*

PROOF. Let $y(t) \in V_k$, $k \in \{1, 2, \dots, n-1\}$. Taking the fact that $\lim_{v\to\infty} |L_0 y(v)| = \infty$ into consideration, we see that for $c > 0$ there exists $v_1 > u \geqq T_y$ such that $|L_0 y(v)| > c$ for all $v > v_1$. Then from (24) (or (29)) and (33) we get

$$1 \geqq R_k(u, v) \int_{m(v)}^{\infty} a_n^{-1}(s)a_0^{-1}(\varphi(s)) \frac{G(s, a_0^{-1}(\varphi(s))c)}{a_0^{-1}(\varphi(s))c} ds$$

for all $v > v_1$. But this leads to a contradiction with (34).

THEOREM 4. *Let all assumptions of Theorem 1 be satisfied. Moreover, suppose that for all fixed* $t \geqq t_0$

$$(35) \qquad y^{-1}G(t, y) \quad \text{nonincreasing for} \quad y > 0$$

and that for $k \in \{1, 2, \dots, n-1\}$

$$(36) \qquad \limsup_{v\to\infty} \int_{m(v)}^{\infty} a_n^{-1}(s)c^{-1}G(s, R_k(v, u)a_0^{-1}(\varphi(s))c)ds > 1$$

for some $c > 0$. *Then the set* V_k *is empty.*

PROOF. Let $y(t) \in V_k$, $k \in \{1, 2, \dots, n-1\}$. Taking into consideration that $\lim_{v\to\infty} p(v) = 0$ ($\lim_{v\to\infty} q(v) = 0$) and $p(v) > 0$ ($q(v) > 0$) for all $v > u$, we see that to $c > 0$ there exists $v_2 > u \geqq T_y$ such that $c > p(v)$ ($c > q(v)$) for all $v > v_2$. Then from (26) ((31)) we get

$$\begin{aligned} 1 &\geqq \int_{m(v)}^{\infty} a_n^{-1}(s)a_0^{-1}(\varphi(s))R_k(v, u)\frac{G(s, a_0^{-1}(\varphi(s))R_k(v, u)p(v))}{a_0^{-1}(\varphi(s))R_k(v, u)p(v)} ds \\ &\geqq \int_{m(v)}^{\infty} a_n^{-1}(s)a_0^{-1}(\varphi(s))R_k(v, u)\frac{G(s, a_0^{-1}(\varphi(s))R_k(v, u)c)}{a_0^{-1}(\varphi(s))R_k(v, u)c} \end{aligned}$$

for all $v > v_2$. But this leads to a contradiction with (36).

DEFINITION 3. We will say that the equation (E) has property A if in the case that n is even all solutions of (E) are oscillatory and in the case that n is odd each solution $y(t)$ of (E) is either oscillatory or $\lim_{t\to\infty} L_i y(t) = 0$, $i = 0, 1, \ldots, n-1$.

DEFINITION 4. We will say that the equation (E) has property B if for n even each solution $y(t)$ of (E) is either oscillatory or $\lim_{t\to\infty} L_i y(t) = 0$, $i = 0, 1, \ldots, n-1$ or belongs to the class V_n, i.e. $\lim_{t\to\infty} |L_i y(t)| = \infty$, $i = 0, 1, \ldots, n-1$, and if for n odd each solution $y(t)$ of (E) is oscillatory or belongs to the class V_n

Now we can state the summary result.

THEOREM 5. *Let all assumptions of Theorem* 1 *be satisfied.* a) *If* 2. *holds true and if* (33) *and* (34) (*or* (35) *and* (36)) *hold for* $k = 1, 2, \ldots, n-1$, *then the equation* (E) *has property* A.

b) *If* 3. *holds true and if* (33) *and* (36) (*or* (35) *and* (36)) *hold for* $k = 1, 2, \ldots, n-1$, *then the equation* (E) *has property* B.

References

[1] M. Švec, Behavior of nonoscillatory solutions of some nonlinear differential equations, Acta Mathematica U.C. XXXIX-1980, 115–130.

[2] V. Šeda, Nonoscillatory solutions of differential equations with deviating argument, Czech. Math. J., **36** (111) (1986), 93–107.

[3] J. Ohriska, Sufficient conditions for the oscillation of n-th order nonlinear delay differential equations, Czech. Math. J., **30** (105) (1980), 490–497.

[4] Y. Kitamura, Oscillation of functional differential equations with general deviating arguments, Hiroshima Math. J., **15** (1985), 445–491.

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