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A limit theorem of symmetric statistics for non-identically distributed independent random elements

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§1. Introduction

In paper [14], Mustafid and Kubo have obtained a description of the asymptotic distribution for symmetric statistics based on samples of identically distributed independent random elements $X_{n,1}, \ldots, X_{n,n}$,

$$\sum_{k=0}^{n} \sum_{1 \leq s_1 < \cdots < s_k \leq n} h_k(X_{n,s_1}, \ldots, X_{n,s_k}), \tag{1.1}$$

in terms of multiple Poisson-Wiener-Ito integrals, where h_k is a symmetric function. Central limit theorems of symmetric statistics (1.1) have been obtained by several authors. See e.g., Dynkin and Mandelbaum [7] and Mandelbaum and Taqqu [13]. They obtained the asymptotic distribution of (1.1) in terms of multiple Wiener-Ito integrals under a suitable normalization. A problem of limiting distribution of symmetric statistics is closely related to that of U-statistics or von Mises statistics. An alternative approach to the limiting distribution is due to Dehling, Denker and Philipp [5], [6] and Dehling [4]. They described limiting distributions in terms of multiple stochastic integrals with respect to Kiefer processes.

The aim of this paper is to generalize the result of Mustafid and Kubo's paper [14] to the non-identically distributed independent random elements case. Even in the simple case when the symmetric statistics are symmetric polynomials, investigations in this direction have been considered by some authors. Teicher [16] described the asymptotic distribution of symmetric polynomials in terms of Hermite polynomials and in the other limiting distribution, Avram and Taqqu [1] described the asymptotic distribution in terms of a multiple integral with respect to a Lévy process. The results of this paper are stated in Section 3. In Section 2, we recall some results on convergence properties of Radon measures and Poisson random measures. In Section 4, we present examples.

§2. Convergence in distribution

Let $\mathfrak{X}$ be a locally compact second countable Hausdorff space. Let $\mathscr{A}$ denote the topological Borel field in $\mathfrak{X}$ and $\mathscr{B}$ the ring of all bounded (i.e.

relatively compact) sets in $\mathscr{A}$. A measure λ on $(\mathfrak{X}, \mathscr{A})$ is called *a Radon measure* if $\lambda(B) < \infty$ for all $B \in \mathscr{B}$. Let $\mathscr{M}(\mathfrak{X})$ be the family of all Radon measures on $\mathfrak{X}$. Let $\mathscr{C}(\mathfrak{X})$ be the space of all bounded continuous functions on $\mathfrak{X}$ and $\mathscr{K}(\mathfrak{X})$ be the space of all continuous functions on $\mathfrak{X}$ with compact support. By a vague topology of $\mathscr{M}(\mathfrak{X})$ we mean the weakest topology under which

$$\lambda \in \mathscr{M}(\mathfrak{X}) \to \lambda(f) \equiv \int_{\mathfrak{X}} f(x)\, d\lambda(x) \in \boldsymbol{R}$$

is continuous for $f \in \mathscr{K}(\mathfrak{X})$. We use some basic properties of Radon measures written in [14] and refer to Kallenberg [12] for further details.

Let λ be a Radon measure on $(\mathfrak{X}, \mathscr{A})$. We define a class of bounded sets $\mathscr{B}_\lambda$ by

$$\mathscr{B}_\lambda \equiv \{B \in \mathscr{B};\ \lambda(\partial B) = 0\}\,,$$

where ∂B denotes the boundary of B. A random measure $\{P_\lambda(B) = P_\lambda(\omega, B), B \in \mathscr{B}\}$ is called *a Poisson random measure with intensity* λ, if for any natural number p, any disjoint sets $B_1, \ldots, B_p \in \mathscr{B}$ and any non-negative integers $q_1, \ldots, q_p$,

$$\begin{aligned}\Pr(P_\lambda(B_1) &= q_1, \ldots, P_\lambda(B_p) = q_p) \\ &= \frac{1}{q_1! \ldots q_p!} \lambda(B_1)^{q_1} \ldots \lambda(B_p)^{q_p} \exp[-\lambda(B_1) - \cdots - \lambda(B_p)]\,.\end{aligned}$$

A sequence of random elements X_n converges to X in distribution sense and is denoted as $X_n \xrightarrow{d} X$, if the distribution ν_n of X_n converges weakly to the distribution ν of X, that is,

$$\lim_{n\to\infty} \int f(x)\, d\nu_n(x) = \int f(x)\, d\nu(x)\,,$$

for any $f \in \mathscr{C}(\mathfrak{X})$.

Let $X_{n,1}, \ldots, X_{n,k_n}$ $(1 \le k_n \le \infty)$, $n = 1, 2, \ldots$, be sequences of independent random elements on $\mathfrak{X}$ with distributions $\nu_{n,1}, \ldots, \nu_{n,k_n} \in \mathscr{M}(\mathfrak{X})$, respectively. We assume the following:

(A.1) $\lambda_n \equiv \sum_{i=1}^{k_n} \nu_{n,i}$ converges vaguely to a $\lambda \in \mathscr{M}(\mathfrak{X})$ without atoms as $n \to \infty$,
(A.2) $\lim_{n\to\infty} \max_i \nu_{n,i}(K) = 0$ for any compact set K.

LEMMA 2.1 ([12]). *For $\nu, \nu_1, \nu_2, \ldots \in \mathscr{M}(\mathfrak{X})$, the following two statements are equivalent*:

(i) *ν_n converges vaguely to ν as $n \to \infty$,*
(ii) *$\lim_{n\to\infty} \nu_n(B) = \nu(B)$ for all $B \in \mathscr{B}_\nu$.*

We define a sequence of random point measures $\{M_n\}$ by

(2.1) $$M_n \equiv \sum_{i=1}^{k_n} \delta_{X_{n,i}},$$

where δ_x is the Dirac measure at x. Then the following theorem is seen by Corollary 7.5 of Kallenberg [12] and Theorem of Polak [15].

THEOREM 2.2. *In order that M_n given by* (2.1) *converges in distribution to a Poisson random measure with intensity λ as $n \to \infty$, it is necessary and sufficient that*

$$\lim_{n\to\infty} \sum_{i=1}^{k_n} \Pr\{\delta_{X_{n,i}}(B) = 1\} = \lambda(B), \qquad B \in \mathscr{B}_\lambda,$$

$$\lim_{n\to\infty} \max_i \Pr\{\delta_{X_{n,i}}(B) = 1\} = 0, \qquad B \in \mathscr{B}_\lambda.$$

§3. The asymptotic distribution of symmetric statistics

In this section we generalize the results of Mustafid and Kubo's paper [14] to the case of non-identically distributed independent random elements. Let $X_{n,1}, \dots, X_{n,k_n}$ $(1 \le k_n \le \infty)$, $n = 1, 2, \dots$, be sequences of independent random elements on $\mathfrak{X}$ as in Section 2 which satisfy the assumptions (A.1) and (A.2). For a symmetric function $h_k(x_1, \dots, x_k)$, we define symmetric statistics based on samples $X_{n,1}, \dots, X_{n,k_n}$, $n = 1, 2, \dots$, by

$$\sigma_k^n(h_k) \equiv \begin{cases} \displaystyle\sum_{1 \le s_1 < \cdots < s_k \le k_n} h_k(X_{n,s_1}, \dots, X_{n,s_k}) & \text{for } k \le k_n, \\ 0 & \text{for } k > k_n. \end{cases}$$

NOTATION 1. Denote by $\mathscr{K}(\mathfrak{X})$ the set of all sequences $h = \{h_k\}_{k \ge 0}$ of continuous symmetric functions which satisfy the following conditions:
(K.1) there exists a compact set K such that for any $k \ge 1$

$$h_k(x_1, \dots, x_k) = 0 \qquad \text{if } (x_1, \dots, x_k) \notin K^k,$$

(K.2) there exists a constant $H > 1$ such that for any $k \ge 0$

$$h_k(x_1, \dots, x_k) < H^{k+1}.$$

By the same method as in [14], we investigate the asymptotic distribution of the symmetric statistics

(3.1) $$Y_n(h) \equiv \sum_{k=0}^{k_n} \sigma_k^n(h_k),$$

for $h = (h_0, h_1, \dots) \in \mathscr{K}(\mathfrak{X})$. We will show that the limiting distribution is expressed in terms of multiple Poisson-Wiener-Ito integrals with respect to a Poisson random measure P_λ with intensity λ;

$$W(h) = \sum_{k=0}^{\infty} \frac{1}{k!} \int \cdots \int h_k(x_1, \ldots, x_k)\, dP_\lambda(x_1) \cdots dP_\lambda(x_k). \tag{3.2}$$

For the Radon measure λ, let $\bar{\mathscr{E}}(\mathfrak{X})$ be the space of all sequences of symmetric step functions $h = \{h_k\}_{k \geq 0}$ of special forms as follow: there exist a natural number $p \geq 1$ and disjoint sets $B_1, \ldots, B_p \in \mathscr{B}_\lambda$ such that each h_k is expressed in the form

$$h_k(x^k) = \begin{cases} h_0 & k = 0, \\ \displaystyle\sum_{1 \leq i_1, \ldots, i_k \leq p} h_{i_1, \ldots, i_k} \chi_{B_{i_1}}(x_1) \cdots \chi_{B_{i_k}}(x_k) & 1 \leq k \leq p, \\ 0 & p < k, \end{cases}$$

where coefficients $h_{i_1, \ldots, i_k}$, $1 \leq i_1, \ldots, i_k \leq p$, are symmetric and $h_{i_1, \ldots, i_k} = 0$, if $i_s = i_t$ with some $s \neq t$. We denote

$$x^k = (x_1, \ldots, x_k) \in \mathfrak{X}^k \quad \text{and} \quad d\lambda^k(x^k) = d\lambda(x_1) \cdots d\lambda(x_k).$$

As in [14], for a given Radon measure ν, we define a norm $\|h\|_\nu$ of a sequence of symmetric functions $h = \{h_k\}_{k=0}^{\infty}$ by

$$\|h\|_\nu^2 \equiv \sum_{k,l=0}^{\infty} \sum_{j=0}^{k \wedge l} \frac{1}{(k-j)!} \frac{1}{(l-j)!} \frac{1}{j!}$$
$$\times \int \cdots \int |h_k(x^j, y^{k-j}) h_l(x^j, z^{l-j})| d\nu^j(x^j) d\nu^{k-j}(y^{k-j}) d\nu^{l-j}(z^{l-j}),$$

where $k \wedge l$ is the minimum of k and l. We define a norm $\|\cdot\|$ by

$$\|h\| \equiv \varlimsup_{n \to \infty} \|h\|_{\lambda_n} \left(\geq \varliminf_{n \to \infty} \|h\|_{\lambda_n} \geq \|h\|_\lambda \right). \tag{3.3}$$

NOTATION 2. Denote by $\bar{\mathscr{H}}(\mathfrak{X})$ the set of all sequences $h = \{h_k\}_{k \geq 0}$ which can be approximated by elements of $\bar{\mathscr{E}}(\mathfrak{X})$ with respect to the norm $\|\cdot\|$, that is, for any $h \in \bar{\mathscr{H}}(\mathfrak{X})$ and any $\varepsilon > 0$, there exists an $h^\varepsilon \in \bar{\mathscr{E}}(\mathfrak{X})$ such that

$$\|h - h^\varepsilon\| < \varepsilon.$$

By the same way as in [14], we have the following estimation of the covariance,

$$
\begin{aligned}
E|\sigma_k^n(h_k)\sigma_l^n(g_l)| &\le \sum_{1\le s_1<\cdots<s_k\le k_n}\ \sum_{1\le r_1<\cdots<r_l\le k_n} \\
&\quad\times E|h_k(X_{n,s_1},\dots,X_{n,s_k})g_l(X_{n,r_1},\dots,X_{n,r_l})| \\
&= \textstyle\sum_{j=0}^{k\wedge l}\sum^{\#} E|h_k(X_{n,s_1},\dots,X_{n,s_j},X_{n,s_{j+1}},\dots,X_{n,s_k}) \\
&\quad\times g_l(X_{n,s_1},\dots,X_{n,s_j},X_{n,s_{k+1}},\dots,X_{n,s_{k+l-j}})| \\
&\le \textstyle\sum_{j=0}^{k\wedge l}\dfrac{1}{(k-j)!}\dfrac{1}{(l-j)!}\dfrac{1}{j!}\displaystyle\sum_{1\le s_1,\dots,s_{k+l-j}\le k_n} \\
&\quad\times E|h_k(X_{n,s_1},\dots,X_{n,s_j},X_{n,s_{j+1}},\dots,X_{n,s_k}) \\
&\quad\times g_l(X_{n,s_1},\dots,X_{n,s_j},X_{n,s_{k+1}},\dots,X_{n,s_{k+l-j}})| \\
&\le \textstyle\sum_{j=0}^{k\wedge l}\dfrac{1}{(k-j)!}\dfrac{1}{(l-j)!}\dfrac{1}{j!} \\
&\quad\times \displaystyle\int\cdots\int |h_k(x^j,y^{k-j})g_l(x^j,z^{l-j})|\,d\lambda_n^j(x^j)\,d\lambda_n^{k-j}(y^{k-j})\,d\lambda_n^{l-j}(z^{l-j}),
\end{aligned}
$$

for $k_n \ge k$, $k_n \ge l$ and that

$$E|\sigma_k^n(h_k)\sigma_l^n(g_l)| = 0\,,$$

for $k_n < k$ or $k_n < l$, where the sum $\sum^{\#}$ is extended over all different s_i, $1 \le i \le k+l-j$ such that $1 \le s_1 < \cdots < s_j \le k_n$, $1 \le s_{j+1} < \cdots < s_k \le k_n$ and $1 \le s_{k+1} < \cdots < s_{k+l-j} \le k_n$. Then we have

$$
\begin{aligned}
(3.4)\quad E|Y_n(h)|^2 &\le \textstyle\sum_{k,l=0}^{k_n}\sum_{j=0}^{k\wedge l}\dfrac{1}{(k-j)!}\dfrac{1}{(l-j)!}\dfrac{1}{j!} \\
&\quad\times \displaystyle\int\cdots\int |h_k(x^j,y^{k-j})h_l(x^j,z^{l-j})|\,d\lambda_n^j(x^j)\,d\lambda_n^{k-j}(y^{k-j})\,d\lambda_n^{l-j}(z^{l-j}) \\
&\le \|h\|_{\lambda_n}^2\,.
\end{aligned}
$$

From the definition of multiple Poisson-Wiener-Ito integrals, for $h \in \bar{\mathscr{E}}(\mathfrak{X})$, (3.2) can be expressed by

$$W(h) = \textstyle\sum_{k=1}^{p}\dfrac{1}{k!}\displaystyle\sum_{1\le i_1,\dots,i_k\le p} h_{i_1,\dots,i_k}P_\lambda(B_{i_1})\cdots P_\lambda(B_{i_k})\,,$$

(cf. [10], [11] and [14]). The symmetric statistics $Y_n(h)$ for $h \in \bar{\mathscr{E}}(\mathfrak{X})$ can be also expressed by

$$Y_n(h) = \textstyle\sum_{k=1}^{p}\displaystyle\sum_{1\le s<\cdots<s_k\le k_n}\ \sum_{1\le i_1,\dots,i_k\le p} h_{i_1,\dots,i_k}\chi_{B_{i_1}}(X_{n,s_1})\cdots\chi_{B_{i_k}}(X_{n,s_k})\,.$$

LEMMA 3.1. *For* $h \in \bar{\mathscr{E}}(\mathfrak{X})$,

$$Y_n(h) \xrightarrow{d} W(h) = \sum_{k=1}^{p} \frac{1}{k!} \sum_{1 \le i_1, \dots, i_k \le p} h_{i_1, \dots, i_k} P_\lambda(B_{i_1}) \cdots P_\lambda(B_{i_k})$$

as $n \to \infty$.

PROOF. We consider the family of indicator functions $\chi_{B_1}, \dots, \chi_{B_p}$. Then for each $n \ge 1$, $X_{n,i} = (\chi_{B_1}(X_{n,i}), \dots, \chi_{B_p}(X_{n,i}))$, $1 \le i \le k_n$, is an independent p-dimensional Bernoulli array which satisfies the following

$$\lim_{n\to\infty} \sum_{i=1}^{k_n} \Pr(\chi_{B_j}(X_{n,i}) = 1) = \lim_{n\to\infty} \sum_{i=1}^{k_n} \nu_{n,i}(B_j) = \lambda(B_j)\,, \qquad j = 1, \dots, p\,,$$

$$\lim_{n\to\infty} \max_{j,i} \Pr(\chi_{B_j}(X_{n,i}) = 1) = \lim_{n\to\infty} \max_{j,i} \nu_{n,i}(B_j) = 0\,,$$

by Lemma 2.1 together with (A.1) and (A.2). Therefore, by Theorem 2.2,

$$\left(\sum_{i=1}^{k_n} \chi_{B_1}(X_{n,i}), \dots, \sum_{i=1}^{k_n} \chi_{B_p}(X_{n,i})\right) \xrightarrow{d} (P_\lambda(B_1), \dots, P_\lambda(B_p))$$

as $n \to \infty$, where $P_\lambda(B_j)$ are independent Poisson random variables with means $\lambda(B_j)$, $j = 1, \dots, p$. The assertion of the lemma follows from Corollary 5.1 of Billingsley [2].

Similarly to the proof of Theorem 4.1 in [14], we have the following theorem.

THEOREM 3.2. *For* $h \in \bar{\mathscr{H}}(\mathfrak{X})$, $Y_n(h) \xrightarrow{d} W(h)$ *as* $n \to \infty$.

PROOF. By Lemma 3.1 and (3.4), the estimation

$$\begin{aligned}
&\overline{\lim_{n\to\infty}} |E\{\exp[itY_n(h)]\} - E\{\exp[itW(h)]\}| \\
&\quad \le \overline{\lim_{n\to\infty}} |E\{\exp[itY_n(h)]\} - E\{\exp[itY_n(h^\varepsilon)]\}| + \overline{\lim_{n\to\infty}} |E\{\exp[itY_n(h^\varepsilon)]\} \\
&\qquad - E\{\exp[itW(h^\varepsilon)]\}| + |E\{\exp[itW(h^\varepsilon)]\} - E\{\exp[itW(h)]\}| \\
&\quad \le \overline{\lim_{n\to\infty}} |t|\,\|h - h^\varepsilon\|_{\lambda_n} + |t| E|W(h^\varepsilon) - W(h)| \\
&\quad \le 2|t|\,\|h - h^\varepsilon\| \le 2|t|\varepsilon
\end{aligned}$$

is shown, for $h^\varepsilon \in \bar{\mathscr{E}}(\mathfrak{X})$ with $\|h - h^\varepsilon\| < \varepsilon$. Since $\varepsilon > 0$ can be chosen aribitrary, the characteristic function of $Y_n(h)$ converges to that of $W(h)$. Thus we see the assertion.

Let $h = \{h_k\}_{k \geq 0}$ be in $\bar{\mathscr{K}}(\mathfrak{X})$. Then we can choose a compact set K with $\lambda(\partial K) = 0$ which satisfies (K.1). For a given ε $(0 < \varepsilon < 1)$, let L be so large as

$$\sum_{k=L}^{\infty} \sum_{j=0}^{k} \frac{1}{(k-j)!\,j!} H^{k+j+2}[\lambda(K) + 1]^k e^{H(\lambda(K)+1)} < \varepsilon \,. \tag{3.5}$$

We may suppose that the topology of $\mathfrak{X}$ is given by a metric $d(\cdot, \cdot)$. Since h_k, $1 \leq k \leq L$, are continuous and have compact supports, there exists a $\delta > 0$ such that

$$|h_k(\boldsymbol{x}^k) - h_k(\boldsymbol{y}^k)| < \varepsilon(\lambda(K) + 1)^{-2L} \quad \text{if} \quad \max_{1 \leq i \leq k} d(x_i, y_i) < \delta$$

holds for any k, $1 \leq k \leq L$. Then there exist disjoint sets $B_i \in \mathscr{B}_\lambda$, $1 \leq i \leq p$, such that $\operatorname{diam}(B_i) < \delta$, $\lambda(B_i) < \frac{\varepsilon}{2} L^{-2}(\lambda(K) + 1)^{-2L} H^{-2L-2}$ and $\bigcup_{i=1}^{p} B_i = K$ (cf. [14]). Furthermore, by (A.1), there exists a natural number N such that for any $n \geq N$, $1 \leq i \leq p$,

$$\lambda_n(K) < \lambda(K) + 1 \quad \text{and} \quad \lambda_n(B_i) < \varepsilon L^{-2}(\lambda(K) + 1)^{-2L} H^{-2L-2} \,. \tag{3.6}$$

Choose an element $x^{(i)} \in B_i$ for each i. For k, $1 \leq k \leq p \wedge L$, put

$$h_{i_1,\dots,i_k} \equiv \begin{cases} h(x^{(i_1)}, \dots, x^{(i_k)}) & \text{if } i_j\text{'s are all different}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

For $k > p \wedge L$, put $h_{i_1,\dots,i_k} \equiv 0$. Define h_k^ε for $k \geq 1$ by

$$h_k^\varepsilon(\boldsymbol{x}^k) \equiv \sum_{1 \leq i_1,\dots,i_k \leq p} h_{i_1,\dots,i_k} \chi_{B_{i_1}}(x_1) \cdots \chi_{B_{i_k}}(x_k) \tag{3.7}$$

and define $h_0^\varepsilon \equiv h_0$. According to the proof of Theorem 4.3 in [14], we have the following lemma.

LEMMA 3.3. *Let $h = \{h_k\}_{k \geq 0}$ be in $\bar{\mathscr{K}}(\mathfrak{X})$. Then the function $h_k^\varepsilon(\boldsymbol{x}^k)$ given by* (3.7) *satisfies that for $\boldsymbol{x}^j \in B_{i_1} \times \cdots \times B_{i_j}$,*

$$\int \cdots \int |(h_k - h_k^\varepsilon)(\boldsymbol{x}^j, \boldsymbol{y}^{k-j})| \, d\lambda_n^{k-j}(\boldsymbol{y}^{k-j}) \leq \begin{cases} 2\varepsilon(\lambda(K) + 1)^{-L} & \text{distinct } i_l\text{'s}\,, \\ H^{k+1} \lambda_n(K)^{k-j} & \text{otherwise}\,. \end{cases}$$

THEOREM 3.4. *The space $\bar{\mathscr{K}}(\mathfrak{X})$ is included in $\bar{\mathscr{H}}(\mathfrak{X})$. Hence,*

$$Y_n(h) \xrightarrow{d} W(h) \quad \text{as} \quad n \to \infty \,,$$

for any $h \in \bar{\mathscr{K}}(\mathfrak{X})$.

PROOF. For a given $h \in \bar{\mathscr{K}}(\mathfrak{X})$, let $h^\varepsilon = \{h_k^\varepsilon\}_{k=0}^{\infty}$ be given by (3.7). By (3.5), (3.6) and Lemma 3.3, we have

$$\begin{aligned}\|h - h^{\varepsilon}\|_{\lambda_n}^2 &= \textstyle\sum_{k,l=0}^{\infty} \sum_{j=0}^{k\wedge l} \dfrac{1}{(k-j)!(l-j)!j!} \\ &\quad \times \displaystyle\int\cdots\int |(h_k - h_k^{\varepsilon})(x^j, y^{k-j})(h_l - h_l^{\varepsilon})(x^j, z^{l-j})| \\ &\quad \times d\lambda_n^j(x^j)d\lambda_n^{k-j}(y^{k-j})d\lambda_n^{l-j}(z^{l-j}) \\ &\le \textstyle\sum_{k,l=0}^{L} \sum_{j=0}^{k\wedge l} \dfrac{(\lambda(K)+1)^{-2L}}{(k-j)!(l-j)!j!}(4\varepsilon^2\lambda_n(K)^j + \varepsilon H^{k+l-2L}\lambda_n(K)^{k+l-j-1}) \\ &\quad + 2\textstyle\sum_{k=L+1}^{\infty} \sum_{l=0}^{\infty} \sum_{j=0}^{k\wedge l} \dfrac{1}{(k-j)!(l-j)!j!} H^{k+l+2}(\lambda_n(K))^{k+l-j} \\ &\le (4\varepsilon^2 + \varepsilon)e^3 + 2\varepsilon\,.\end{aligned}$$

Thus we see the first assertion in the theorem. The second assertion is obvious from Theorem 3.2.

REMARK. *Let* $X_{n,1}, \dots, X_{n,k_n}$ $(1 \le k_n \le \infty)$, $n = 1, 2, \dots$, *be sequences of non-identically distributed independent random elements on* $\mathfrak{X}$ *such that* M_n *given by* (2.1) *converges in distribution to a Poisson random measure with intensity* λ *without atoms as* $n \to \infty$. *Then for* $h \in \bar{\mathscr{K}}(\mathfrak{X})$, $Y_n(h) \xrightarrow{d} W(h)$ *as* $n \to \infty$.

PROOF. By Theorem 2.2 and Lemma 2.1, we have the assumptions (A.1) and (A.2). Hence, the assertion of the remark follows from Theorem 3.4.

§4. Examples

EXAMPLE 4.1. *Symmetric polynomials for indicator functions.* Let $X_{n,1}, \dots, X_{n,k_n}$ $(1 \le k_n \le \infty)$, $n = 1, 2, \dots$, be sequences of non-identically distributed independent random elements on $\mathfrak{X}$ as in Section 2 which satisfy the assumptions (A.1) and (A.2). For a given set $B \in \mathscr{B}_\lambda$, define an $h = \{h_k\}_{k\ge 0}$ by

$$\begin{cases} h_0 \equiv 1 \\ h_k(x^k) \equiv \omega^k \chi_B(x_1)\cdots\chi_B(x_k) \qquad \text{for} \quad \omega > -1 \quad \text{and} \quad k \ge 1\,. \end{cases}$$

The symmetric statistics (3.1) can be written as the sums of symmetric polynomials

$$Y_n(h) = \textstyle\sum_{k=1}^{k_n} \omega^k \displaystyle\sum_{1\le s_1<\cdots<s_k\le k_n} \chi_B(X_{n,s_1})\cdots\chi_B(X_{n,s_k})\,.$$

By the representation of $Y_n(h)$ in the form

$$Y_n(h) = \textstyle\prod_{i=1}^{k_n} \{1 + \omega\chi_B(X_{n,i})\}$$

(cf. [7] and [8]), we have

$$\log Y_n(h) = \textstyle\sum_{i=1}^{k_n} \log\{1 + \omega\chi_B(X_{n,i})\} = \log(1+\omega)M_n(B)\,,$$

where M_n is the random measure given by (2.1). Then by Lemma 2.1 and Theorem 2.2,

$$\log Y_n(h) \xrightarrow{d} \log(1+\omega)P_\lambda(B) \quad \text{as} \quad n \to \infty\,.$$

Therefore,

$$Y_n(h) \xrightarrow{d} (1+\omega)^{P_\lambda(B)} = \textstyle\sum_{k=0}^{\infty} \dfrac{\omega^k}{k!} P_\lambda(B)(P_\lambda(B)-1)\cdots(P_\lambda(B)-k+1)$$

and

$$\sum_{1 \le s_1 < \cdots < s_k \le k_n} \chi_B(X_{n,s_1})\cdots\chi_B(X_{n,s_k}) \xrightarrow{d} \frac{1}{k!} P_\lambda(B)(P_\lambda(B)-1)\cdots(P_\lambda(B)-k+1)$$

as $n \to \infty$.

EXAMPLE 4.2. *Non-interacting particle system.* Consider a stochastic non-interacting particles travelling in d-dimensional Euclidean space $\boldsymbol{R}^d$. The particles can overtake and meet each other without delay. Let X_i be the initial position (non-random) of the ith particle and V_i be its constant velocity (but random) which are identically distributed independent d-dimensional random variables (cf. [3], [17] and [18]). Put $X_i(t) = X_i + V_i t$, the position of the ith particle at time t. We assume the following:

(4.1) simultaneously expand the cube I and increase the number of particles keeping the density λ constant $(0 < \lambda < \infty)$, i.e. for any cube I, $\lim_{|I|\to\infty}$ {the number of $X_i \in I\}/|I| = \lambda$, where $|I|$ is the volume of I.

(4.2) the distribution of V_i is absolutely continuous with respect to the Lebesgue measure on $\boldsymbol{R}^d$ with the density $g(u)$ which is almost everywhere continuous and bounded on every bounded d-dimensional interval.

For any given d-dimensional interval J, define

$$\nu_{t,i}(J) = \nu[(J - X_i)/t] \equiv \Pr[V_i \in J - X_i)/t]\,.$$

By the assumptions (4.1) and (4.2), we know that the sequence $\{X_i(t)\}$ satisfies the following

$$\lim_{t\to\infty} \textstyle\sum_i \nu_{t,i}(I) = \lambda|I|\,, \tag{4.3}$$

$$\lim_{t\to\infty} \max_i \nu_{t,i}(I) = 0\,, \tag{4.4}$$

for any cube I (see [3] and [17]). Then by Lemma 2.1, (4.3) implies that

$$\sum_i \nu_{t,i} \text{ converges to } \lambda \cdot \text{Lebesgue measure vaguely} .$$

Therefore, the sequence $\{X_i(t)\}$ satisfies the assumptions (A.1) and (A.2) in Section 2. Thus we have the following theorem:

THEOREM 4.3. *Under the assumptions* (4.1) *and* (4.2), *we have for* $h = (h_0, h_1, h_2, \dots) \in \mathscr{K}(\boldsymbol{R}^d)$

$$\sum_{k=0}^{\infty} \sum_{1 \le s_1 < \cdots < s_k < \infty} h_k(X_{s_1}(t), \dots, X_{s_k}(t))$$
$$\xrightarrow{d} \sum_{k=0}^{\infty} \frac{1}{k!} \int \cdots \int h_k(x_1, \dots, x_k) dP_\lambda(x_1) \cdots dP_\lambda(x_k)$$

as $t \to \infty$.

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