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Boundary limits of locally n-precise functions

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1. Introduction

In this note we investigate the existence of boundary limits of locally n-precise functions u on a domain G in R^n which satisfy a condition of the form:

$$\int_G \Psi(|\mathrm{grad}\, u(x)|)\omega(x)dx < \infty \tag{1}$$

with a nonnegative measurable function ω on G and a positive nondecreasing function Ψ on the interval $(0, \infty)$; for the definition and basic properties of locally p-precise functions, see Ohtsuka [4] and Ziemer [5]. The function $\Psi(r)$ is assumed to be of the form $r^n\psi(r)$, where $\psi(r)$ is a positive nondecreasing function on the interval $(0, \infty)$ satisfying the following conditions:

(ψ_1) There exists $A > 0$ such that

$$A^{-1}\psi(r) \leqq \psi(r^2) \leqq A\psi(r) \qquad \text{for any} \quad r > 0\,.$$

(ψ_2) $\int_0^1 \psi(r^{-1})^{-1/(n-1)}r^{-1}dr < \infty$.

For example,

$$\psi(r) = [\log(2 + r)]^\alpha,\ [\log(2 + r)]^{n-1}[\log(2 + (\log(2 + r)))]^\alpha, \dots,$$

satisfy the above conditions, as long as $\alpha > n - 1$.

We shall first show that if $\int_G \Psi(|\mathrm{grad}\, u(x)|)dx < \infty$, then there exists a continuous function u^* on G such that $u^* = u$ a.e. on G, and furthermore, in case G is a Lipschitz domain, u^* can be extended to a continuous function on $G \cup \partial G$.

Next, in section 3, we are concerned with the existence of limits at a given boundary point ξ, in the case where u satisfies (1) with $\omega(x) = \lambda(|x - \xi|)$ for a positive nondecreasing function λ on the interval $(0, \infty)$. Then, in the next section, we study the existence of boundary limits along certain subsets of G for a function u satisfying (1) with $\omega(x) = \lambda(\rho(x))$, where λ is as above and $\rho(x)$ denotes the distance of x from the boundary ∂G.

In the last section, we discuss the existence of limits at infinity, in case G is unbounded and $\omega \equiv 1$.

2. Continuity of locally n-precise functions

First we give several properties on ψ, which follow from condition (ψ_1).

$(\psi_1)'$ There exists $A' > 0$ such that $\psi(2r) \leqq A'\psi(r)$ on $(0, \infty)$.

$(\psi_1)''$ For each $\gamma > 0$, there exists $A_\gamma > 0$ such that

$$A_\gamma^{-1}\psi(r) \leqq \psi(r^\gamma) \leqq A_\gamma\psi(r) \qquad \text{on} \quad (0, \infty).$$

$(\psi_1)'''$ If $\varepsilon > 0$, then $s^\varepsilon\psi(s^{-1}) \leqq At^\varepsilon\psi(t^{-1})$ whenever $0 < s < t < A^{-1/\varepsilon}$.

For the sake of convenience, we introduce the function

$$\tilde{\psi}(r) = \left(\int_0^r \psi(t^{-1})^{-1/(n-1)}t^{-1}dt\right)^{1-1/n}.$$

Then $\tilde{\psi}$ satisfies condition (ψ_1), too, and

$$(\psi_3) \qquad \tilde{\psi}(r) \geqq M\psi(r^{-1})^{-1/n} \qquad \text{for any} \quad r > 0$$

with a positive constant M.

Our first aim is to establish the following result.

THEOREM 1. *If u is a locally n-precise function on G satisfying*

$$(2) \qquad \int_G \Psi(|\mathrm{grad}\, u(x)|)dx < \infty,$$

then there exists a continuous function on G which equals u a.e. on G.

For a proof of Theorem 1, we use the following results.

LEMMA 1 (cf. [3; Theorem 1], [4; Theorem 9.11]). *Let $1 < p < \infty$. If u is a p-precise function on R^n with compact support, then*

$$u(x) = c\sum_{i=1}^n \int (x_i - y_i)|x - y|^{-n}(\partial/\partial y_i)u(y)dy \qquad \textit{a.e. on } R^n,$$

where c is a constant independent of u.

LEMMA 2. *Let E be a measurable set in R^n, and let g, ω be nonnegative measurable functions on E. Then, for any δ with $0 < \delta < 1$ and $\alpha > 0$,*

$$\int_E |x-y|^{1-n}g(y)dy \leqq A_\delta^{1/n}\left(\int_E \Psi(g(y))\omega(y)dy\right)^{1/n}$$
$$\times\left(\left(\int_E |x-y|^{-n}[\psi(\alpha^{-1}|x-y|^{-1})\omega(y)]^{-1/(n-1)}dy\right)^{1-1/n} + \alpha^{-\delta}\int_E |x-y|^{1-n-\delta}dy\right).$$

PROOF. Let $E_1 = \{y \in E; g(y) \geqq (\alpha|x-y|)^{-\delta}\}$ and $E_2 = E - E_1$. Then, $\psi(g(y)) \geqq \psi((\alpha|x-y|)^{-\delta}) \geqq A_\delta^{-1}\psi((\alpha|x-y|)^{-1})$ on E_1 and $g(y) \leqq \alpha^{-\delta}|x-y|^{-\delta}$ on E_2. Hence, Hölder's inequality implies the required inequality.

COROLLARY. *If E, g, δ and α are as in Lemma 2, then*

$$\int_E |x-y|^{1-n}g(y)dy \leqq M\left(\int_E \Psi(g(y))dy\right)^{1/n}\tilde{\psi}(\alpha R) + M\alpha^{-\delta}|E|^{(1-\delta)/n},$$

where $|E|$ denotes the measure of E, $R = \sup\{|x-y|; y \in E\}$ and M is a positive constant independent of α, x, g, E.

PROOF. Taking $\omega \equiv 1$ in Lemma 2 and remarking that

$$\int_E |x-y|^{1-n-\delta}dy \leqq Mr^{1-\delta}$$

for $r \geqq 0$ such that $|E| = |B(x,r)|$, $B(x,r)$ denoting the open ball with center x and radius r, we obtain the Corollary.

PROOF OF THEOREM 1. Let $B(x_0, 2r_0) \subset G$, and take $\varphi \in C_0^\infty(G)$ such that $\varphi = 1$ on $B(x_0, r_0)$. Then, by Lemma 1, φu is equal a.e. to

$$v(x) = c\sum_{i=1}^n \int (x_i - y_i)|x-y|^{-n}(\partial/\partial y_i)(\varphi u)(y)dy.$$

Thus it suffices to show that v is continuous on $B(x_0, r_0)$. We write

$$v(x) = c\sum_{i=1}^n \int (x_i - y_i)|x-y|^{-n}[(\partial/\partial y_i)\varphi(y)]u(y)dy$$

$$+ c\sum_{i=1}^n \int (x_i - y_i)|x-y|^{-n}\varphi(y)[(\partial/\partial y_i)u(y)]dy = u_1(x) + u_2(x).$$

We first note that u_1 is continuous on $B(x_0, r_0)$. Let x_1 be any point of $B(x_0, r_0)$. For $r > 0$, we set

$$u_{2,r}(x) = c\sum_{i=1}^n \int_{B(x_1,r)} (x_i - y_i)|x-y|^{-n}\varphi(y)[(\partial/\partial y_i)u(y)]dy.$$

For simplicity, put

$$f(y) = \sum_{i=1}^n |\varphi(y)[(\partial/\partial y_i)u(y)]|.$$

We note that $\int_{R^n}\Psi(f(y))dy < \infty$, by condition (2). For $x \in B(x_1, r)$, we derive from the Corollary to Lemma 2

$$|u_{2,r}(x)| \leqq M_1 \int_{B(x_1,r)} |x-y|^{1-n} f(y)dy$$
$$\leqq M_2 \left(\int_{B(x_1,r)} \Psi(f(y))dy \right)^{1/n} \tilde{\psi}(r) + M_2 r^{1-\delta},$$

where $0 < \delta < 1$ and M_1, M_2 are positive constants independent of x and r. Consequently, $\lim_{r \to 0} \sup_{x \in B(x_1,r)} u_{2,r}(x) = 0$. Since $u_2 - u_{2,r}$ is continuous at x_1, it follows that u_2 is continuous at x_1. Therefore, v is continuous on $B(x_0, r_0)$, and hence Theorem 1 is established.

REMARK. If $\Psi(r) = r^p$ and $p > n$, then the same conclusion as in Theorem 1 is true.

Let λ be a positive nondecreasing function on $(0, \infty)$ such that $\lambda(2r) \leqq B\lambda(r)$ on $(0, \infty)$ with a positive constant B, and consider

$$\kappa'_\lambda(r) = \left(\int_r^1 [\psi(s^{-1})\lambda(s)]^{-1/(n-1)} s^{-1} ds \right)^{1-1/n}.$$

THEOREM 2. *Let G be a Lipschitz domain in R^n, and u be a locally n-precise function on G satisfying*

$$\int_G \Psi(|\text{grad}\, u(x)|)\lambda(\rho(x))dx < \infty, \tag{3}$$

where $\rho(x)$ denotes the distance of x from the boundary ∂G. If $\kappa'_\lambda(0) < \infty$, then there exists a continuous function on $G \cup \partial G$ which equals u a.e. on G.

REMARK. If $\lim_{r \downarrow 0} \lambda(r) > 0$ (in particular, if $\lambda \equiv 1$), then $\kappa'_\lambda(0) < \infty$ by assumption (ψ_2).

For a proof of Theorem 2, we need the following result, which is a key lemma in the discussions throughout this paper.

LEMMA 3. *If u is a locally n-precise continuous function on G, then for any x, $x_0 \in G$ and $r_0 > 0$ such that $E(x, x_0, r_0) = \{tx + (1-t)y; 0 < t < 1, y \in B(x_0, r_0)\} \subset G$,*

$$\left| u(x) - |B(x_0, r_0)|^{-1} \int_{B(x_0,r_0)} u(y)dy \right|$$
$$\leqq M r_0^{-n} (|x - x_0| + r_0)^n \int_{E(x,x_0,r_0)} |x-z|^{1-n} |\text{grad}\, u(z)| dz,$$

where M is a positive constant depending only on the dimension n.

REMARK. If $x \in B(x_0, r_0)$, then

$$\left| u(x) - |B(x_0, r_0)|^{-1} \int_{B(x_0, r_0)} u(y)dy \right| \leqq 2^n M \int_{B(x_0, r_0)} |x - z|^{1-n} |\mathrm{grad}\, u(z)| dz .$$

PROOF OF LEMMA 3. If $0 < \varepsilon < 1$, then, in view of Example 1 given after Theorem 3.21 in [4], we have

$$|u(x + \varepsilon(y - x)) - u(y)| \leqq \int_\varepsilon^1 |x - y| |\mathrm{grad}\, u(tx + (1 - t)y)| dt$$

for almost every $y \in B(x_0, r_0)$. Letting $\varepsilon \to 0$, we obtain

$$|u(x) - u(y)| \leqq |x - y| \int_0^1 |\mathrm{grad}\, u(tx + (1 - t)y)| dt$$

for almost every $y \in B(x_0, r_0)$. Hence

$$\begin{aligned} &\left| u(x) - |B(x_0, r_0)|^{-1} \int_{B(x_0, r_0)} u(y)dy \right| \\ &\leqq |B(x_0, r_0)|^{-1} \int_{B(x_0, r_0)} |x - y| \left(\int_0^1 |\mathrm{grad}\, u(tx + (1 - t)y)| dt \right) dy \\ &\leqq |B(x_0, r_0)|^{-1} \int_{E(x, x_0, r_0)} |x - z| |\mathrm{grad}\, u(z)| \\ &\quad \times \left(\int_{\{1 - t \geqq |x - z|(|x - x_0| + r_0)^{-1}\}} (1 - t)^{-1-n} dt \right) dz \\ &\leqq M r_0^{-n} (|x - x_0| + r_0)^n \int_{E(x, x_0, r_0)} |x - z|^{1-n} |\mathrm{grad}\, u(z)| dz , \end{aligned}$$

since for $z = tx + (1 - t)y$, $|x - z| = (1 - t)|x - y| \leqq (1 - t)(|x - x_0| + r_0)$, where M is a positive constant independent of x, x_0, r_0 and u.

PROOF OF THEOREM 2. By Theorem 1 we may assume that u is continuous on G. We shall prove that u has a finite limit at any $\xi \in \partial G$. Since G is a Lipschitz domain, there is a cylindrical neighborhood U of ξ such that, by a suitable orthogonal coordinate system, we can write

$$\xi = 0 , \qquad U \cap G = \{x = (x_1, x'); \varphi(x') < x_1 < h, |x'| < \rho\} ,$$

where $h > 0$, $\rho > 0$ and φ is a Lipschitz function on $\{x' \in R^{n-1}; |x'| < \rho\}$ such that $\varphi(0) = 0$. Let K be the Lipschitz constant of φ. For any $r > 0$ with $r < \min\{h/2, 2(K + 1)\rho\}$, let $e_r = (0, r)$ and $\sigma_r = r/3(K + 1)$. Then, for any

$x \in B(0, \sigma_r) \cap G$, $E(x, e_r, \sigma_r) \subset U \cap G$. Hence, by Lemmas 3 and 2, we have

$$\left| u(x) - |B(e_r, \sigma_r)|^{-1} \int_{B(e_r, \sigma_r)} u(y) dy \right|$$

$$\leqq M \sigma_r^{-n} (|x - e_r| + \sigma_r)^n \int_{E(x, e_r, \sigma_r)} |x - z|^{1-n} |\operatorname{grad} u(z)| dz$$

$$\leqq M_1 r^{1-\delta} + M_1 \left(\int_{E(x, e_r, \sigma_r)} \Psi(|\operatorname{grad} u(z)|) \lambda(\rho(z)) dz \right)^{1/n}$$

$$\times \left(\int_{E(x, e_r, \sigma_r)} |x - z|^{-n} [\psi(|x - z|^{-1}) \lambda(\rho(z))]^{-1/(n-1)} dz \right)^{1-1/n}$$

for any $x \in B(0, \sigma_r) \cap G$, where $0 < \delta < 1$ and M_1 is a positive constant independent of r. If $x \in B(0, \sigma_r) \cap G$ and $z \in E(x, e_r, \sigma_r)$, then $|x - z| \leqq M_2 \rho(z)$ with a positive constant M_2, so that

$$\left(\int_{E(x, e_r, \sigma_r)} |x - z|^{-n} [\psi(|x - z|^{-1}) \lambda(\rho(z))]^{-1/(n-1)} dz \right)^{1-1/n}$$

$$\leqq M_3 \left(\int_0^{2r} [\psi(t^{-1}) \lambda(t)]^{-1/(n-1)} t^{-1} dt \right)^{1-1/n} \leqq M_4 \kappa'_\lambda(0)$$

with positive constants M_3 and M_4. Therefore,

$$|u(x) - u(y)| \leqq 2 M_1 M_4 \kappa'_\lambda(0) \left(\int_{G \cap B(0, 2r)} \Psi(|\operatorname{grad} u(z)|) \lambda(\rho(z)) dz \right)^{1/n} + 2 M_1 r^{1-\delta}$$

whenever $x, y \in G \cap B(0, \sigma_r)$. This implies that u has a finite limit at $\xi = 0$.

REMARK. Theorem 2 fails to hold if G is not a Lipschitz domain. For example, consider the set $G_a = \{(x, y); 0 < x < 1, -x^a < y < x^a\}$, where $a > 1$. If $u(x, y) = x^{-\beta}$ and $-\beta + (a - 1)/2 > 0$, then u satisfies condition (3) with $G = G_a$ and $\lambda \equiv 1$.

3. Boundary limits, I

Let λ be a positive nondecreasing function on $(0, \infty)$ such that $\lambda(2r) \leqq B\lambda(r)$ on $(0, \infty)$ with a positive constant B, and let

$$\kappa_\lambda(r) = \kappa'_\lambda(r) + \lambda(r)^{-1/n} \tilde{\psi}(r).$$

Recall that

$$\kappa'_\lambda(r) = \left(\int_r^1 [\psi(t^{-1}) \lambda(t)]^{-1/(n-1)} t^{-1} dt \right)^{1-1/n}$$

and

$$\tilde{\psi}(r) = \left(\int_0^r \psi(t^{-1})^{-1/(n-1)} t^{-1} dt \right)^{1-1/n} .$$

REMARK. (i) It is easy to see that $\kappa'_\lambda(0) < \infty$ if and only if κ_λ is bounded on $(0, 1)$. In fact, if $\kappa'_\lambda(0) < \infty$, then $\lim_{r \downarrow 0} \lambda(r)^{-1/n} \tilde{\psi}(r) = 0$.

(ii) If $\lambda(r) = r^\beta$ $(\beta > 0)$, then $\kappa_\lambda(r) \sim r^{-\beta/n} \tilde{\psi}(r)$ (cf. the Appendix) and $\kappa'_\lambda(0) = \infty$.

In this section, we are concerned with the existence of limits at a given boundary point ξ, for functions u satisfying

$$\int_G \Psi(|\text{grad}\, u(x)|) \lambda(|\xi - x|) dx < \infty . \tag{4}$$

THEOREM 3. *Let $\xi \in \partial G$, and suppose there exist $x_0 \in G$, $r_0 > 0$ and $\varepsilon_0 > 0$ such that $E(x, x_0, r_0) \subset G$ for all $x \in G \cap B(\xi, \varepsilon_0)$. If u is a locally n-precise continuous function on G satisfying* (4) *and if $\kappa'_\lambda(0) = \infty$, then*

$$\lim_{x \to \xi, x \in G} [\kappa_\lambda(|x - \xi|)]^{-1} u(x) = 0 .$$

PROOF. We may assume that $\xi = 0$ and $\varepsilon_0 < |x_0| - r_0$. First, we note that there is $a > 0$ (depending only on x_0, r_0 and ε_0) such that

$$|z| > a|x| \qquad \text{and} \qquad |z| > a|x - z|$$

whenever $x \in G \cap B(0, \varepsilon_0)$ and $z \in E(x, x_0, r_0/2)$.

For $x \in G \cap B(0, \varepsilon_0)$, by Lemma 3, we have

$$\begin{aligned} \left| u(x) - |B(x_0, r_0/2)|^{-1} \int_{B(x_0, r_0/2)} u(y) dy \right| &\leq M_1 \int_{E(x, x_0, r_0/2)} |x - z|^{1-n} f(z) dz \\ &= M_1(I_1 + I_2) , \end{aligned}$$

where $f(z) = |\text{grad}\, u(z)|$, M_1 is a positive constant independent of x,

$$I_1 = \int_{E(x, x_0, r_0/2) \cap B(x, r)} |x - z|^{1-n} f(z) dz$$

and

$$I_2 = \int_{E(x, x_0, r_0/2) - B(x, r)} |x - z|^{1-n} f(z) dz$$

for r with $|x| < r < \varepsilon_0$. In view of Lemma 2, we obtain

$$I_1 \leqq M_2\left(\int_{E(x,x_0,r_0/2)\cap B(x,r)} \Psi(f(z))\lambda(|z|)dz\right)^{1/n}$$
$$\times\left(\int_{E(x,x_0,r_0/2)} |x-z|^{-n}[\psi(|x-z|^{-1})\lambda(|z|)]^{-1/(n-1)}dz\right)^{1-1/n} + M_2$$

for a positive constant M_2 independent of x and r. Now, let

$$E_1 = E(x, x_0, r_0/2) - B(x, |x|)$$

and

$$E_2 = E(x, x_0, r_0/2) \cap B(x, |x|)\,.$$

For $z \in E_1$, we use the inequality $|z| > a|x-z|$ and obtain

$$\left(\int_{E_1} |x-z|^{-n}[\psi(|x-z|^{-1})\lambda(|z|)]^{-1/(n-1)}dz\right)^{1-1/n}$$
$$\leqq M_3\left(\int_{|x|}^{r_1} [\psi(t^{-1})\lambda(at)]^{-1/(n-1)}t^{-1}dt\right)^{1-1/n} \leqq M_4\kappa'_\lambda(|x|)\,,$$

where $r_1 = |x_0| + r_0/2 + \varepsilon_0$ and M_3, M_4 are positive constants independent of x. For $z \in E_2$, we use the inequality $|z| > a|x|$ and obtain

$$\left(\int_{E_2} |x-z|^{-n}[\psi(|x-z|^{-1})\lambda(|z|)]^{-1/(n-1)}dz\right)^{1-1/n}$$
$$\leqq M_5[\lambda(a|x|)]^{-1/n}\left(\int_0^{|x|} [\psi(t^{-1})]^{-1/(n-1)}t^{-1}dt\right)^{1-1/n}$$
$$\leqq M_6[\lambda(|x|)]^{-1/n}\tilde{\psi}(|x|)$$

with positive constants M_5 and M_6. Hence

$$I_1 \leqq M_7\kappa_\lambda(|x|)\left(\int_{E(x,x_0,r_0/2)\cap B(x,r)} \Psi(f(z))\lambda(|z|)dz\right)^{1/n} + M_2$$

with a positive constant M_7 independent of x and r. Similarly, by using the inequality $|z| > a|x-z|$, we obtain

$$I_2 \leqq M_8\kappa_\lambda(r)\left(\int_{E(x,x_0,r_0/2)-B(x,r)} \Psi(f(z))\lambda(|z|)dz\right)^{1/n} + M_8$$

with a positive constant M_8. Thus we establish

$$\begin{aligned}&\left|u(x) - |B(x_0, r_0/2)|^{-1}\int_{B(x_0,r_0/2)} u(y)dy\right| \\ &\leqq M_9\kappa_\lambda(|x|)\left(\int_{G\cap B(x,r)} \Psi(f(z))\lambda(|z|)dz\right)^{1/n} \\ &+ M_9\kappa_\lambda(r)\left(\int_G \Psi(f(z))\lambda(|z|)dz\right)^{1/n} + M_9\end{aligned}$$

with a positive constant M_9. Since $\kappa_\lambda(|x|) \to \infty$ as $x \to 0$, it follows that

$$\limsup_{x\to 0, x\in G} [\kappa_\lambda(|x|)]^{-1}|u(x)| \leqq M_9\left(\int_{G\cap B(0,r)} \Psi(f(z))\lambda(|z|)dz\right)^{1/n}$$

for any r with $0 < r < \varepsilon_0$, which implies the required result.

Now we consider a special domain

$$G_a = \{x = (x_1, x') \in R^1 \times R^{n-1}; 0 < x_1 < 1, |x'| < x_1^a\}.$$

If $a > 1$, then G_a is not a Lipschitz domain, and it does not satisfy the condition in Theorem 3 at $\xi = 0$. However, we have the following result for this domain.

PROPOSITION 1. *Let λ be a positive monotone function on the interval* $(0, \infty)$ *such that $B^{-1}\lambda(r) \leqq \lambda(2r) \leqq B\lambda(r)$ for any $r > 0$ with a positive constant B. For $a > 1$, let*

$$\lambda_a(r) = \left(\int_r^1 \lambda(s)^{-1/(n-1)}s^{-a}ds\right)^{-n+1}.$$

If u is a locally n-precise continuous function on G_a satisfying condition (4), *then*

(i) *$u(x)$ has a finite limit as $x_1 \to 0$, $x \in G_a$, in case $\kappa'_{\lambda_a}(0) < \infty$;*

(ii) $\lim_{x_1\to 0, x\in G_a} [\kappa_{\lambda_a}(x_1)]^{-1}u(x) = 0$ *in case $\kappa'_{\lambda_a}(0) = \infty$.*

PROOF. For each positive integer $j \geqq j_0$, let $r_j = Mj^{1/(1-a)}$. Here j_0 and M are taken so large that $0 < r_j < 1/2$ and $r_j - r_{j+1} < \rho(e(j))$ for $j \geqq j_0$, where $e(j) = (r_j, 0)$. For simplicity, set $\Delta(j) = B(e(j), \rho(e(j)))$, $j \geqq j_0$. We shall show the existence of $N > 0$ such that the number of $\Delta(m)$ with $\Delta(m) \cap \Delta(j) \neq \emptyset$ is at most N for any j. Letting β and γ be positive numbers, we assume that $r_j - \beta r_j^a \leqq r_{j+k} + \gamma r_{j+k}^a$. Then

$$j[1 - (j/(j+k))^{1/(a-1)}] \leqq M^{a-1}[\beta + \gamma(j/(j+k))^{a/(a-1)}].$$

Since $K = \inf_{0<x<1} (1 - x^{1/(a-1)})/(1 - x) > 0$, we derive

$$jk/(j+k) \leqq K^* \qquad \text{with} \quad K^* = [M^{a-1}(\beta + \gamma)]/K,$$

so that

$$k \leqq K^*j/(j-K^*) \qquad \text{when} \quad j > K^* .$$

From this fact we can readily find $N > 0$ with the required property.

For $0 < r < 1/2$, let $X(r) = (r, 0) \in G_a$ and $B_r = B(X(r), \rho(X(r)))$. If $x \in B_r$, then Lemmas 2 and 3 imply

$$\begin{aligned}\left| u(x) - |B_r|^{-1} \int_{B_r} u(z)dz \right| &\leqq M_1 \int_{B_r} |x-z|^{1-n} |\mathrm{grad}\, u(z)|\, dz \\ &\leqq M_2 \left(\int_{B_r} \Psi(|\mathrm{grad}\, u(z)|)\lambda(|z|)dz \right)^{1/n} \lambda(r)^{-1/n} \tilde{\psi}(r^a) \\ &\quad + M_2 r^{a(1-\delta)} ,\end{aligned}$$

so that

$$\begin{aligned}(5) \qquad |u(x) - u(X(r))| \leqq 2M_2 &\left(\int_{B_r} \Psi(|\mathrm{grad}\, u(z)|)\lambda(|z|)dz \right)^{1/n} \\ &\times \lambda(r)^{-1/n} \tilde{\psi}(r^a) + 2M_2 r^{a(1-\delta)}\end{aligned}$$

with positive constants M_1 and M_2 independent of x, y and r, where δ is a positive number so chosen that $a\delta < 1$. Since $\tilde{\psi}(r^a) \leqq M(a)\tilde{\psi}(r)$ for $r > 0$ with a positive constant $M(a)$, we obtain

$$\begin{aligned}|u(e(j)) - u(e(j+k))| &\leqq |u(e(j)) - u(e(j+1))| + |u(e(j+1)) - u(e(j+2))| + \cdots \\ &\quad + |u(e(j+k-1)) - u(e(j+k))| \\ &\leqq M_3 \left(\int_{\Delta(j,j+k)} \Psi(|\mathrm{grad}\, u(z)|)\lambda(|z|)dz \right)^{1/n} \\ &\quad \times \left(\textstyle\sum_{m=j}^{j+k-1} \tilde{\psi}(m^{-1})^{n'} [\lambda(m^{1/(1-a)})]^{-n'/n}\right)^{1/n'} \\ &\quad + M_3 \textstyle\sum_{m=j}^{\infty} m^{-a(1-\delta)/(a-1)} ,\end{aligned}$$

where $1/n + 1/n' = 1$, $\Delta(j, j+k) = \bigcup_{j \leqq m \leqq j+k} \Delta(m)$ and M_3 is a positive constant independent of j and k. Here note that

$$\begin{aligned}&\textstyle\sum_{m=j}^{j+k-1} \tilde{\psi}(m^{-1})^{n'} \lambda(m^{1/(1-a)})^{-n'/n} \\ &\leqq M_4 \int_j^{j+k} \tilde{\psi}(t^{-1})^{n'} \lambda(t^{1/(1-a)})^{-n'/n} dt \\ &\leqq M_5 \int_{(j+k)^{-1}}^{j^{-1}} \psi(s^{-1})^{-1/(n-1)} s^{-1} \left(\int_j^{s^{-1}} \lambda(t^{1/(1-a)})^{-1/(n-1)} dt \right) ds \\ &\quad + M_5 \left(\int_0^{(j+k)^{-1}} \psi(s^{-1})^{-1/(n-1)} s^{-1} ds \right) \left(\int_j^{j+k} \lambda(t^{1/(1-a)})^{-1/(n-1)} dt \right)\end{aligned}$$

for sufficiently large j, where M_4 and M_5 are positive constants independent of j and k. Since $\int_j^{s^{-1}} \lambda(t^{1/(1-a)})^{-1/(n-1)}dt \leqq [(a-1)\lambda_a(s^{1/(a-1)})]^{-1/(n-1)}$, we find, by $(\psi_1)''$ and change of variables, that

$$(\textstyle\sum_{m=j}^{j+k-1} \tilde{\psi}(m^{-1})^{n'}\lambda(m^{1/(1-a)})^{-n'/n})^{1/n'} \leqq M_6\kappa_{\lambda_a}((j+k)^{1/(1-a)}) \leqq M_7\kappa_{\lambda_a}(r_{j+k})$$

with positive constants M_6 and M_7 independent of j and k.

First suppose $\kappa'_{\lambda_a}(0) = \infty$. Then

$$\limsup_{k\to\infty} [\kappa_{\lambda_a}(r_{j+k})]^{-1}|u(e(j+k))| \leqq M_3M_7\left(\int_{\Delta(j,\infty)} \Psi(|\mathrm{grad}\, u(z)|)\lambda(|z|)dz\right)^{1/n},$$

which implies

$$\lim_{j\to\infty} [\kappa_{\lambda_a}(r_j)]^{-1}u(e(j)) = 0\,.$$

If $x \in B_r$ and $r_{j+1} < r \leqq r_j$, then $e(j) \in B_r$ and $x_1 < r \leqq r_j$. Hence, by (5),

$$\begin{aligned}[\kappa_{\lambda_a}(x_1)]^{-1}|u(x)| &\leqq M_8[\kappa_{\lambda_a}(r_j)]^{-1}(|u(e(j))| + r_j^{a(1-\delta)}) \\ &\quad + M_8[\lambda_a(r_j)]^{1/n}\lambda(r_j)^{-1/n}\left(\int_{G_a} \Psi(|\mathrm{grad}\, u(z)|)\lambda(|z|)dz\right)^{1/n}\end{aligned}$$

with a positive constant M_8. Since

$$\lambda_a(r)^{1/n}\lambda(r)^{-1/n} \leqq \left([B^{-1}\lambda(r)]^{-1/(n-1)}\int_r^{2r} s^{-a}ds\right)^{-1/n'}\lambda(r)^{-1/n} \leqq M_9r^{(a-1)/n'}$$

with a positive constant M_9 independent of r, we see that $[\kappa_{\lambda_a}(x_1)]^{-1}|u(x)|$ tends to zero as $x \to 0$, $x \in G_a$.

If $\kappa'_{\lambda_a}(0) < \infty$, then κ_{λ_a} is bounded and the above arguments imply that $\{u(e(j))\}$ is a Cauchy sequence and

$$\lim_{j\to\infty} (\sup\{|u(x) - u(e(j))|;\, x \in \textstyle\bigcup_{r_{j+1}<r\leqq r_j} B_r\}) = 0\,.$$

From these facts it follows readily that $u(x)$ has a finite limit as $x \to 0$, $x \in G_a$.

REMARK 1. Let $\lambda(r) = r^\gamma$ for a number γ. If $\gamma < -(n-1)(a-1)$, then $\kappa'_{\lambda_a}(0) < \infty$. If $\gamma > -(n-1)(a-1)$, then $\kappa'_{\lambda_a}(0) = \infty$ and $\kappa_{\lambda_a}(r) \sim r^{-[\gamma+(n-1)(a-1)]/n}$.

REMARK 2. Proposition 1 is best possible as to the order of infinity in the following sense: if $\varepsilon > 0$, then we can find a locally n-precise continuous function u on G_a satisfying condition (4) such that

$$\lim_{x_1\to 0, x\in G_a} x_1^{-\varepsilon}[\kappa_{\lambda_a}(x_1)]^{-1}u(x_1, x') = \infty\,. \tag{6}$$

In fact, let $\psi(r) = [\log(2+r)]^{\beta}$ and $\lambda(r) = r^{\gamma}$, where $\beta > n-1$ and $\gamma + (n-1)(a-1) > 0$. Then $\tilde{\psi}(r) \sim [\log(2+r^{-1})]^{(n-1-\beta)/n}$ and $\lambda_a(r) \sim r^{\gamma+(n-1)(a-1)}$ for $r \in (0, 1)$. Consider the function

$$u(x_1, x') = x_1^{-[\gamma+(n-1)(a-1)]/n}[\log(2 + x_1^{-1})]^{(n-1-\beta)/n-\delta}$$

for $\delta > 1$. Since $\kappa_{\lambda_a}(r) \leqq M_1 \tilde{\psi}(r)\lambda_a(r)^{-1/n}$ with a positive constant M_1, (6) is satisfied. On the other hand, we have

$$|(\partial/\partial x_1)u| \leqq M_1 x_1^{-1-[\gamma+(n-1)(a-1)]/n}[\log(2 + x_1^{-1})]^{(n-1-\beta)/n-\delta},$$

so that

$$\Psi(|\mathrm{grad}\, u(x_1, x')|) \leqq M_2 x_1^{-[1+\gamma+(n-1)a]}[\log(2 + x_1^{-1})]^{n-1-n\delta}.$$

Hence we obtain

$$\int_{G_a} \Psi(|\mathrm{grad}\, u(x)|)|x|^{\gamma} dx \leqq M_3 \int_0^1 x_1^{-[1+\gamma+(n-1)a]}[\log(2 + x_1^{-1})]^{n-1-n\delta} x_1^{\gamma+(n-1)a} dx_1$$
$$< \infty.$$

Thus u satisfies (4), and it is the required function.

4. Boundary limits, II

In this section we discuss the existence of boundary limits along a set in G, for locally n-precise continuous functions u on G satisfying (3). Here λ is a positive nondecreasing function on $(0, \infty)$ such that $\lim_{r \downarrow 0} \lambda(r) = 0$ and $\lambda(2r) \leqq B\lambda(r)$ for $r > 0$ with a positive constant B.

Let h be a nonnegative nondecreasing function on $(0, \infty)$ such that $h(2r) \leqq Mh(r)$ for any $r > 0$ with a positive constant M, and denote by H_h the Hausdorff mesure with the measure function h.

For $\xi \in \partial G$ and a set T, suppose there exist positive numbers c and C satisfying the following conditions:

(T_1) $\xi \in \partial T$;

(T_2) for sufficiently small $r > 0$, there exist $x_r \in G$ and $d_r > 0$ such that $x_r \in B(\xi, r)$, $cr < d_r < r$ and $E(x, x_r, d_r) \subset T$ whenever $x \in T \cap B(\xi, r)$;

(T_3) $\kappa_{\xi,\lambda}(x) \leqq Ch(|x - \xi|)^{-1/n}$ if $x \in T$, where

$$\kappa_{\xi,\lambda}(x) = \left(\int_{G \cap B(\xi, 2|\xi - x|)} |x - y|^{-n}[\psi(|x - y|^{-1})\lambda(\rho(y))]^{-1/(n-1)} dy \right)^{1-1/n}$$

A typical example of T is a set of the form

$$\{x = (x_1, x') \in R^1 \times R^{n-1};\ \varphi(|x'|) < ax_1\}$$

or a set similar to this set, where $a>0$ and φ is a positive nondecreasing function on the interval $(0, \infty)$ such that $\limsup_{t\to 0} \varphi(t)/t < \infty$.

REMARK. If G is a Lipschitz domain and $\lambda(r)=r^{\beta}$ with $0<\beta<n-1$, then we can prove that $\kappa_{\xi,\lambda}(x) \sim \kappa_{\lambda}(\rho(x))$ (see the Appendix).

THEOREM 4. *Let u be a locally n-precise continuous function on G, and suppose*

$$\int_{G\cap B(\xi,r)} |\xi-y|^{1-n}|\text{grad}\, u(y)|\,dy < \infty \qquad \text{for some} \quad r>0\,, \tag{7}$$

$$\limsup_{r\downarrow 0} h(r)^{-1} \int_{G\cap B(\xi,r)} \Psi(|\text{grad}\, u(y)|)\lambda(\rho(y))\,dy = 0\,. \tag{8}$$

Then, for a set $T\subset G$ satisfying the above conditions (T_1), (T_2) *and* (T_3), *$u(x)$ has a finite limit as $x\in G$ tends to ξ along T.*

REMARK. Let E_0 (resp. E_h) be the set of $\xi\in\partial G$ for which (7) (resp. (8)) does not hold. If u satisfies condition (3), then we can show that $H_h(E_h)=0$; moreover, in case $\lambda(r)=r^{\beta}$ and G is a Lipschitz domain, then, in view of [2; Section 5], we see that $B_{1-\beta/n,n}(E_0)=0$, where $B_{\gamma,p}$ denotes the Bessel capacity of index (γ, p) (see [1] for the definition of Bessel capacities).

PROOF OF THEOREM 4. Let $r_0>0$ be sufficiently small, and take $x_0=x_{r_0}$ and $d_0=d_{r_0}$ having the properties in condition (T_2). By Lemma 3, we have

$$\left| u(x) - |B(x_0,d_0)|^{-1}\int_{B(x_0,d_0)} u(y)\,dy \right| \leqq M_1 \int_{E(x,x_0,d_0)} |x-z|^{1-n} f(z)\,dz$$

for $x\in T\cap B(\xi,r_0)$, where $f(z)=|\text{grad}\, u(z)|$ and M_1 is a positive constant independent of x. Thus it follows that

$$\sup_{x\in T\cap B(\xi,r_0)} |u(x)-u(x_0)| \leqq 2M_1 \sup_{x\in T\cap B(\xi,r_0)} \int_{E(x,x_0,d_0)} |x-z|^{1-n} f(z)\,dz\,.$$

If $z\in T(\xi,a)-B(\xi,2|\xi-x|)$, then $|x-z|\geqq|\xi-z|-|x-\xi|\geqq|\xi-z|/2$, so that

$$\int_{E(x,x_0,d_0)-B(\xi,2|\xi-x|)} |x-z|^{1-n} f(z)\,dz \leqq 2^{n-1}\int_{G\cap B(\xi,2r_0)} |\xi-z|^{1-n} f(z)\,dz\,.$$

On the other hand, by Lemma 2 and condition (T_3), we have

$$\int_{E(x,x_0,r_0)\cap B(\xi,2|\xi-x|)} |x-z|^{1-n}f(z)dz$$
$$\leqq M_2 h(|\xi-x|)^{-1/n}\left(\int_{G\cap B(\xi,2|\xi-x|)} \Psi(f(y))\lambda(\rho(y))dy\right)^{1/n} + M_2|\xi-x|^{1-\delta}$$

with a positive constant M_2, where $0<\delta<1$. Thus,

$$\sup_{x\in T\cap B(\xi,r_0)}|u(x)-u(x_0)| \leqq M_3 r_0^{1-\delta} + M_3\int_{G\cap B(\xi,2r_0)}|\xi-z|^{1-n}f(z)dz$$
$$+ M_3 \sup_{0<r<2r_0}\left(h(r)^{-1}\int_{G\cap B(\xi,r)}\Psi(f(y))\lambda(\rho(y))dy\right)^{1/n}$$

with a positive constant M_3 independent of r_0. In view of conditions (7) and (8), it follows that $u(x)$ has a finite limit as $x\in G$ tends to ξ along T.

For $\alpha>1$, $a\in R^1$ and $b\geqq 0$, set

$$S_\alpha(a,b)=\{x=(x_1,x')\in R^1\times R^{n-1}; x_1>a|x'|+b|x'|^\alpha\}.$$

If G is a bounded Lipschitz domain and $\alpha>0$ is given, then, for each $\xi\in\partial G$ we can find $a_\xi\in R^1$, $b_\xi\geqq 0$, $r_\xi>0$ and an orthogonal transformation Ξ_ξ such that

$$\{\xi+\Xi_\xi x; x\in S_\alpha(a_\xi,b_\xi)\}\cap B(\xi,r_\xi)\subset G.$$

For $b>b_\xi$, put

$$T_\alpha(\xi,b)=\{\xi+\Xi_\xi x; x\in S_\alpha(a_\xi,b)\}\cap B(\xi,r_\xi).$$

COROLLARY. *Let G be a bounded Lipschitz domain and let $\alpha>1$. Let $\{T_\alpha(\xi,b); \xi\in\partial G, b>b_\xi\}$ be given as above. If u is a locally n-precise continuous function on G satisfying*

$$\int_G \Psi(|\mathrm{grad}\, u(x)|)\rho(x)^\beta dx<\infty$$

with $0<\beta<n-1$, then there exists a set $E\subset\partial G$ such that

(i) *$H_h(E)=0$ for $h(r)=\sup_{0<t<r} t^{\alpha\beta}[\tilde{\psi}(t)]^{-n}$;*
(ii) *if $\xi\in\partial G-E$, then $u(x)$ has a finite limit as $x\to\xi$ along $T_\alpha(\xi,b)$ for any $b>b_\xi$.*

PROOF. It is easy to see that, for fixed $\xi\in\partial G$, $T=T_\alpha(\xi,b)$ satisfies conditions (T_1) and (T_2). By Remark (ii) before Theorem 3 and the Remark before Theorem 4, we see that $\kappa_{\xi,\lambda}(x) \sim \rho(x)^{-\beta/n}\tilde{\psi}(\rho(x))$ if $\lambda(r) = r^{\beta}$. Since $\rho(x) \geqq c|x-\xi|^{\alpha}$ for $x \in T_{\alpha}(\xi, b)$ with some $c > 0$ (depending on ξ, α, b; but not on x), condition (T_3) is satisfied with $T = T_{\alpha}(\xi, b)$ and the function h given in (i).

Let $E = E_0 \cup E_h$ in the notation given in the Remark after Theorem 4. Since $B_{1-\beta/n,n}(E_0) = 0$ implies that E_0 has Hausdorff dimension at most β (cf. [1; Theorem 22]) and since $\lim_{r\downarrow 0} h(r)/r^{\beta} = 0$, we see that $H_h(E_0) = 0$. Hence $H_h(E) = 0$, and the Corollary follows from Theorem 4.

REMARK. If $\beta = 0$ in the Corollary, then u can be extended to a continuous function on $G \cup \partial G$, on account of Theorem 2.

5. Limits at infinity

In this section, we discuss the existence of limits at infinity of n-precise functions on unbounded domains R^n and $G = \{x = (x_1, x') \in R^1 \times R^{n-1}; |x'| < 1\}$.

THEOREM 5. *If u is a locally n-precise continuous function on R^n satisfying condition* (2) *with $G = R^n$, then $[\tilde{\psi}(|x|)]^{-1}u(x) \to 0$ as $|x| \to \infty$.*

PROOF. By Lemma 3 we have

$$\left| u(x) - |B(0,r)|^{-1} \int_{B(0,r)} u(y)dy \right| \leqq M_1 \int_{B(0,r)} |x-z|^{1-n} f(z)dz$$

with a positive constant M_1 independent of x, where $r = |x|$ and $f(z) = |\mathrm{grad}\, u(z)|$. For fixed $r_0 > 0$, taking $\alpha = r^{\gamma}$ with $\delta(\gamma+1) > 1$ in the Corollary to Lemma 2, we have

$$\int_{B(0,r)-B(0,r_0)} |x-z|^{1-n} f(z)dz \leqq M_2 \left(\int_{R^n - B(0,r_0)} \Psi(f(z))dz \right)^{1/n} \tilde{\psi}(r) + M_2 r^{1-\delta-\delta\gamma}$$

with a positive constant M_2 independent of r. Here, note that $\tilde{\psi}(r) \geqq \psi(1)^{-1/n}(\log r)^{1-1/n}$ for $r > 1$, so that $\lim_{r\to\infty} \tilde{\psi}(r) = \infty$. Hence

$$\limsup_{|x|\to\infty} \tilde{\psi}(|x|)^{-1} \int_{B(0,|x|)} |x-z|^{1-n} f(z)dz \leqq M_2 \left(\int_{R^n - B(0,r_0)} \Psi(f(z))dz \right)^{1/n},$$

which implies that the left hand side is equal to zero. Similarly,

$$\left| u(0) - |B(0,r)|^{-1} \int_{B(0,r)} u(y)dy \right| \leqq M_1 \int_{B(0,r)} |z|^{1-n} f(z)dz$$

$$\leqq M_1 \int_{B(0,r_0)} |z|^{1-n} f(z)dz + M_3 \left(\int_{R^n - B(0,r_0)} \Psi(f(z))dz \right)^{1/n} \tilde{\psi}(r) + M_3 ,$$

where M_3 is a positive constant independent of r and r_0, so that

$$\lim_{r\to\infty} \tilde{\psi}(r)^{-1}\left(|B(0, r)|^{-1} \int_{B(0,r)} u(y)dy\right) = 0 .$$

Consequently, $\lim_{|x|\to\infty} [\tilde{\psi}(|x|)]^{-1}u(x) = 0$.

PROPOSITION 2. *Let $G = \{(x_1, x'); |x'| < 1\}$, and let u be a locally n-precise continuous function on G satisfying condition* (2). *Then*

$$\lim_{x_1\to\infty,(x_1,x')\in G} [\tilde{\psi}(x_1)]^{-1}x_1^{1/n-1}u(x_1, x') = 0 .$$

REMARK. Proposition 2 is best possible as to the order of infinity, in the same sense as in Remark 2 given after Proposition 1.

PROOF OF PROPOSITION 2. Let $X(r) = (r, 0)$, $r \in R^1$. If x and y belong to $\Delta(r) = B(X(r), 1)$, then, as in the proof of Theorem 5, we have

$$\begin{aligned}|u(x) - u(y)| &\leqq M_1 \sup_{z\in\Delta(r)} \int_{\Delta(r)} |z - w|^{1-n}|\text{grad } u(w)|\, dw \\ &\leqq M_2\left(\int_{\Delta(r)} \Psi(|\text{grad } u(w)|)dw\right)^{1/n} \tilde{\psi}(r) + M_2 r^{-2}\end{aligned}$$

with positive constants M_1, M_2 independent of x, y and r, where we used the Corollary to Lemma 2 with $\alpha = r^{2/\delta}$ in the second inequality. For any fixed r_0, let $r_j = r_0 + j/2$. If $r_k \leqq x_1 < r_k + 2^{-1}$, then

$$\begin{aligned}|u(x) - u(X(r_0))| &\leqq |u(x) - u(X(x_1))| + |u(X(x_1)) - u(X(r_k))| + \cdots \\ &\quad + |u(X(r_1)) - u(X(r_0))| \\ &\leqq M_3\left(\int_{E(r_0,x_1)} \Psi(|\text{grad } u(w)|)dw\right)^{1/n} (\tilde{\psi}(x_1)^{n'} + \textstyle\sum_{j=0}^{k} \tilde{\psi}(r_j)^{n'})^{1/n'} \\ &\quad + M_2(x_1^{-2} + \textstyle\sum_{j=0}^{k} r_j^{-2}) \\ &\leqq M_4\left(\int_{E(r_0,x_1)} \Psi(|\text{grad } u(w)|)dw\right)^{1/n} \tilde{\psi}(x_1)x_1^{1/n'} + M_5 r_0^{-1} ,\end{aligned}$$

where $1/n + 1/n' = 1$, $E(s, t) = \bigcup_{s<r<t} \Delta(r)$ and M_3, M_4 are positive constants independent of x and r_0. It follows that

$$\limsup_{x_1\to\infty, x\in G} [\tilde{\psi}(x_1)]^{-1}x_1^{-1/n'}|u(x)| \leqq M_4\left(\int_{E(r_0,\infty)} \Psi(|\text{grad } u(w)|)dw\right)^{1/n}$$

for any r_0, which implies that the left hand side equals zero.

Appendix

We now give a proof of $\kappa_{\xi,\lambda}(x) \sim \kappa_\lambda(\rho(x))$ given in the Remark before Theorem 4. By a change of coordinate system by a Lipschitz transformation, we may assume that G is the half space $\{x = (x_1, x'); x_1 > 0\}$ and ξ is the origin. For $x = (x_1, x') \in G \cap B(0, 1)$, let

$$E_1 = \{y = (y_1, y'); y_1 > x_1/2\} \cap B(0, 2|x|) - B(x, x_1/2),$$

$$E_2 = B(x, x_1/2),$$

$$E_3 = \{y = (y_1, y'); 0 < y_1 \leqq x_1/2\} \cap B(0, 2|x|)$$

and write

$$I_j(x) = \int_{E_j} |x - y|^{-n} [\psi(|x - y|^{-1}) \lambda(y_1)]^{-1/(n-1)} dy$$

for $j = 1, 2, 3$. Since $y_1 \geqq |y_1 - x_1|$ on E_1 and $\lambda(r) = r^\beta$ with $0 < \beta < n - 1$, we have by properties $(\psi_1)'''$ and (ψ_3)

$$\begin{aligned} I_1(x) &\leqq M_1 \int_{x_1/2}^{3|x|} [\psi(r^{-1}) \lambda(r)]^{-1/(n-1)} r^{-1} dr \\ &\leqq M_2 [x_1^\varepsilon \psi(x_1^{-1})]^{-1/(n-1)} \int_{x_1/2}^{3|x|} [r^{-\varepsilon} \lambda(r)]^{-1/(n-1)} r^{-1} dr \\ &\leqq M_3 [\psi(x_1^{-1}) \lambda(x_1)]^{-1/(n-1)} \leqq M_4 [\tilde{\psi}(x_1) \lambda(x_1)^{-1/n}]^{n'}, \end{aligned}$$

where $0 < \varepsilon < \beta$. If $y \in E_2$, then $y_1 > x_1/2$, so that

$$I_2(x) \leqq M_5 \lambda(x_1)^{-1/(n-1)} \int_0^{x_1/2} [\psi(r^{-1})]^{-1/(n-1)} r^{-1} dr \leqq M_6 [\tilde{\psi}(x_1) \lambda(x_1)^{-1/n}]^{n'}.$$

If $y \in E_3$, then $|(0, x') - y|^2 + (x_1/2)^2 \leqq 2|x - y|^2 \leqq 2[|(0, x') - y|^2 + x_1^2]$. Hence, letting $r = |(0, x') - y|$, by a computation similar to the above, we have

$$\begin{aligned} I_3(x) &\leqq M_7 \int_0^{3|x|} (r + x_1)^{-n} [\psi((r + x_1)^{-1}) \lambda(r)]^{-1/(n-1)} r^{n-1} dr \\ &\leqq M_8 \int_{x_1}^{3|x|} [\psi(r^{-1}) \lambda(r)]^{-1/(n-1)} r^{-1} dr \\ &\quad + M_8 x_1^{-n} [\psi(x_1^{-1})]^{-1/(n-1)} \int_0^{x_1} [\lambda(r)]^{-1/(n-1)} r^{n-1} dr \\ &\leqq M_9 [\tilde{\psi}(x_1) \lambda(x_1)^{-1/n}]^{n'}. \end{aligned}$$

Consequently, we establish

$$\kappa_{\xi,\lambda}(x) \leqq M_{10}\tilde{\psi}(x_1)\lambda(x_1)^{-1/n}.$$

Conversely, we obtain

$$\begin{aligned} I_2(x) &\geqq M_{11}\lambda(x_1)^{-1/(n-1)} \int_0^{x_1/2} [\psi(r^{-1})]^{-1/(n-1)} r^{-1} dr \\ &\geqq M_{12}[\tilde{\psi}(x_1)\lambda(x_1)^{-1/n}]^{n'}, \end{aligned}$$

and hence

$$\kappa_{\xi,\lambda}(x) \geqq M_{13}\tilde{\psi}(x_1)\lambda(x_1)^{-1/n}.$$

On the other hand, we find, in view of $(\psi_1)'''$, that

$$\begin{aligned} [\kappa'_\lambda(r)]^{n'} &\leqq M_{14}[r^\varepsilon\psi(r^{-1})]^{-1/(n-1)} \int_r^1 [r^{-\varepsilon}\lambda(r)]^{-1/(n-1)} r^{-1} dr \\ &\leqq M_{15}[\tilde{\psi}(r)\lambda(r)^{-1/n}]^{n'}, \end{aligned}$$

so that

$$\kappa_\lambda(x_1) \leqq M_{16}\tilde{\psi}(x_1)\lambda(x_1)^{-1/n} \leqq M_{16}\kappa_\lambda(x_1).$$

Since the constants $M_1 \sim M_{16}$ do not depend on $x \in G \cap B(0, 1)$, the required result has been derived.

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