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Planar Navier-Stokes flows in a bounded domain with measures as initial vorticities

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Introduction

Let D be a simply connected bounded domain in $\mathbb{R}^2$ with smooth boundary S. In this paper we consider the two-dimensional Navier-Stokes equations of the following form:

$$
\text{(NS)}\qquad
\begin{aligned}
&\frac{\partial u}{\partial t} + u\cdot\nabla u = \lambda\Delta u - \nabla p && (x\in D,\ t>0\\
&\nabla\cdot u = 0 && (x\in D,\ t\geqq 0)\\
&u\cdot\nu|_S = 0\,;\ \nabla\times u|_S = 0\,;\ u|_{t=0} = a,
\end{aligned}
$$

and discuss the existence and uniqueness of strong solutions when the initial vorticity $\nabla\times a$ is very singular. Here, $\lambda>0$ is the kinematic viscosity; ν is the unit outward normal to the boundary; $u=(u^1, u^2)$ and p are, respectively, unknown velocity and pressure; a is a given initial velocity; and $\nabla\cdot u = \Sigma_j\partial_j u^j$, $u\cdot\nabla u = \Sigma_j u_j\partial_j u$, $\nabla\times u = \partial_1 u^2 - \partial_2 u^1$, $\partial_j = \partial/\partial x_j$. Our goal is to establish the existence of a smooth global solution in the case where $\nabla\times a$ is a finite Borel measure on D. Our result extends those of [4, 10] obtained for the Cauchy problem to the case of simply connected bounded domains. The boundary condition for u described above not only appears in a free-boundary problem for the Navier-Stokes equations, but also is well known as a standard boundary condition for the magnetic field in the theory of magnetohydrodynamics [18].

As a byproduct we obtain an existence result for the Euler equations

$$
\text{(E)}\qquad
\begin{aligned}
&\frac{\partial u}{\partial t} + u\cdot\nabla u + \nabla p = 0 && (x\in D,\ t>0)\\
&\nabla\cdot u = 0 && (x\in D,\ t\geqq 0)\\
&u\cdot\nu|_S = 0\,;\ u|_{t=0} = a,
\end{aligned}
$$

in the case where $\nabla\times a$ belongs to L^q for some $q>1$, by investigating the behavior of solutions u_λ to (NS) as λ goes to 0. A similar result was obtained by Bardos [2] in L^2-framework, and our result can be regarded as an L^p-version of that of [2].

This paper is organized as follows. In Section 1 we introduce necessary notation and definitions and state our results. Our main results are stated in Theorems 1.4 and 1.5. Theorem 1.4 asserts that for each a such that the associated vorticity $\nabla \times a$ is a finite Borel measure, there exists a smooth global solution u to problem (NS). The uniqueness is proved only when the variation of $\nabla \times a$ is small compared with the viscosity $\lambda > 0$. The existence result is deduced in the standard manner, namely, we first construct approximate solutions and then show their convergence with the aid of a-priori estimates which are uniform in apporoximation. The main difficulty consists in finding these a-priori estimates. Indeed, the standard method of parabolic evolution equations is not applicable, mainly because in our case the initial vorticities form a Banach space in which the smooth elements are not dense. We can overcome this difficulty by appealing to the estimate of Nash [14, 15] for the fundamental solution of the Dirichelt-Cauchy problem for a second order parabolic equation with discontinuous coefficients. Theorem 1.5 is concerned with the existence of global L^p-solutions to the Euler equations (E).

Section 2 is devoted to the construction of approximate solutions to (NS) under the assumtion that the initial vorticity is a finite Borel measure. To this end, we need to consider (NS) in general L^r-spaces. So, we first prove that for each $a \in L^r$, $r > 2$, with $\nabla \times a \in L^q$, $1/q = 1/2 + 1/r$, there exists a unique smooth solution defined for all $t \geqq 0$, first by constructing a local solution and then extending it to a global one with the aid of the vorticity transport equation for $\omega = \nabla \times u$:

$$\text{(V)} \qquad \begin{aligned} &\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = \lambda \Delta \omega \qquad (x \in D,\ t > 0) \\ &\omega|_S = 0\ ;\ \omega|_{t=0} = \nabla \times a. \end{aligned}$$

The result is then applied to construct approximate solutions for problem (NS) when $\nabla \times a$ is a finite Borel measure.

In Section 3 we deduce a-priori estimates for approximate solutions u of problem (NS), applying the estimate of Nash for the fundamental solution to the linear (in ω) parabolic problem (V). As will be shown in Section 4, these a-priori estimates ensure the convergence of (a subsequence of) the approximate solutions and lead us to the conclusions stated in Theorem 1.4. In Section 5 we prove Theorem 1.5 by the so-called vanishing viscosity method. Namely, we consider problem (NS), assuming that $\nabla \times a \in L^q$ for some $1 < q < 2$, and prove the convergence of solutions u_λ of (NS) to a solution of (E) as $\lambda \to 0$. Our proof of Theorem 1.5 also shows the Hölder continuity in $x \in \bar{D}$ of the solution u in case $\nabla \times a \in L^q$ for some $q > 2$, and this generalizes the classical result of Yudovich [23] which deals with the case $q=\infty$. The uniqueness problem remains open.

1. Preliminaries and results

First we introduce necessary function spaces. By $L^r(D)$, $1 \leqq r \leqq \infty$, we denote the usual Lebesgue spaces of real-valued functions with norm $\|\cdot\|_r$, and $W^{s,r}(D)$ denotes the usual L^r Sobolev spaces [1]. Moreover, $L^{r,q}(D)$, $1<r \leqq \infty$, $1 \leqq q \leqq \infty$, denotes the Lorentz spaces, with norm $\|\cdot\|_{r,q}$ which are obtained by applying the real interpolation method [20] to L^r-spaces. As is well known [19]. $L^r(D)=L^{r,r}(D)$ and we have the duality relation :

$$L^{r,q}(D)^* = L^{r',q'}(D), \quad 1/r' = 1-1/r, \quad 1/q' = 1-1/q,$$

for $1<r<\infty$ and $1 \leqq q < \infty$. We also notice that [11] a function f is in $L^{r,\infty}(D)$ for some $1<r<\infty$ if and only if

$$\|f\|_{r,\infty} \equiv \sup_E |E|^{1-1/r} \int_E |f| dx < +\infty$$

Where $|E|$ denotes the Lebesgue measure of a measurable set $E \subset D$. Now, consider the Helmenholtz decomposition [8] :

$$L^r(D)^2 = X_r \oplus G_r, \qquad (1<r<\infty),$$

where

$$X_r = \{u \in L^r(D)^2 \,;\, \nabla \cdot u = 0,\ u \cdot \nu|_S = 0\} \,;\quad G_r = \{\nabla p \,;\, p \in W^{1,r}(D)\},$$

and the associated bounded projector $P=P_r$ onto X_r. The standard interpolation argument shows that the operator P defines bounded projector on $L^{r,\infty}(D)^2$, the range of which will be denoted by $X_{r,\infty}$.

LEMMA 1.1. *Let* $1<p<q<\infty$, $0<\theta<1$; *then*

$$X_{r,\infty} = (X_p, X_q)_{\theta,\infty}, \quad \text{for} \quad 1/r = (1-\theta)/p + \theta/q, \tag{1.1}$$

where the right-hand side denotes the real interpolation space. Moreover,

$$X_{r,\infty} = \{u \in L^{r,\infty}(D)^2 \,;\, \nabla \cdot u = 0,\ u \cdot \nu|_S = 0\}. \tag{1.2}$$

PROOF. It follows from [19, Sect. 1.2.4] that interpolating between the operators

$$P\colon L^p(D)^2 \to X_p \quad \text{and} \quad P\colon L^q(D)^2 \to X_q$$

gives the surjection

$$P: (L^p(D)^2, L^q(D)^2)_{\theta,\infty} \to (X_p, X_q)_{\theta,\infty}.$$

But, since the left-hand side above equals $L^{r,\infty}(D)^2$ and since P is a bonded projector in $L^{r,\infty}(D)^2$ by [19, Sect. 1.2.4], we obtain the first assertion (1.1). The second assertion (1.2) immediately follows from the definition of the so-called J-method for constructing interpolation spaces [19, Sect. 1.6]. This proves Lemma 1.1.

We next consider the Laplace operator $B = B_r = -\Delta$ in $L^r(D)^2$, $1 < r < \infty$, defined on

$$D(B_r) = \{u \in W^{2,r}(D)^2\,;\ u\cdot v|_S = 0,\ \nabla \times u|_S = 0\}.$$

As shown in [13], the restriction A_r of B_r to the subspace X_r is a densely defined closed operator. Furthermore, since D is simply connected, $\|A^m u\|_r$ is equivalent to the $W^{2m,r}$-norm for every integer $m \geqq 1$ and so A_r is boundedly invertible. Consequently, $-A_r$ generates a bounded analytic C_0-semigroup $\{e^{-tAr};\ t \geqq 0\}$ [16] and therefore the fractional powers A_r^α, $\alpha \geqq 0$, are well defined. The result of Seeley [17] on the domains of fractional powers implies that, for all $1 < r < \infty$ and $0 < \alpha < 1$,

$$D(A_r^\alpha) = D(B_r^\alpha) \cap X_r \subset H^{2\alpha,r}(D)^2 \cap X_r \qquad \text{with continuous injection,}$$

where $H^{s,r}$ stands for the space of Bessel potentials [20]. Thus, the standard Sobolev embedding yields the following L^r-L^q estimeates :

$$(1.3)\qquad \begin{aligned} &\|e^{-tA}a\|_q \leqq Ct^{-(1/r-1/q)}\|a\|_r \qquad && a \in X_r,\\ &\|\nabla e^{-tA}a\|_q \leqq Ct^{-1/2-(1/r-1/q)}\|a\|_r \qquad && a \in X_r, \end{aligned}$$

provided either $1 < r \leqq q < \infty$ or $1 < r < q \leqq \infty$. It then follows from Lemma 1.1 that $\{e^{-tA};\ t \geqq 0\}$ is also bounded and analytic (but not strongly continuous) in $X_{r,\infty}$, and we have the estimates

$$\begin{aligned} &\|e^{-tA}a\|_{q,\infty} \leqq Ct^{-(1/r-1/q)}\|a\|_{r,\infty} \qquad && a \in X_r,\\ &\|\nabla e^{-tA}a\|_{q,\infty} \leqq Ct^{-1/2-(1/r-1/q)}\|a\|_{r,\infty} \qquad && a \in X_r, \end{aligned}$$

provided that $1 < r \leqq q < \infty$. Now, Iet $1 < p < q < r < \infty$. Since $L^{p,\infty} \cap L^{r,\infty} \subset L^q$ with the relation

$$(1.4)\qquad \|f\|_q \leqq C\|f\|_{p,\infty}^{1-\theta}\|f\|_{r,\infty}^{\theta}, \quad 1/q = (1-\theta)/p + \theta/r,$$

we obtain the estimates

$$(1.5)\qquad \begin{aligned} &\|e^{-tA}a\|_q \leqq Ct^{-(1/r-1/q)}\|a\|_{r,\infty}, \qquad && a \in X_r,\\ &\|\nabla e^{-tA}a\|_q \leqq Ct^{-1/2-(1/r-1/q)}\|a\|_{r,\infty}, \qquad && a \in X_r, \end{aligned}$$

provided that $1 < r < q < \infty$. Estimate (1.4) is deduced in the following way: Let

$$\lambda(\alpha) = |D(|f| > \alpha)| \quad \text{for} \quad \alpha > 0.$$

By the standard definition of $L^{r,\infty}$-spaces,

$$\lambda(\alpha) \leqq (\|f\|_{p,\infty}^*)^p \alpha^{-p}, \quad \text{and} \quad \lambda(\alpha) \leqq (\|f\|_{r,\infty}^*)^r \alpha^{-r},$$

where $\|f\|_{r,\infty}^*$ is the standard quasinorm defining the $L^{r,\infty}$-topology; see [11, 19, 20]. Thus, the definition of the Lebesgue integral gives

$$\begin{aligned}
\int_D |f|^q dx &= -\int_0^\infty \alpha^q d\lambda(\alpha) = q\int_0^\infty \alpha^{q-1}\lambda(\alpha)d\alpha \\
&= q\left(\int_0^M + \int_M^\infty\right)\alpha^{q-1}\lambda(\alpha)d\alpha \\
&\leqq C\left[(\|f\|_{p,\infty}^*)^p \int_0^M \alpha^{q-p-1}d\alpha + (\|f\|_{r,\infty}^*)^r \int_M^\infty \alpha^{q-r-1}\,d\alpha\right] \\
&= C\left[(\|f\|_{p,\infty}^*)^p M^{q-p} + (\|f\|_{r,\infty}^*)^r M^{q-r}\right].
\end{aligned}$$

Since $\|f\|_{r,\infty}^*$ and $\|f\|_{r,\infty}$ are equivalent [11], taking the minimum in $M > 0$ yields (1.4).

The next two lemmas (Lemmas 1.2 and 1.3) clarify the reason why we need the Lorentz spaces $L^{r,\infty}(D)$ in our study of planar Navier-Stokes flows.

LEMMA 1.2. *Let $K(x, y)$ be a measurable function defined on $D \times D$ such that*

$$|K(x, y)| \leqq C|x - y|^{-1}.$$

Given a finite Borel measure μ on D, the function

$$(K\mu)(x) = \int_D K(x, y)\mu(dy)$$

belongs to $L^{2,\infty}(D)$ and satisfies the estimate

$$\|K\mu\|_{2,\infty} \leqq C\|\mu\|_1$$

with C independent of μ, where $\|\mu\|_1$ denotes the total variation of the measure μ.

PROOF. Without loss of generality we may assume that the measure μ is nonnegative. Let $\tilde{\mu}$ be the finite Borel measure on $\mathbb{R}^2$ defined by

$$\tilde{\mu}(E) = \mu(E \cap D).$$

Since

$$|K\mu|(x) \leqq C \int_{R^2} |x-y|^{-1} \tilde{\mu}(dy),$$

we get, for any Borel set $E \subset D$,

$$\begin{aligned}\int_E |K\mu| dx &\leqq C \iint_{R^2 \times R^2} 1_E(x)|x-y|^{-1} \tilde{\mu}(dy), \\ &= \int \tilde{\mu}(dy) \int 1_E(x+y)|x|^{-1} dx,\end{aligned}$$

where 1_E is the indicator function for $E \subset \mathbb{R}^2$. Since

$$\int 1_F(x)|x|^{-1} dx \leqq C_0 |F|^{1/2},$$

with $C_0 > 0$ independent of Borel sets $F \subset \mathbb{R}^2$, we see that

$$\int_E |k\mu| dx \leqq CC_0 \|\tilde{\mu}\|_1 |E-y|^{1/2} = CC_0 \|\mu\|_1 |E|^{1/2}.$$

This proves Lemma 1.2.

Suppose now we are given a solenoidal vector field a such that $\nabla \times a$ is a finite Borel measure on D and $a \cdot \nu$ vanishes on S. Since D is simply connected by assumption, we may assume that

$$a = (a^1, a^2) = \nabla^{\perp}\psi \equiv \left(-\frac{\partial\psi}{\partial x_2}, \frac{\partial\psi}{\partial x_1}\right)$$

for some distribution ψ on D. Direct calculation then shows that $\Delta\psi = \nabla \times a$ in D and that we may assume $\psi = 0$ on S. Hence

$$\psi(x) = \int_D G(x, y)(\nabla \times a)(dy)$$

in terms of the Green function $G(x, y)$ of the Dirichlet problem for the Laplacian in D. We thus have

$$a(x) = \int_D K(x, y)(\nabla \times a)(dy)$$

where

$$K(x, y) = \nabla_x^{\perp} G(x, y) \equiv \left(-\frac{\partial G}{\partial x_2}(x, y), \frac{\partial G}{\partial x_1}(x, y)\right) \tag{1.6}$$

and therefore

$$|K(x, y)| \leqq C|x-y|^{-1}$$

for some constant $C>0$ (see [12]). Lemma 1.2 then implies that $a \in L^{2,\infty}(D)$ and hence $a \in X_{2,\infty}$ by Lemma 1.1. On the other hand, the simple-connectivity of D implies that if

$$\nabla \cdot a = 0, \ \nabla \times a = 0, \quad \text{and} \quad a \cdot \nu|_S = 0,$$

then $a = 0$ in D; see [3]. We thus obtain the following

LEMMA 1.3. (i) *If $a \in X_{2,\infty}$ and $\nabla \times a$ is a finite Borel measure on D, then*

$$a(x) = \int_D K(x, y)(\nabla \times a)(dy)$$

in terms of the function K introduced in (1.6) ; *moreover, the estimate*

$$\|a\|_{2,\infty} \leqq C\|\nabla \times a\|_1 \tag{1.7}$$

holds with C independent of a.

(ii) *For any finite Borel measure μ on D, the vector function*

$$a(x) = \int_D K(x, y)\mu(dy)$$

belongs to $X_{2,\infty}$, satisfies estimate (1.7) *and solves the equation*

$$\nabla \times a = \mu \qquad in \quad D$$

in the sense of ditributions.

Now let $a \in X_{2,\infty}$, and let $\nabla \times a$ be a finite Borel measure on D. In view of Lemma 1.3 we have to find a function $u(t)$ of $t \geqq 0$ with values in $X_{2,\infty}$ which solves the integral equation

$$u(t) = e^{-\lambda t A}a - \int_0^t e^{-\lambda(t-s)A}P(u \cdot \nabla)u(s)ds \tag{1.8}$$

in an appropriate sense. To solve (1.8) we mainly follow the arguments developed in [4, 10] which deal with the case of the Cauchy problem. Namely, we first solve (1.8) in Section 2 in the spaces X_r, $2<r<\infty$, and then apply the obtained result to construct approximate solutions. The convergence of the approximate solutions to a desired solution is then discussed in Sections 3 and 4 with the aid of the estimate of Nash [14, 15] for the fundamental solution to the Dirichlet-Cauchy problem for a second-order parabolic equation with discontinuous coefficients. Our results are stated as follows.

THEOREM 1.4. *Given* $a \in X_{2,\infty}$ *such that* $\nabla \times a$ *is afinite Borel measure on* D, *there exists a function* $u(t)$ *defined for all* $t \geqq 0$ *with the following properties.*

(i) u *is bounded and continuous from* $[0, \infty)$ *to* $X_{2,\infty}$ *in the weak* topology of* $L^{2,\infty}(D)$, and satisfies $u(0) = a$.

(ii) $\nabla \times u$ *is bounded and continuous from* $[0, \infty)$ *to the space* M *of finite Borel measure on* D *in the vague topology of* M, *and* $(\nabla \times u)(0) = \nabla \times a$.

(iii) *There exist positive constants* C_j, $j = 1, 2, 3$, *depending only on* r *and* D, *so that*

$$\|(\nabla \times u)(t)\|_r \leqq C_1(\lambda t)^{-1+1/r}\|\nabla \times a\|_1 \quad \text{for} \quad 1 \leqq r \leqq \infty,$$

$$\|u(t)\|_r \leqq C_2(\lambda t)^{1/r-1/2}\|\nabla \times a\|_1 \quad \text{for} \quad 2 < r \leqq \infty,$$

$$\|\nabla u(t)\|_r \leqq C_3(\lambda t)^{-1+1/r}\|\nabla \times a\|_1 \quad \text{for} \quad 1 < r < \infty.$$

(iv) u *satisfies the integral equation* (1.8) *in the weak* topology of* $L^{2,\infty}(D)$.

(v) u *is unique in case* $\|\nabla \times a\|_1/\lambda$ *is sufficiently small.*

THEOREM 1.5. *Let* $a \in X_r$, $2 < r < \infty$, *and* $\nabla \times a \in L^q$ *with* $1/q = 1/r + 1/2$. *Then there exists a function* u *with the following properties:*

(i) u *is bounded and continuous from* $[0, \infty)$ *to* X_r *in the weak topology of* X_r, *and satisfies* $u(0) = a$. *Furthermore,* u *is continuous from* $[0, \infty)$ *to* X_q.

(ii) $\nabla \times u$ *is bounded and continuous from* $[0, \infty)$ *to* L^q *the weak topology of* $L^q(D)$, *and* $(\nabla \times u)(0) = \nabla \times a$.

(iii) *The function* u *solves the Euler equations* (E) *in the sense that the identity*

$$\frac{d}{dt}(u, \phi) - (u \otimes u, \nabla\phi) = 0, \qquad \text{in} \quad t > 0$$

holds for all $\phi \in C^1(\bar{D})^2$ *with* $\nabla \cdot \phi = 0$, *and* $\phi \cdot v|_S = 0$, *where* $(u \otimes u)_{ij} = u_i u_j$, $i, j = 1, 2$, *and* $(\cdot,\cdot)$ *denotes the standard* L^2*-inner product.*

(iv) *If in addition* $\nabla \times a \in L^s$ *for some* $2 < s \leqq \infty$, *then for any fixed* $t > 0$, $u(\cdot, t)$ *is Hölder-continuous with exponent* $\alpha = 1 - 2/s$ *when* $s < \infty$, *and with an arbitrary exponent* $0 < \alpha < 1$ *when* $s = \infty$.

Part (iv) of Theorem 1.5 is originally due to Yudovich [23], in which is discussed the case $s = \infty$. As shown in Section 2, the function u given in Theorem 1.4 is smooth on $\bar{D} \times (0, \infty)$ and solves (NS) in the classical sense for $t > 0$. Theorem 1.4 is proved in Section 4 after preparing necessary material in Sections 2 and 3. Theorem 1.5 will be proved in Section 5 by letting $\lambda \to 0$ for the solution u_λ of problem (NS) given in Theorem 1.4.

2. Smooth solutions X_r, $r > 2$

The standard iteration technique as developed in [7, 9] does not apply to equation (1.8) unless $a \in X_{2,\infty}$ is small. This is because of the fact that smooth functions are not dense in $X_{2,\infty}$. So, we employ the following approach, due to [10]: First we consider the regularized initial data $a_\varepsilon = e^{-\varepsilon\lambda A}a$, which are smooth over $\bar{D}$ and so belong to X_r for all $1 < r < \infty$, and try to find a solution u_ε of equation (1.8) with $u_\varepsilon(0) = a_\varepsilon$. We then discuss convergence of u_ε, as $\varepsilon \to 0$, to get a desired solution u of (1.8). To carry out this procedure, we discuss in this section the solvability of (1.8) with $a \in X_r$ for some $r > 2$, employing the standard iteration scheme:

$$
\begin{aligned}
u_{k+1}(t) &= u_0(t) - \int_0^t e^{-\lambda(t-s)A} P(u_k \cdot \nabla) u_k(s) ds, \qquad k = 0, 1, 2, \dots, \\
u_0(t) &= e^{-\lambda t A} a.
\end{aligned} \tag{2.1}
$$

THEOREM 2.1. *Given $a \in X_r$, $2 < r < \infty$, there exist a number $T > 0$ and a unique solution $u \in C([0, T); X_r)$ of (1.8) with the following properties:*

(i) $\quad \|u(t)\|_r \leqq C\|a\|_r; \qquad (\lambda t)^{1/2}\|\nabla u(t)\|_r \leqq C\|a\|_r.$

(ii) *The number $T > 0$ is bounded below as*

$$T \geqq C_r \lambda^{-1+1/\sigma} / \|a\|_r^{1/\sigma}, \qquad \sigma = 1/2 - 1/r.$$

Theorem 2.1 can be proved in the same way as in [10, Sect. 1] if we apply the following Lemma 2.2 to estimate the bilinear operator

$$
\begin{aligned}
S_\lambda[v, w](t) &= -\int_0^t e^{-\lambda(t-s)A} P(v \cdot \nabla) w(s) ds \\
&= -\int_0^t A^{1/2} e^{-\lambda(t-s)A} A^{-1/2} P \nabla \cdot (v \otimes w)(s) ds.
\end{aligned}
$$

LEMMA 2.2. *Let $0 < T < \infty$ and*

$$|u|_{r,T} = \sup_{t \in [0,T)} \|u(t)\|_r.$$

Then we have the estimates

$$|S_\lambda[v, w]|_{r,T} \leqq M(\lambda T)^\sigma |v|_{r,T} |w|_{r,T} / \lambda,$$

and

$$|(\lambda \cdot)^{1/2} \nabla S_\lambda[v, w]|_{r,T} \leqq M(\lambda T)^\sigma |v(\cdot)|_{r,T} |(\lambda \cdot)^{1/2} \nabla w(\cdot)|_{r,T} / \lambda,$$

where $r > 2$, $\sigma = 1/2 - 1/r$; and $M > 0$ depends only on r and D.

Lemma 2.2 is easily obtained with the aid of estimates (1.3) and the fact that the linear operator $A_r^{-1/2}P\nabla\cdot$ is continuous in L^r-topology, $1<r<\infty$. The latter follows from the interpolation result:

$$D(A_r^{1/2})=D(B_r^{1/2})\cap X_r=W^{1,r}(D)^2\cap X_r \text{ with equivalent norms}$$

and the Poincaré inequality

$$\|v\|_r\leqq C\|\nabla v\|_r \quad \text{for} \quad v\in W^{1,r}(D)^2\cap X_r$$

which is valid since we assume that D is simply connected. By the standard argument as developed in [9] the solution u given in Theorem 2.1 is smooth on $\bar{D}\times(0,T)$. When $a\in D(A_r)$, u is regular up to the initial time $t=0$. More precisely, we have

THEOREM 2.3. *Let $a\in D(A_r)$, $r>2$; then the solution u given in Theorem 2.1 lies in $C^\alpha(\bar{D}\times[0,T))\cap C^\infty(\bar{D}\times(0,T))$ for some $0<\alpha<1$.*

This regularity result is important in Section 3 in discussing the passage to the limit $\varepsilon\to 0$ for approximate solution u_ε. A proof of Theorem 2.3 is found in [7. 9]. The Hölder continuity of u up to $t=0$ in Theorem 2.3 is proved in [7] only for $r=2$; but, the proof given there applies also to the case of general r.

We now extend the local solution u of Theorem 2.1 to a global one. To do so, it suffices in view of Theorem 2.1 (iii) to show the boundedness of the norm $\|u(t)\|_r$ as t approaches T. To this purpose we consider the vorticity transport equation for $\omega=\nabla\times u$:

$$\text{(V)}\qquad \begin{aligned}&\frac{\partial\omega}{\partial t}+u\cdot\nabla\omega=\lambda\Delta\omega \qquad (x\in D,\ t>0)\\ &\omega|_S=0;\ \omega|_{t=0}=\nabla\times a.\end{aligned}$$

Since $\nabla\cdot u=0$, a standard argument shows

$$\|\omega(t)\|_q\leqq\|\omega(s)\|_q \quad \text{for} \quad 1\leqq q\leqq 2 \quad \text{and} \quad 0\leqq s\leqq t.$$

Since

$$u(x,t)=\int_D K(x,y)\omega(y,t)dy$$

in terms of the function K introduced in (1.6), the Hardy-Littlewood-Sobolev inequality [16] yields

$$\|u(t)\|_r\leqq C\|\omega(t)\|_q\leqq C\|\omega(s)\|_q$$

provided $1<q<2$ and $1/r=1/q-1/2$. These arguments, together with Theorem 2.1, imply the following

THEOREM 2.4 *The local solution u of* (1.8) *given in Theorem* 2.1 *extends uniquely to a global solution.*

3. A-priori estimates for approximate solutions

Let $a \in X_{2,\infty}$. Then, estimate (1.5) shows that the function $a_\varepsilon = e^{-\varepsilon A}a$, $\varepsilon > 0$, belongs to X_r for all $r > 2$. Thus, by the results in Section 2 there exists a global smooth solution u_ε to the integral equation (1.8) with $u_\varepsilon(0) = a_\varepsilon$. We shall prove Theorem 1.4 by showing that, as $\varepsilon \to 0$, a subsequence of u_ε converges to the desired solution with initial data a. To this end, we establish in this section a-priori estimates for approximate solutions u_ε which are uniform in $\varepsilon > 0$. These estimates will then be applied in Section 4 in discussing the passage to the limit $\varepsilon \to 0$.

Consider the vorticity transport equation (V). Since $a_\varepsilon \in D(A_r)$ for all $r > 2$, u_ε is in $C^\alpha(\bar{D} \times [0, \infty))$ by Theorems 2.3 and 2.4. So there exists [5] the fundamental solution

$$\Gamma_{\varepsilon\lambda}(x, t; y, s), \quad 0 \leqq s < t$$

to the Dirichlet-Cauchy problem for the linear parabolic operator

$$L_{\varepsilon\lambda} = \frac{\partial}{\partial t} - \lambda\Delta + u_\varepsilon \cdot \nabla.$$

By the maximum principle for parabolic equations we see that $\Gamma_{\varepsilon\lambda} > 0$ for x, $y \in D$ and $0 < s < t$, and

$$\int_D \Gamma_{\varepsilon\lambda}(x, t; y, s)ds \leqq 1\,; \quad \int_D \Gamma_{\varepsilon\lambda}(x, t; y, s)dx \leqq 1. \tag{3.1}$$

Notice that the latter estimate of (3.1) is valid since the function

$$v(y, s) = \int_D \Gamma_{\varepsilon\lambda}(x, t; y, s)dx, \quad s < t,$$

solves the backward problem

$$\frac{\partial v}{\partial s} + \lambda\Delta v + \nabla \cdot (u_\varepsilon v) = 0 \quad (s < t)$$

$$v|_S = 0;\ v|_{s=t} = 1$$

and since $\nabla \cdot (u_\varepsilon v) = u_\varepsilon \cdot \nabla v$ because $\nabla \cdot u_\varepsilon = 0$. Moreover, we have the following result of Nash [14]; see also [15].

THEOREM 3.1. *There is a constant $C > 0$ independent of $\varepsilon > 0$ and $\lambda > 0$ so* *that*

$$0 < \Gamma_{\varepsilon\lambda}(x, t; y, s) \leqq C[\lambda(t-s)]^{-1}$$

for all $x, y \in D$ *and* $0 \leqq s < t$.

PROOF. We give a complete for the reader's convenience. It suffices to show that, for all $s < t$ and $x, y \in D$,

$$\begin{aligned} &\int_D \Gamma_{\varepsilon\lambda}(x, t; y, s)^2 dy \leqq C_1[\lambda(t-s)]^{-1}; \\ &\int_D \Gamma_{\varepsilon\lambda}(x, t; y, s)^2 dx \leqq C_2[\lambda(t-s)]^{-1}, \end{aligned} \tag{3.2}$$

with C_1 and C_2 independent of s, t, x, y and $\varepsilon > 0$. Indeed, the result follows by applying the Schwarz inequality to the Chapman-Kolmogorov relation:

$$\Gamma_{\varepsilon\lambda}(x, t; y, s) = \int_D \Gamma_{\varepsilon\lambda}(x, t; z, \tau)\Gamma_{\varepsilon\lambda}(z, \tau; y, s)dz$$

for $s < \tau < t$. To show (3.2) we follow Nash [14]. We can assume $(y, s) = (0, 0)$ and consider

$$E(t) = \int_D \Gamma_{\varepsilon\lambda}(x, t; 0, 0)^2 dx.$$

Then, since $\nabla \cdot u_\varepsilon = 0$ in D and $u_\varepsilon \cdot \nu = 0$ on S, we get

$$E_t = 2\lambda \int_D \Gamma_{\varepsilon\lambda} \Delta \Gamma_{\varepsilon\lambda} dx = -2\lambda \int_D |\nabla \Gamma_{\varepsilon\lambda}|^2 dx.$$

Using the inequality

$$\|f\|_2 \leqq C\|f\|_1^{1/2}\|\nabla f\|_2^{1/2}$$

which follows by combining

$$\|f\|_2 \leqq \|f\|_1^{1/3}\|f\|_4^{2/3} \quad \text{and} \quad \|f\|_4 \leqq C\|f\|_1^{1/2}\|\nabla f\|_2^{1/2},$$

we see from (3.1) that

$$\begin{aligned} -E_t = 2\lambda\|\nabla\Gamma_{\varepsilon\lambda}\|_2^2 &\geqq 2\lambda\|\Gamma_{\varepsilon\lambda}\|_1^2\|\nabla\Gamma_{\varepsilon\lambda}\|_2^2 \\ &\geqq \lambda C\|\Gamma_{\varepsilon\lambda}\|_2^4 = \lambda C E^2. \end{aligned}$$

Integrating this yields

$$E(t) \leqq C(\lambda t)^{-1}$$

and this shows the first inequality of (3.2). The second one is similarly obtained if we note that the function $\Gamma_{\varepsilon\lambda}(\cdot, \cdot; y, s)$ is the fundamental solution to the backward Dirichelet-Cauchy problem for the operator

$$\frac{\partial}{\partial s} + \lambda\Delta + \nabla\cdot(u_\varepsilon\cdot) = \frac{\partial}{\partial s} + \lambda\Delta + u_\varepsilon\cdot\nabla.$$

This proves Theorem 3.1.

Applying (3.1) and Theorem 3.1 to the relation

$$\omega_\omega(x, t) = \int_D \Gamma_{\varepsilon\lambda}(x, t; y, 0)(\nabla \times a_\varepsilon)(y)dy,$$

we easily obtain

$$\text{(3.3)} \qquad \begin{aligned} \|\omega_\varepsilon(t)\|_q &\leqq C_q(\lambda t)^{-1+1/q}\|\nabla \times a_\varepsilon\|_1 \\ &\leqq C_q(\lambda t)^{-1+1/q}\|\nabla \times a\|_1 \end{aligned}$$

for $1 \leqq q \leqq \infty$, with $C_q > 0$ inedependent of $\varepsilon > 0$. Notice that here we have used the fact that

$$\nabla \times a_\varepsilon = \nabla \times e^{-\lambda\varepsilon A}a = e^{\lambda\varepsilon\Delta}(\nabla \times a),$$

where Δ is the Laplace operator with zero boundary condition, and therefore

$$\|\nabla \times a_\varepsilon\|_1 = \|e^{\lambda\varepsilon\Delta}(\nabla \times a)\|_1 \leqq \|\nabla \times a\|_1.$$

Furthermore, since

$$|K(x, y)| \leqq C|x - y|^{-1},$$

applying (1.7), (3.3) and the Hardy-Littlewood-Sobolev inequality [16] to

$$u_\varepsilon(x, t) = \int_D K(x, y)\omega_\varepsilon(y, t)dy$$

gives

$$\text{(3.4)} \qquad \|u_\varepsilon(t)\|_{2,\infty} \leqq C\|\nabla \times a\|_1,$$

$$\text{(3.5)} \qquad \|u_\varepsilon(t)\|_r \leqq C_r(\lambda t)^{1/r-1/2}\|\nabla \times a\|_1 \quad \text{for} \quad 2 < r < \infty,$$

with C_r independent of $\varepsilon > 0$. Furthermore, we have

$$\text{(3.6)} \qquad \|\nabla u_\varepsilon(t)\|_q \leqq C_q(\lambda t)^{-1+1/q}\|\nabla \times a\|_1 \quad \text{for} \quad 1 < q < \infty,$$

with C_q independent of $\varepsilon > 0$, which follows from the fact that function $\nabla_x K(x, y)$ is decomposed into the sum of a Calderón-Zygmund kernel [12] and a smooth function on $\bar{D} \times \bar{D}$ and hence defines a bounded linear operator in

$L^r(D)$, $1 < r < \infty$. Since D is simply connected, $\|\nabla v\|_q$ is equivalent to the norm of $D(B_q^{1/2})$. So, (3.5), (3.6) and the Gagliardo-Nirenberg inequality [6]:

$$\|f\|_\infty \leqq C\|f\|_4^{1/2}\|\nabla f\|_4^{1/2}$$

implies that (3.5) is valid also for $r = \infty$.

LEMMA 3.2. *The functions $\omega_\varepsilon(t)$, $\varepsilon > 0$, are uniformly bounded and equicontinuous in $\varepsilon > 0$ on each bounded interval $[0, T]$ in the vague topology of measures.*

PROOF. It suffices to show the existence of $0 < \alpha \leqq 1$ such that

$$|(\omega_\varepsilon(t), \phi) - (\omega_\varepsilon(s), \phi)| \leqq C|t-s|^\alpha$$

uniformly $\varepsilon > 0$, for each fixed $\phi \in C^2(\bar{D})$ with $\phi|_S = 0$. Indeed, the set of such functions ϕ is dense in a predual $\{\psi \in C(\bar{D}); \psi|_S = 0\}$ of M. We write

$$|(\omega_\varepsilon(t), \phi) - (\omega_\varepsilon(s), \phi)| = \left|\int_s^t \left(\frac{\partial \omega_\varepsilon}{\partial \tau}, \phi\right) d\tau\right| = I_1 + I_2, \tag{3.7}$$

where

$$I_1 = \lambda\left|\int_s^t (\Delta\omega_\varepsilon, \phi) d\tau\right| = \lambda\left|\int_s^t (\omega_\varepsilon, \Delta\phi) \mathrm{d}\tau\right|;$$

$$I_2 = \left|\int_s^t (u_\varepsilon\cdot\nabla\omega_\varepsilon, \phi) d\tau\right| = \left|\int_s^t (u_\varepsilon \otimes \omega_\varepsilon, \nabla\phi) d\tau\right|.$$

An integration by parts gives, for $t \geqq s$,

$$I_1 \leqq \lambda\|\Delta\phi\|_\infty \int_s^t \|\omega_\varepsilon\|_1 d\tau \leqq C_\varphi(t-s), \tag{3.8}$$

while (3.5) and (3.6) together imply that, for $t \geqq s$,

$$I_2 \leqq \int_s^t \|\omega_\varepsilon\|_3\|u_\varepsilon\|_3\|\nabla\phi\|_3 d\tau \leqq C\|\nabla\phi\|_3 \int_s^t \tau^{-2/3-1/6} d\tau. \tag{3.9}$$

Combining (3.7)–(3.9) yields the desired result.

4. Proof of Theorem 1.4

By (3.3)–(3.6) and Lemma 3.2 we may asuume that, as $\varepsilon \to 0$, u_ε converges weakly in an appropriate sense to a function u satisfying

$$\|u(t)\|_{2,\infty} \leqq C\|\nabla \times a\|_1; \ \|u(t)\|_r \leqq C_r(\lambda t)^{1/r-1/2}\|\nabla \times a\|_1 \quad \text{for all } 2 < r \leqq \infty; \tag{4.1}$$

(4.2) $$\|(\nabla \times u)(t)\|_q \leqq C_q(\lambda t)^{-1+1/q}\|\nabla \times a\|_1 \quad \text{for all } 1 \leqq q \leqq \infty;$$

(4.3) $$\|\nabla u(t)\|_r \leqq C_r(\lambda t)^{-1+1/r}\|\nabla \times a\|_1 \quad \text{for all } 1 < r < \infty;$$

and

(4.4) $$\nabla \times u\colon [0, \infty) \to M \text{ is vaguely continuous.}$$

In this section we complete the proof of Theorem 1.4 by showing that the function u is the desired solution of (1.8). We begin by establishing the following

LEMMA 4.1. *The function u is continuous from $[0, \infty)$ to $L^{2,\infty}(D)$ in the weak* topology.*

PROOF. Since the predual $L^{2,1}(D)$ of $L^{2,\infty}(D)$ is a separable Banach space, the unit ball of $L^{2,\infty}(D)$ is sequentially compact in the weak* topology [22]. So, we need only show that

(4.5) $$u(t_m) \to u(t) \quad \text{weakly* in } L^{2,\infty}(D)^2$$

whenever $t_m \to t$ as $m \to \infty$. Since the sequence $u(t_m)$ is bounded in $L^{2,\infty}(D)$, there is a subsequence, denoted again $u(t_m)$, which converges weakly* to some function v. Since the weak* convergence implies the convergence in the distribution topology, we see that $(\nabla \times u)(t_m) \to \nabla \times v$ in the distribution topology. Hence, the vague continuity of $\nabla \times u$ ensured by (4.4) shows $\nabla \times v = (\nabla \times u)(t)$. On the other hand, the boundedness of $u(t_m)$ in $L^{2,\infty}(D)$ implies its boundednesss in $L^q(D)$, $1 < q < 2$. Since the smooth functions are dense in both the preduals of $L^{2,\infty}(D)$ and $L^q(D)$, it follows from the Banach-Steinhaus theorem that $u(t_m) \to v$ weakly in $L^q(D)$, $1 < q < 2$. Hence $v \in X_q$, $1 < q < 2$, and therefore $v \cdot \nu|_S = 0$. By Lemma 1.1, $v \in X_{2,\infty}$, and so we get $v = u(t)$ by Lemma 1.3. This proves (4.5).

We now show that the limit function u obtained above solves our original problem. First observe that estimates (4.1)–(4.3) for the approximation u_ε together imply

$$\|P(u_\varepsilon \cdot \nabla)u_\varepsilon\|_q \leqq C_q\|u_\varepsilon\|_{2q}\|\nabla u_\varepsilon\|_{2q} \leqq C_q\|\nabla \times a\|_1^2(\lambda t)^{1/q-3/2}$$

for all $1 < q < \infty$ with C_q independent of $t > 0$ and $\varepsilon > 0$. Thus, if we write

$$u_\varepsilon(t) = e^{-\lambda(t-\eta)A}u_\varepsilon(\eta) - \int_\eta^t e^{-\lambda(t-s)A}P(u_\varepsilon \cdot \nabla)u_\varepsilon(s)ds$$

for any fixed $0 < \eta < T/2$ and notice that $u_\varepsilon(\eta) \in X_q$ with norms bounded uniformly in $\varepsilon > 0$, then we can apply the standard arguments as given in [7, 9] to conclude that the functions $P(u_\varepsilon \cdot \nabla)u_\varepsilon$ are Hölder continuous from $[2\eta, T]$ to

X_q, $q > 2$, and the Hölder seminorms are bounded above uniformly in $\varepsilon > 0$. Hence, by [7, Lemma 2.14] we conclude that, for any $2 < q < \infty$,

$$\left\| \frac{du_\varepsilon}{dt} \right\|_q \text{ is bounded on } [2\eta, T] \text{ uniformly in } \varepsilon > 0.$$

This, together with (4.3) and the compact embedding $X_q \cap W^{1,q}(D)^2 \subset X_q$, implies the existence of a subsequence, again denoted u_ε, which converges as $\varepsilon \to 0$ to u in the topology of $C([2\eta, T]; X_q)$. Since

$$u_\varepsilon(t) = e^{-\lambda(t-2\eta)A} u_\varepsilon(2\eta) - \int_{2\eta}^t e^{-\lambda(t-s)A} P(u_\varepsilon \cdot \nabla) u_\varepsilon(s) ds, \tag{4.6}$$

multiplying both sides of (4.6) by an arbitrary test function $v \in X_{q'} \cap W^{1,q'}(D)^2$ with $1/q' = 1 - 1/q$ and then passing to the limit $\varepsilon \to 0$ yields

$$u(t) = e^{-\lambda(t-2\eta)A} u(2\eta) - \int_{2\eta}^t e^{-\lambda(t-s)A} P(u \cdot \nabla) u(s) ds \quad \text{in } X_q, \quad 2 < q < \infty, \tag{4.7}$$

for all $t \geqq 2\eta$. Using the expression

$$e^{-\lambda(t-s)A} P(u \cdot \nabla) u = A^{1/2} e^{-\lambda(t-s)A} A^{-1/2} P\nabla \cdot (u \otimes u)$$

and the boundedness of the operator $A^{-1/2} P\nabla \cdot$ in L^2, we can estimate the integral of (4.7) in X_2 as

$$\leqq C_\lambda \int_0^t (t-s)^{-1/2} \| u(s) \|_4^2 ds \leqq C_\lambda \int_0^t (t-s)^{-1/2} s^{-1/2} ds = C_\lambda.$$

Hence,

$$\int_{2\eta}^t e^{-\lambda(t-s)A} P(u \cdot \nabla) u(s) ds \to \int_0^t e^{-\lambda(t-s)A} P(u \cdot \nabla) u(s) ds \quad \text{as } \eta \to 0$$

in the weak topology of X_2. Since $e^{-\lambda(t-2\eta)A} u(2\eta) \to e^{-\lambda t A} a$ as $\eta \to 0$ weakly* in $L^{2,\infty}(D)^2$, we conclude that function u is a desired solution.

We finally show the uniqueness of our solutions, assuming that $\| \nabla \times a \|_1 / \lambda$ is small. Let u and v be two solutions with the same initial data a, and $w = u - v$. Since

$$w(t) = - \int_0^t A^{1/2} e^{-\lambda(t-s)A} A^{-1/2} P\nabla \cdot (w \otimes u + v \otimes w)(s) ds,$$

we obtain, for $2 < r < \infty$,

$$\| w \|_r(t) \leqq C \int_0^t [\lambda(t-s)]^{-1/2-1/r} (\| u \|_r + \| v \|_r \| w \|_r(s) ds.$$

This, together with estimate (4.1) for u and v, implies

$$|w|_{r,T} \equiv \sup_{0<t\leq T} (\lambda t)^{1/2-1/r} \|w\|_r(t) \leqq C_r \|\nabla \times a\|_1 |w|_{r,T}/\lambda$$

for all $T > 0$. Hence

$$|w|_{r,T} = 0 \quad \text{provided} \quad C_r \|\nabla \times a\|_1/\lambda < 1.$$

This completes the proof of Theorem 1.4.

5. Proof of Theorem 1.5

This section establishes Theorem 1.5 by applying the vanishing viscosity argument: $\lambda \to 0$ to the solutions of the Navier-Stokes equations (NS). Our arguments go back to Bardos [2] and our result extends [10. Corollary 2.7], which deal with the problem in the entire plane $\mathbb{R}^2$, to the case of simply connected bounded domains. Let u_λ denote the global solution to (NS) given in section 2, and let $\omega_\lambda = \nabla \times u_\lambda$. A standard argument using (3.1) and Lemma 1.3 together imply that if $2 < r < \infty$ and $1/q = 1/r + 1/2$, then

$$\|\omega_\lambda(t)\|_q \leqq \|\nabla \times a\|_q; \tag{5.1}$$

$$\|u_\lambda(t)\|_r \leqq C_r \|\nabla \times a\|_q; \tag{5.2}$$

and

$$\|\nabla u_\lambda(t)\|_s \leqq C_s \|\nabla \times a\|_s, \quad 1 < s < \infty, \tag{5.3}$$

provided $\nabla \times a \in L^s$. Let V^q denote the Banach space

$$V_q = X_q \cap W^{1,q}(D)^2$$

equipped with the usual $W^{1,q}$-topology, any let V_q^* be its dual space. Since A_q gives rise to a linear isomorphism between V_q and $V_{q'}^*$, $1/q' = 1 - 1/q$, and since

$$\begin{aligned}|(u_\lambda \otimes u_\lambda, \nabla\phi)| &\leqq \|u_\lambda\|_r \|u_\lambda\|_2 \|\nabla\phi\|_{q'} \\ &\leqq C\|u_\lambda\|_r^2 \|\nabla\phi\|_{q'},\end{aligned}$$

we see that $P(u_\lambda \cdot \nabla)u_\lambda$ are bounded in $L^\infty(0, T; V_{q'}^*)$. It thus follows that the time-derivatives u'_λ are bounded in $L^\infty(0, T; V_{q'}^*)$. Since

$$V_q \subset X_q \subset V_{q'}^*$$

with continuous injections and since the first inclusion is compact, we can apply the compactness theorem in [21, Chap.III] to extract a subsequence, denoted again u_λ, such that

$$u_\lambda \to u \quad \text{in} \quad L^s(0, T; X_q)$$

for any fixed $1 < s < \infty$, and

$$u_\lambda \to u \quad a.e. \text{ on } D \times (0, \infty). \tag{5.4}$$

By (5.1)–(5.3) we may assume also that the limit function u satisfies

$$u \in L^\infty(0, \infty; X_r) \cap L^\infty(0, \infty; V_q), \quad u' \in L^\infty(0, \infty; V_{q'}^*).$$

Now fix $\phi \in C^1(\bar{D})^2$ with $\nabla \cdot \phi = 0$ and $\phi \cdot \nu|_S = 0$. For any $T > 0$ and $\psi \in C^1([0, T]; \mathbb{R})$ with $\psi(T) = 0$ the functions u_λ satisify

$$-\int_0^T (u_\lambda, \phi)\psi' dt + \lambda \int_0^T (\nabla u_\lambda, \nabla \phi)\psi dt - \int_0^T (u_\lambda \otimes u_\lambda, \nabla \phi)\psi dt = (a, \phi)\psi(0), \tag{5.5}$$

where $\psi' = d\psi/dt$. Obviously, (5.2) and (5.3) imply

$$\lambda \int_0^T (\nabla u_\lambda, \nabla \phi)\psi dt \to 0 \tag{5.6}$$

and

$$\int_0^T (u_\lambda, \phi)\psi' dt \to \int_0^T (u, \phi)\psi' dt \tag{5.7}$$

as $\lambda \to 0$. It therefore remains to treat the nonlinear term of (5.5). By (5.4) and Egoroff's theorem there exists for each $\eta > 0$ a measurable set $E \subset D \times [0, T]$ so that

$$|E^c| < \eta \text{ and } u_\lambda \to u \textit{ uniformly on } E.$$

Hence we have

$$\iint_E (\mathrm{u}_\lambda \otimes u_\lambda) \cdot (\nabla \phi)\psi dx dt \to \iint_E (u \otimes u) \cdot (\nabla \phi)\psi dx dt. \tag{5.8}$$

On the other hand, since $\| u_\lambda \otimes u_\lambda \|_{r/2}$ are bounded in $L^\infty(0, T)$ and since the number $\eta > 0$ can be chosen arbitraly small, we get

$$\iint_{E^c} |u_\lambda \otimes u_\lambda| \cdot |\nabla \phi| \cdot |\psi| dx dt \leqq M \left[\iint_{E^c} (|\nabla \phi| \cdot |\psi|)^{r/(r-2)} dx dt \right]^{1-2/r}$$

and the right-hand side can be made arbitrarily small by the absolute countinuity of the Lebesgue integral. This, together with (5.8), implies

(5.9) $$\int_0^T (u_\lambda \otimes u_\lambda, \nabla\phi)\psi dxdt \to \int_0^T (u \otimes u, \nabla\phi)\psi dxdt$$

as $\lambda \to 0$. It thus follows from (5.5)–(5.7) and (5.9) that

$$\int_0^T (u, \phi)\psi' dt + \int_0^T (u \otimes u. \nabla\phi)\psi dt + (a, \phi)\psi(0) = 0$$

and part (iii) is proved. This shows in particular that

$$u' = -P(u \cdot \nabla)u \in L^\infty(0, T; V_{q'}^*).$$

Since $u \in L^\infty(0, T; V_q) \cap L^\infty(0, T; X_r)$ by (5.1) and (5.3) with $s = q$, and since $X_r \subset X_q \subset V_{q'}^*$ with continuous injections, it follows that u is continuous from $[0, T]$ to X_r in the weak topology, and in view of the compactness of the embedding $V_q \subset X_q$, that u is continuous from $[0, T]$ to X_q in the strong topology. This shows part (i). Now, part (i) implies $(\nabla \times u)(t_m) \to (\nabla \times u)(t)$ in $W^{-1,q}$-topology, whenever $t_m \to t$. Since $\|(\nabla \times u)(t)\|_q$ is bounded by (5.1), part (ii) is proved in a way similar to the proof of Lemma 4.1. Part (iv) is obtained from (5.3) and the Sobolev embedding

$$W^{1,s}(D) \subset C^{1-2/s}(\bar{D}), \quad 2 < s < \infty.$$

The proof is complete.

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