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Note on a Characterization of Solvable Lie Algebras

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By a Lie algebra we mean a finite-dimensional Lie algebra over a field of characteristic 0. Our main purpose in this note is to establish the following Theorem.

THEOREM.[1] *Let* $\mathfrak{g}$ *be a Lie algebra. If there exist nilpotent subalgebras* $\mathfrak{n}_1$ *and* $\mathfrak{n}_2$ *with* $\mathfrak{n}_1+\mathfrak{n}_2=\mathfrak{g}$, *then* $\mathfrak{g}$ *is solvable, and vice versa.*

The proof of the Theorem depends on the following Lemma.

LEMMA. *Let* $\mathfrak{s}$ *be a semi-simple Lie algebra, and let* $\mathfrak{n}$ *be a nilpotent subalgebra of* $\mathfrak{s}$. *Let* r *be the dimension of* $\mathfrak{s}$, *and let* l *be the rank of* $\mathfrak{s}$. *Then we have the following inequality*:

$$2 \dim \mathfrak{n} \leq r - l.$$

About the concept and results in the theory of Lie algebras, used in this note, the reader may refer: e.g. C. Chevalley, Théorie des groupes de Lie, t. III, Hermann, Paris, 1955.

(A) *Proof of the Lemma*

Let $\mathfrak{s}$ be a semi-simple Lie algebra over an algebraically closed field. Let $\mathfrak{n}$ be a nilpotent subalgebra of $\mathfrak{s}$. We use the notation $\mathrm{ad}(X)Y=[X, Y]$ for $X, Y \in \mathfrak{s}$. An element X is said to be *semi-simple* if $\mathrm{ad}(X)$ is a semi-simple linear transformation on $\mathfrak{s}$. Let A be a semi-simple element in $\mathfrak{n}$. Since $\mathrm{ad}(X)$, restricted on $\mathfrak{n}$, is nilpotent for any X in $\mathfrak{n}$, we have $\mathrm{ad}(A)=0$ on $\mathfrak{n}$, i.e. A is in the center of $\mathfrak{n}$. We denote by $\mathfrak{a}$ the set of all semi-simple elements in $\mathfrak{n}$. Then $\mathfrak{a}$ is a central ideal of $\mathfrak{n}$.

Let $\mathfrak{m}$ be a maximal solvable subalgebra of $\mathfrak{s}$ containing $\mathfrak{n}$. We can find a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{s}$ such that

$$\mathfrak{m} \supset \mathfrak{h} \supset \mathfrak{a}.$$

Since any element of $\mathfrak{h}$ is semi-simple, we have $\mathfrak{n} \cap \mathfrak{h} = \mathfrak{a}$. Next, because $\mathfrak{a}$ is a commutative subalgebra composed of semi-simple elements, we obtain a decomposition of $\mathfrak{m}$ into submoduli:

$$\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2,\ \mathfrak{m}_1 \cap \mathfrak{m}_2 = \{0\},$$

$$[\mathfrak{a}, \mathfrak{m}_1] = \{0\}, \text{ and } [\mathfrak{a}, \mathfrak{m}_2] = \mathfrak{m}_2.$$

1) It seems that the question whether the theorem is true or not had been raised originally by Otto H. Kegel and the author was inquired about this by S. Tôgô.

It is to be noted that both $\mathfrak{h}$ and $\mathfrak{n}$ are subalgebras of $\mathfrak{m}_1$, the centralizer of $\mathfrak{a}$ in $\mathfrak{m}$.

Let l be the rank of $\mathfrak{S}$. Then $\mathfrak{h}$ is of dimension l. We can find a linearly independent system of l linear forms (root vectors) $\alpha_1, \alpha_2 \cdots, \alpha_l$ on $\mathfrak{h}$, and a linearly independent system of l elements $E_1, E_2, \cdots, E_l$ in $\mathfrak{m}$, with

$$[H, E_i] = \alpha_i(H)E_i \quad \text{for } H \in \mathfrak{h},\ i = 1, 2, \cdots, l.$$

Let k be the dimension of $\mathfrak{a}$. Since $\mathfrak{n} + \mathfrak{h} \subset \mathfrak{m}_1$ and $\mathfrak{n} \cap \mathfrak{h} = \mathfrak{a}$, we have

$$\dim \mathfrak{m}_1 \geq \dim \mathfrak{n} + \dim \mathfrak{h} - \dim \mathfrak{a},$$

i.e.

$$\dim \mathfrak{m}_1 - \dim \mathfrak{n} \geq l - k. \tag{1}$$

On the other hand, since $\alpha_1, \alpha_2, \cdots, \alpha_l$ is linearly independent, there can be at most $l-k$ forms α_i with $\alpha_i(A) = 0$ for all $A \in \mathfrak{a}$, and if $\alpha_i(A) \neq 0$ for some $A \in \mathfrak{a}$ then $E_i \in \mathfrak{m}_2$. Thus we have

$$\dim \mathfrak{m}_2 \geq k. \tag{2}$$

Adding (1) and (2), we obtain

$$\dim \mathfrak{m} - \dim \mathfrak{n} \geq l.$$

Since $\dim \mathfrak{S} = 2 \dim \mathfrak{m} - l$, we have the Lemma in the special case when the coefficient field is algebraically closed.

Next, we shall consider the general case. Let $\mathfrak{S}$ be a semi-simple Lie algebra and $\mathfrak{n}$ a nilpotent subalgebra of $\mathfrak{S}$. Let us extend the coefficient field F into the algebraic closure $\bar{F}$ and construct the tensor product $\bar{\mathfrak{S}} = \mathfrak{S} \otimes \bar{F}$. $\bar{\mathfrak{S}}$ is regarded as a (semi-simple) Lie algebra over $\bar{F}$, and the dimension (rank) of $\bar{\mathfrak{S}}$ over $\bar{F}$ is equal to the dimension (rank) of $\mathfrak{S}$. $\bar{\mathfrak{n}} = \mathfrak{n} \otimes \bar{F}$ is naturally imbedded in $\bar{\mathfrak{S}}$. Since $\mathfrak{n}$ is nilpotent, so is $\bar{\mathfrak{n}}$. Hence $2 \dim \mathfrak{n} = 2 \dim \bar{\mathfrak{n}} \leq \dim \bar{\mathfrak{S}} - \operatorname{rank} \bar{\mathfrak{S}} = \dim \mathfrak{S} - \operatorname{rank} \mathfrak{S}$.

(B) *The Lemma implies the Theorem*

Let $\mathfrak{g}$ be a Lie algebra which has nilpotent subalgebras $\mathfrak{n}_1$ and $\mathfrak{n}_2$ satisfying $\mathfrak{n}_1 + \mathfrak{n}_2 = \mathfrak{g}$. Suppose that $\mathfrak{g}$ is not solvable. Let $\mathfrak{r}$ be the radical of $\mathfrak{g}$, and let φ be the natural projection from $\mathfrak{g}$ onto the factor algebra $\mathfrak{g}/\mathfrak{r} = \mathfrak{S}$. Then we have $\varphi(\mathfrak{n}_1) + \varphi(\mathfrak{n}_2) = \mathfrak{S}$, where $\varphi(\mathfrak{n}_i)$'s are nilpotent subalgebras of $\mathfrak{S}$. Hence the Lemma implies that $\dim \varphi(\mathfrak{n}_i) < 1/2 \dim \mathfrak{S}\ (i = 1, 2)$, which gives a contradiction. Therefore $\mathfrak{g}$ must be solvable.

Next, let us prove that any solvable Lie algebra can be written as a module sum of two nilpotent subalgebras. Let $\mathfrak{g}$ be a Lie algebra, and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then $\mathfrak{h}$ is nilpotent, and we have $\mathfrak{g} = \mathfrak{h} + \mathfrak{g}'$, where $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is the commutator subalgebra of $\mathfrak{g}$. On the other hand, the commutator subalgebra of a solvable Lie algebra is nilpotent by a Theorem of Lie.

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