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The Potential Force Yielding a Periodic Motion whose Period is an Arbitrary Continuously Differentiable Function of the Amplitude

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1. Introduction

In the previous paper [2], the author has given a method to determine the potential force $g(x)$ so that the period of the periodic solution of the equation

$$\frac{d^2x}{dt^2}+g(x)=0 \tag{1.1}$$

may be an arbitrary given continuous function of the amplitude of the velocity.

In the present paper, first, we shall give a method to determine $g(x)$ so that the period of the periodic solution of (1.1) may be an arbitrary given continuously differentiable function $\omega_1(a)$ of the positive maximum displacement a of x whose derivative $\omega_1'(a)$ with respect to a satisfies the Lipschitz condition. Our method is based on solution of a certain integral equation to which the problem is reduced by the techniques used in the previous paper [2].

Next there will be given a method to determine the desired potential force $g(x)$, namely $g(x)$ such that the period of the periodic solution of (1.1) may be an arbitrary given continuously differentiable function $\omega(A)$ of the amplitude A whose derivative $\omega'(A)$ with respect to A satisfies the Lipschitz condition. By the same techniques as in the first problem, the present problem is reduced to solution of an integral equation which is of a particular type of the integral equation solved already in the first problem.

Lastly, in illustration of our method, there will be given a potential force $g(x)$ such that the period ω of the periodic solution of (1.1) is a linear function of the amplitude A.

Since the work of the present paper is based on the main theorem in the previous paper [2], it is restated here for the convenience of the readers.

Theorem 0. *In case $g(x)$ is continuous in the neighborhood of $x=0$ and differentiable at $x=0$, if any solution of the equation* (1.1) *near $x=\dot{x}=0$ $(\cdot = d/dt)$ oscillates around $x=\dot{x}=0$ with a bounded period, then*

1° *the period* ω ($\geqq 0$) *is expressed as*

$$\omega = \Omega(R), \tag{1.2}$$

where R *is the maximum velocity (namely the amplitude of the velocity)*

$$R = \left| \left[\frac{dx}{dt} \right]_{x=0} \right| \tag{1.3}$$

and

$$\Omega(R) \in C_R, \quad \Omega(0) = \omega_0 > 0; \tag{1.4}$$

2° *the function* $g(x)$ *satisfies the functional equation*

$$g\left\{ \frac{\omega_0}{2\pi} \int_0^X [1 + S(u) + T(u)] du \right\} = \frac{2\pi}{\omega_0} \cdot \frac{X}{1 + S(X) + T(X)}, \tag{1.5}$$

where $S(X)$ *is a continuous odd function and* $T(X)$ *is a continuous even function such that* $T(0) = 0$ *and*

$$T(X) = \frac{1}{\omega_0} \cdot \frac{d}{dX} \int_0^X \frac{\Omega(R) - \omega_0}{\sqrt{X^2 - R^2}} R dR \quad \text{for} \quad X > 0. \tag{1.6}$$

Conversely, given any function $\Omega(R)$ *for which* (1.4) *holds, if the even function* $T(X)$ *defined by* (1.6) *is continuous, then the function* $g(x)$ *which is determined by the functional equation* (1.5) *for an arbitrary continuous odd function* $S(X)$ *and for the continuous even function* $T(X)$ *defined by* (1.6), *is continuous in the neighborhood of* $x = 0$ *and is differentiable at* $x = 0$. *Furthermore, for this* $g(x)$, *any solution of the equation* (1.1) *near* $x = \dot{x} = 0$ *oscillates around* $x = \dot{x} = 0$ *with the period* (1.2) *for* R *given by* (1.3).

In case $\Omega(R) \in C_R^1$ *for* $R \geqq 0$, *the relation* (1.6) *can be replaced by*

$$T(X) = \frac{1}{\omega_0} X \int_0^{\pi/2} \Omega'(X \cos \varphi) d\varphi \quad \text{for} \quad X \geqq 0, \tag{1.7}$$

whose right member is continuous. Consequently, for any given $\Omega(R) \in C_R^1$ *with* $\Omega(0) = \omega_0 > 0$, *there always exists a continuous potential force* $g(x)$ *which is differentiable at* $x = 0$ *and for which any solution of the equation* (1.1) *near* $x = \dot{x} = 0$ *oscillates around* $x = \dot{x} = 0$ *with the period* (1.2) *for* R *given by* (1.3).

2. A Lemma for an integral equation

As is stated in the preceding paragraph, the problems in question are reduced to solution of a certain integral equation. In this paragraph, a lemma for asserting the existence and uniqueness of the solution of such an integral equation is proved.

LEMMA. *Given an integral equation*

$$(2.1)\quad T(X)=\frac{X}{2\pi}\int_0^X \frac{1+S(R)+T(R)}{\sqrt{X^2-R^2}}\,F\left[\frac{\omega_0}{2\pi}\int_0^R \{1+S(u)+T(u)\}\,du\right]dR \qquad (\omega_0>0),$$

where $F(a)$ is a given function defined on $I[0, l]$ $(l>0)$ satisfying the Lipschitz condition:

$$(2.2)\qquad |F(a')-F(a'')|\leqq L\,|a'-a''| \quad \text{for} \quad {}^\forall a', a''\in I \quad (L>0)$$

and $S(X)$ is a given continuous function defined on $J[0, \alpha]$ $(\alpha>0)$ satisfying the inequality

$$(2.3)\qquad |S(X)|\leqq K \quad \text{for} \quad {}^\forall X\in J \qquad (0\leqq K<1).$$

Let

$$(2.4)\qquad M=\max_{a\in I}|F(a)|.$$

Then the integral equation (2.1) *has, on $J_0[0, \alpha_0]$ $(\alpha_0\leqq\alpha)$, one and only one continuous solution $T(X)$ such that*

$$(2.5)\qquad |T(X)|\leqq K_0$$

for any $X\in J_0$, where

$$(2.6)\quad \alpha_0=\max_{0\leqq\kappa\leqq1-K}\min\left[\alpha, \frac{2\pi l}{\omega_0(1+K+\kappa)}, \frac{4\kappa}{M(1+K+\kappa)}, \frac{\pi^2 M}{2L\omega_0(1+K+\kappa)}\left\{\sqrt{1+\frac{16kL\omega_0(1+K+\kappa)}{\pi^2M^2}}-1\right\}\right]$$

k being a positive constant less than unity and K_0 is any value of non-negative $\kappa\leqq 1-K$ for which the minimum of the right member of (2.6) *equals α_0.*

Proof. Let us consider the iteration process on J_0 as follows:

$$(2.7)\quad \begin{cases} T_0(X)\equiv 0, \\ T_{n+1}(X)=\dfrac{X}{2\pi}\displaystyle\int_0^X \frac{1+S(R)+T_n(R)}{\sqrt{X^2-R^2}}\,F\left[\frac{\omega_0}{2\pi}\int_0^R\{1+S(u)+T_n(u)\}\,du\right]dR \end{cases} \qquad (n=0, 1, 2,\cdots).$$

First, we prove that this iteration process is actually possible indefinitely on J_0 and that

$$(2.8)\qquad T_n(X)\in C[J_0], \quad |T_n(X)|\leqq K_0 \qquad (n=0, 1, 2,\cdots).$$

The condition (2.8) is evidently valid for $n=0$.

Let us assume that the iteration process is actually possible up to the m-th step and that (2.8) is valid for $n=m$.

Then, by (2.3), (2.8) and (2.6), it holds for $R \in J_0$ that

$$0 \leqq \frac{\omega_0}{2\pi} \int_0^R \{1 + S(u) + T_m(u)\}\, du \leqq \frac{\omega_0}{2\pi}(1 + K + K_0)\alpha_0 \leqq l.$$

Therefore, by (2.4),

$$\left| F\left[\frac{\omega_0}{2\pi} \int_0^R \{1 + S(u) + T_m(u)\}\, du\right] \right| \leqq M$$

for any $R \in J_0$. Then, by (2.7) and (2.6), we see that, for any $X \in J_0$,

$$|T_{m+1}(X)| \leqq \frac{\alpha_0}{2\pi} \cdot M(1 + K + K_0) \int_0^X \frac{dR}{\sqrt{X^2 - R^2}} \leqq K_0.$$

This proves the latter of (2.8) for $n = m+1$.

Since $T_m(X) \in C[J_0]$ by our assumption, it is readily seen by the substitution $R = X \cos \varphi$ that

$$T_{m+1}(X) \in C[J_0].$$

This proves the former of (2.8) for $n = m+1$.

Thus, by the induction, it follows readily that the iteration process is actually possible indefinitely on J_0 and (2.8) holds for any non-negative integer n.

Secondly, let us prove the uniform convergence of the sequence $\{T_n(X)\}$ $(n = 0, 1, 2, \dots)$ on J_0.

The difference $T_{n+1}(X) - T_n(X)$ can be rewritten as follows:

$$\begin{aligned}
&T_{n+1}(X) - T_n(X) \\
&= \frac{X}{2\pi} \int_0^X \left[\frac{1 + S(R) + T_n(R)}{\sqrt{X^2 - R^2}} - \frac{1 + S(R) + T_{n-1}(R)}{\sqrt{X^2 - R^2}}\right] \\
&\qquad\qquad\qquad \times F\left[\frac{\omega_0}{2\pi} \int_0^R \{1 + S(u) + T_n(u)\}\, du\right] dR \\
&+ \frac{X}{2\pi} \int_0^X \frac{1 + S(R) + T_{n-1}(R)}{\sqrt{X^2 - R^2}} \\
&\qquad \times \left[F\left\{\frac{\omega_0}{2\pi} \int_0^R (1 + S(u) + T_n(u))du\right\} - F\left\{\frac{\omega_0}{2\pi} \int_0^R (1 + S(u) + T_{n-1}(u))du\right\}\right] dR.
\end{aligned}$$

Therefore, by (2.4), (2.2), (2.3) and (2.8), it holds for any $X \in J_0$ that

$$\begin{aligned}
(2.9) \qquad & |T_{n+1}(X) - T_n(X)| \\
& \leqq \left[\frac{M}{4}\alpha_0 + \frac{L\omega_0(1 + K + K_0)}{4\pi^2}\alpha_0^2\right] \cdot \max_{X \in J_0} |T_n(X) - T_{n-1}(X)|.
\end{aligned}$$

But, by (2.6),

$$\frac{L\omega_0(1+K+K_0)}{4\pi^2}\alpha_0^2+\frac{M}{4}\alpha_0\leqq k.$$

Therefore, from (2.9), it readily follows that

$$\max_{X\in J_0}|T_{n+1}(X)-T_n(X)|\leqq k\cdot\max_{X\in J_0}|T_n(X)-T_{n-1}(X)| \qquad (n=1,\,2,\,3,\dots). \tag{2.10}$$

Since $|k|<1$, (2.10) implies the uniform convergence of $\{T_n(X)\}$ $(n=0,\,1,\,2,\dots)$ on J_0.

Lastly, let us prove the existence and uniqueness of the continuous solution of the given integral equation (2.1).

By the uniform convergence of $\{T_n(X)\}$ $(n=0,\,1,\,2,\dots)$,

$$\tilde{T}(X)=\lim_{n\to\infty}T_n(X) \tag{2.11}$$

exists and, from (2.8),

$$\tilde{T}(X)\in C[J_0],\quad |\tilde{T}(X)|\leqq K_0. \tag{2.12}$$

Then

$$\frac{X}{2\pi}\int_0^X\frac{1+S(R)+\tilde{T}(R)}{\sqrt{X^2-R^2}}F\Big[\frac{\omega_0}{2\pi}\int_0^R\{1+S(u)+\tilde{T}(u)\}\,du\Big]dR$$

exists and its difference with $\tilde{T}(X)$ can be written by means of (2.7) as follows:

$$\frac{X}{2\pi}\int_0^X\frac{1+S(R)+\tilde{T}(R)}{\sqrt{X^2-R^2}}F\Big[\frac{\omega_0}{2\pi}\int_0^R\{1+S(u)+\tilde{T}(u)\}\,du\Big]dR-\tilde{T}(X)$$

$$=\Big[\frac{X}{2\pi}\int_0^X\frac{1+S(R)+\tilde{T}(R)}{\sqrt{X^2-R^2}}F\Big[\frac{\omega_0}{2\pi}\int_0^R\{1+S(u)+\tilde{T}(u)\}\,du\Big]dR$$

$$-\frac{X}{2\pi}\int_0^X\frac{1+S(R)+T_n(R)}{\sqrt{X^2-R^2}}F\Big[\frac{\omega_0}{2\pi}\int_0^R\{1+S(u)+T_n(u)\}\,du\Big]dR\Big]$$

$$+[T_{n+1}(X)-\tilde{T}(X)].$$

But the first difference in the right member can be estimated quite similarly as $T_{n+1}(X)-T_n(X)$. Thus we have

$$\Big|\frac{X}{2\pi}\int_0^X\frac{1+S(R)+\tilde{T}(R)}{\sqrt{X^2-R^2}}F\Big[\frac{\omega_0}{2\pi}\int_0^R\{1+S(u)+\tilde{T}(u)\}\,du\Big]dR-\tilde{T}(X)\Big|$$

$$\leqq k\cdot\max_{X\in J_0}|\tilde{T}(X)-T_n(X)|+|T_{n+1}(X)-\tilde{T}(X)|.$$

This implies $\tilde{T}(X)$ is a solution of (2.1), because the right member tends to zero as $n\to\infty$ since the sequence $\{T_n(X)\}$ $(n=0,\,1,\,2,\dots)$ converges to $\tilde{T}(X)$ uniformly on J_0. Then, from (2.12), readily follows the existence of a continuous solution of (2.1) on J_0 which satisfies the inequality (2.5).

Let us prove the uniqueness of the solution satisfying (2.5). Let $\hat{T}(X)$ be another continuous solution of (2.1) satisfying (2.5). Then, quite similarly as $T_{n+1}(X)-T_n(X)$, we have

$$|\tilde{T}(X)-\hat{T}(X)|\leqq k\cdot\max_{X\in J_0}|\tilde{T}(X)-\hat{T}(X)|,$$

from which readily follows

$$(1-k)\cdot\max_{X\in J_0}|\tilde{T}(X)-\hat{T}(X)|\leqq 0.$$

Since $1-k>0$, the above inequality implies

$$\tilde{T}(X)=\hat{T}(X),$$

which proves the uniqueness of the solution satisfying (2.5).

REMARK 1. Let K_1 and K_2 be respectively the least and the greatest value of non-negative $\kappa\leqq 1-K$ for which the minimum of the right member of (2.6) equals α_0. Then the Lemma implies

(i) the existence of a solution $T(X)$ such that

$$T(X)\in C[J_0],\quad |T(X)|\leqq K_1,$$

(ii) the uniqueness of a solution $T(X)$ such that

$$T(X)\in C[J_0],\quad |T(X)|\leqq K_2.$$

REMARK 2. As is seen from the proof of the Lemma, the solution of the integral equation (2.1), if wanted, can be obtained approximately by the iteration method starting from $T(X)=0$.

3. The first problem

The answer to the first problem is given by

THEOREM 1. *In case $g(x)$ is continuous in the neighborhood of $x=0$ and differentiable at $x=0$, if any solution of the equation* (1.1) *near $x=\dot{x}=0$ oscillates around $x=\dot{x}=0$ with a bounded period, then*

1° *the period $\omega(\geqq 0)$ is expressed as*

$$\omega=\omega_1(a), \tag{3.1}$$

where a is the positive maximum displacement of x and

$$\omega_1(a)\in C_a,\quad \omega_1(0)=\omega_0>0; \tag{3.2}$$

2° *the function $g(x)$ satisfies the functional equation*

$$g\left\{\frac{\omega_0}{2\pi}\int_0^X [1+S(u)+T(u)]du\right\} = \frac{2\pi}{\omega_0}\cdot\frac{X}{1+S(X)+T(X)}, \tag{3.3}$$

where $S(X)$ is a continuous odd function and $T(X)$ is a continuous even function such that $T(0)=0$ and

$$T(X) = \frac{1}{\omega_0}\cdot\frac{d}{dX}\int_0^X \frac{\omega_1\left[\frac{\omega_0}{2\pi}\int_0^R (1+S(u)+T(u))du\right]-\omega_0}{\sqrt{X^2-R^2}} RdR \tag{3.4}$$

for $X>0$;

3° *if $\omega_1(a) \in C_a^1$, the equation (3.4) can be written as follows:*

$$T(X) = \frac{X}{2\pi}\int_0^X \frac{1+S(R)+T(R)}{\sqrt{X^2-R^2}}\,\omega_1'\left[\frac{\omega_0}{2\pi}\int_0^R \{1+S(u)+T(u)\}\,du\right]dR. \tag{3.5}$$

Conversely, given any function $\omega_1(a) \in C_a^1$ whose derivative satisfies the Lipschitz condition, there exists a continuous potential force $g(x)$ which is differentiable at $x=0$ and for which any solution of the equation (1.1) *near $x=\dot{x}=0$ oscillates around $x=\dot{x}=0$ with the period* (3.1). *The function $g(x)$ is determined by the functional equation* (3.3) *for an arbitrary continuous odd function $S(X)$ and for the continuous even function $T(X)$ determined by the unique solution of the integral equation* (3.5).

Proof. If we write the equation (1.1) in a simultaneous form as

$$\begin{cases} \dfrac{dx}{dt} = y, \\ \dfrac{dy}{dt} = -g(x), \end{cases} \tag{3.6}$$

then the periodic solutions of (1.1) are represented by the closed orbits

$$\frac{1}{2}y^2 + G(x) = \text{const.} \tag{3.7}$$

in the phase plane, where

$$G(x) = \int_0^x g(u)du.$$

Hence the positive maximum displacement a of x is connected with the maximum velocity R (the amplitude of the velocity) as follows:

$$\frac{1}{2}R^2 = G(a). \tag{3.8}$$

Let us transform a to ξ by

$$a = \frac{\omega_0}{2\pi}\int_0^\xi [1+S(u)+T(u)]du, \tag{3.9}$$

where $S(X)$ and $T(X)$ are the functions determined from $g(x)$ in the way stated in Theorem 0. Then, from (3.8), readily follows

$$R = g(a) \cdot \frac{da}{dR}.$$

This can be written by (1.5) and (3.9) as follows:

$$R = \frac{2\pi}{\omega_0} \cdot \frac{\xi}{1+S(\xi)+T(\xi)} \cdot \frac{da}{d\xi} \cdot \frac{d\xi}{dR}$$
$$= \xi \frac{d\xi}{dR},$$

which implies

$$R^2 - \xi^2 = \text{const.}$$

But, from (3.8) and (3.9), $\xi=0$ implies $R=0$. Therefore

$$\xi^2 = R^2. \tag{3.10}$$

But ξ and R are of the same sign provided ξ is sufficiently small, because $S(0) = T(0) = 0$. Then, by (3.10), we have

$$\xi = R,$$

from which, by (3.9), follows

$$a = \frac{\omega_0}{2\pi} \int_0^R [1 + S(u) + T(u)] du. \tag{3.11}$$

Then, comparing (1.2) with (3.1), we have

$$\omega = \Omega(R) = \omega_1 \left[\frac{\omega_0}{2\pi} \int_0^R \left\{ 1 + S(u) + T(u) \right\} du \right],$$

by which, from (1.4) and (1.6), follows (3.2) and (3.4) respectively.

The functional equation (3.3) is nothing but (1.5) itself.

The equation (3.5) is easily derived by integration by parts from (3.4).

The converse part of the theorem is evident from Theorem 0 and the Lemma in §2.

4. The second problem

The answer to the second problem is given by

THEOREM 2. *In case $g(x)$ is continuous in the neighborhood of $x=0$ and differentible at $x=0$, if any solution of the equation* (1.1) *near $x=\dot{x}=0$ oscillates around $x=\dot{x}=0$ with a bounded period, then*

1° *the period* ω ($\geqq 0$) *is expressed as*

$$\omega = \omega(A), \tag{4.1}$$

where A *is the amplitude and*

$$\omega(A) \in C_A, \quad \omega(0) = \omega_0 > 0; \tag{4.2}$$

2° *the function* $g(x)$ *satisfies the functional equation*

$$g\left\{\frac{\omega_0}{2\pi}\int_0^X [1 + S(u) + T(u)]du\right\} = \frac{2\pi}{\omega_0}\cdot\frac{X}{1+S(X)+T(X)}, \tag{4.3}$$

where $S(X)$ *is a continuous odd function and* $T(X)$ *is a continuous even function such that* $T(0)=0$ *and*

$$T(X) = \frac{1}{\omega_0}\cdot\frac{d}{dX}\int_0^X \frac{\omega\left[\frac{\omega_0}{2\pi}\int_0^R \left\{1+T(u)\right\}du\right] - \omega_0}{\sqrt{X^2-R^2}}\cdot RdR \quad \textit{for} \quad X>0; \tag{4.4}$$

3° *if* $\omega(A) \in C_A^1$, *the equation* (4.4) *can be written as follows*:

$$T(X) = \frac{X}{2\pi}\int_0^X \frac{1+T(R)}{\sqrt{X^2-R^2}}\,\omega'\left[\frac{\omega_0}{2\pi}\int_0^R \{1+T(u)\}\,du\right]dR. \tag{4.5}$$

Conversely, given any function $\omega(A) \in C_A^1$ *whose derivative satisfies the Lipschitz condition, there exists a continuous potential force* $g(x)$ *which is differentiable at* $x=0$ *and for which any solution of the equation* (1.1) *near* $x=\dot{x}=0$ *oscillates around* $x=\dot{x}=0$ *with the period* (4.1). *The function* $g(x)$ *is determined by the functional equation* (4.3) *for an arbitrary continuous odd function* $S(X)$ *and for the continuous even function* $T(X)$ *determined by the unique solution of the integral equation* (4.5).

Proof. Let b be the negative minimum displacement of x. Then, by (3.7), b is connected with the maximum velocity R as follows:

$$\frac{1}{2}R^2 = G(b). \tag{4.6}$$

Then, quite similarly as (3.11), we have

$$\begin{aligned} b &= \frac{\omega_0}{2\pi}\int_0^{-R}[1 + S(u) + T(u)]du \\ &= \frac{\omega_0}{2\pi}\int_0^{R}[-1 + S(u) - T(u)]du. \end{aligned} \tag{4.7}$$

Since the amplitude A is one half of the maximum displacement minus the minimum one, from (3.11) and (4.7), we have

$$A = \frac{\omega_0}{2\pi}\int_0^R [1 + T(u)]du. \tag{4.8}$$

Then, comparing (1.2) with (4.1), we have

$$\omega = \Omega(R) = \omega\left[\frac{\omega_0}{2\pi}\int_0^R \{1+T(u)\}\,du\right],$$

from which, quite similarly as the proof of Theorem 1, follows (4.4) and (4.5).

The rest of the theorem is literally same as Theorem 1.

REMARK. The integral equations (4.4) and (4.5) are respectively the particular ones of (3.4) and (3.5) where $S(X)\equiv 0$.

5. The example where $\omega(A)=\omega_0+cA$

In the present example, $\omega'(A)=c$, consequently the integral equation (4.5) becomes

$$T(X)=\frac{c}{2\pi}X\int_0^X \frac{1+T(R)}{\sqrt{X^2-R^2}}\,dR.$$

This can be rewritten as follows:

$$T(X)=\frac{c}{4}X+\frac{c}{2\pi}X\int_0^X \frac{T(R)}{\sqrt{X^2-R^2}}\,dR. \tag{5.1}$$

From the proof of the Lemma in §2, it is readily seen that the solution $\hat{T}(X)$ of (5.1) can be expanded into power series of X as follows:

$$\hat{T}(X)=\frac{c}{4}X\sum_{n=0}^{\infty}\alpha_n X^n. \tag{5.2}$$

Put

$$I_n=\frac{1}{X^n}\int_0^X \frac{R^n}{\sqrt{X^2-R^2}}\,dR \qquad (X>0;\ n=0,\,1,\,2,\dots), \tag{5.3}$$

then it is readily seen that

$$\begin{cases} I_0=\dfrac{\pi}{2}, \quad I_1=1, \\[2mm] I_n=\dfrac{n-1}{n}I_{n-2} \qquad (n=2,\,3,\,4,\dots), \end{cases}$$

from which follows

$$I_n=\begin{cases} \dfrac{1\cdot3\cdot5\cdot\dots\cdot(2m-1)}{2\cdot4\cdot6\cdot\dots\cdot2m}\cdot\dfrac{\pi}{2} & \text{for}\quad n=2m, \\[3mm] \dfrac{2\cdot4\cdot6\cdot\dots\cdot2m}{1\cdot3\cdot5\cdot\dots\cdot(2m-1)}\cdot\dfrac{1}{2m+1} & \text{for}\quad n=2m+1 \end{cases} \tag{5.4}$$

$$(m=0,\,1,\,2,\dots).$$

Now, if we substitute (5.2) into (5.1), then, by means of (5.3), we have

$$\begin{cases} \alpha_0 = 1, \\ \alpha_n = \dfrac{c}{2\pi} I_n \alpha_{n-1} \qquad (n = 1, 2, 3, \dots), \end{cases}$$

from which, by means of (5.4), we have

$$\alpha_n = \begin{cases} \dfrac{1}{m!}\left(\dfrac{c}{4}\right)^{2m} \cdot \dfrac{1}{\pi^m} & \text{for } \; n = 2m, \\ \dfrac{1}{1\cdot 3\cdot 5\cdot \dots \cdot (2m+1)}\left(\dfrac{c}{4}\right)^{2m+1}\left(\dfrac{2}{\pi}\right)^{m+1} & \text{for } \; n = 2m+1 \end{cases} \tag{5.5}$$

$$(m = 0, 1, 2, \dots).$$

Substituting (5.5) into (5.2), we see that

$$\begin{aligned} \tilde{T}(X) &= \frac{c}{4} X \Big[\sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{c^2}{16\pi} X^2\right)^m \\ &\qquad + \frac{\sqrt{2}}{\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{1}{1\cdot 3\cdot 5\cdot \dots \cdot (2m+1)}\left(\frac{c}{2\sqrt{2\pi}} X\right)^{2m+1} \Big] \\ &= \frac{c}{4} X \exp\left(\frac{c^2}{16\pi} X^2\right) + \frac{c}{2\sqrt{2\pi}} X \varphi\left(\frac{c}{2\sqrt{2\pi}} X\right), \end{aligned} \tag{5.6}$$

where

$$\begin{aligned} \varphi(Z) &= e^{\frac{1}{2}Z^2} \int_0^Z e^{-\frac{1}{2}u^2} du \\ &= Z \sum_{p=0}^{\infty} \left\{ \frac{1}{2^p \cdot p!} \left(\sum_{n=0}^{p} \binom{p}{n} \frac{(-1)^n}{2n+1} \right) Z^{2p} \right\} \\ &= Z \sum_{p=0}^{\infty} \left\{ \frac{1}{2^p \cdot p!} \int_0^1 (1-s^2)^p ds \cdot Z^{2p} \right\} \\ &= \sum_{p=0}^{\infty} \frac{1}{1\cdot 3\cdot 5\cdot \dots (2p+1)} Z^{2p+1}. \end{aligned} \tag{5.7}$$

Since the solution of the equation (5.1) is unique by the Lemma of §2, the function $\tilde{T}(X)$ given by (5.6) is the unique solution of (5.1).

The even function $T(X)$ such that $T(0)=0$ and

$$T(X) = \tilde{T}(X) \quad \text{for} \quad X > 0,$$

is evidently given by

$$T(X) = \frac{c}{4} |X| \exp\left(\frac{c^2}{16\pi} X^2\right) + \frac{c}{2\sqrt{2\pi}} X \varphi\left(\frac{c}{2\sqrt{2\pi}} X\right). \tag{5.8}$$

Let us put

$$U(X)=\int_0^X T(u)du, \tag{5.9}$$

then, after elementary calculations, we find

$$U(X)=\frac{2\pi}{c}\cdot\frac{|X|}{X}\left[\exp\left(\frac{c^2}{16\pi}X^2\right)-1\right] + \frac{2\sqrt{2\pi}}{c}\varphi\left(\frac{c}{2\sqrt{2\pi}}X\right)-X, \tag{5.10}$$

where we suppose $\lim_{c\to 0} U(X)$ is taken for $[U(X)]_{c=0}=0$.

Thus, by Theorem 2, the desired potential force $g(x)$ yielding periodic motions with the period $\omega=\omega_0+cA$ is given in the neighborhood of $x=0$ by

$$\begin{cases} g(x)=\dfrac{2\pi}{\omega_0}\cdot\dfrac{X}{1+S(X)+T(X)}, \\ x=\dfrac{\omega_0}{2\pi}\left[X+\displaystyle\int_0^X S(u)du+U(X)\right], \end{cases} \tag{5.11}$$

where $T(X)$ and $U(X)$ are the functions given by (5.8) and (5.10) respectively and $S(X)$ is an arbitrary continuous odd function.

The graphs of $g(x)$ for which $\omega_0=2\pi$ and $S(X)=\alpha X$ are shown in Figs. 1–5 for $c=\pm 1$, ± 5 and $\alpha=0$, ± 0.5, ± 1.0. The computation has been carried out by HIPAC 103 installed at Hiroshima University.

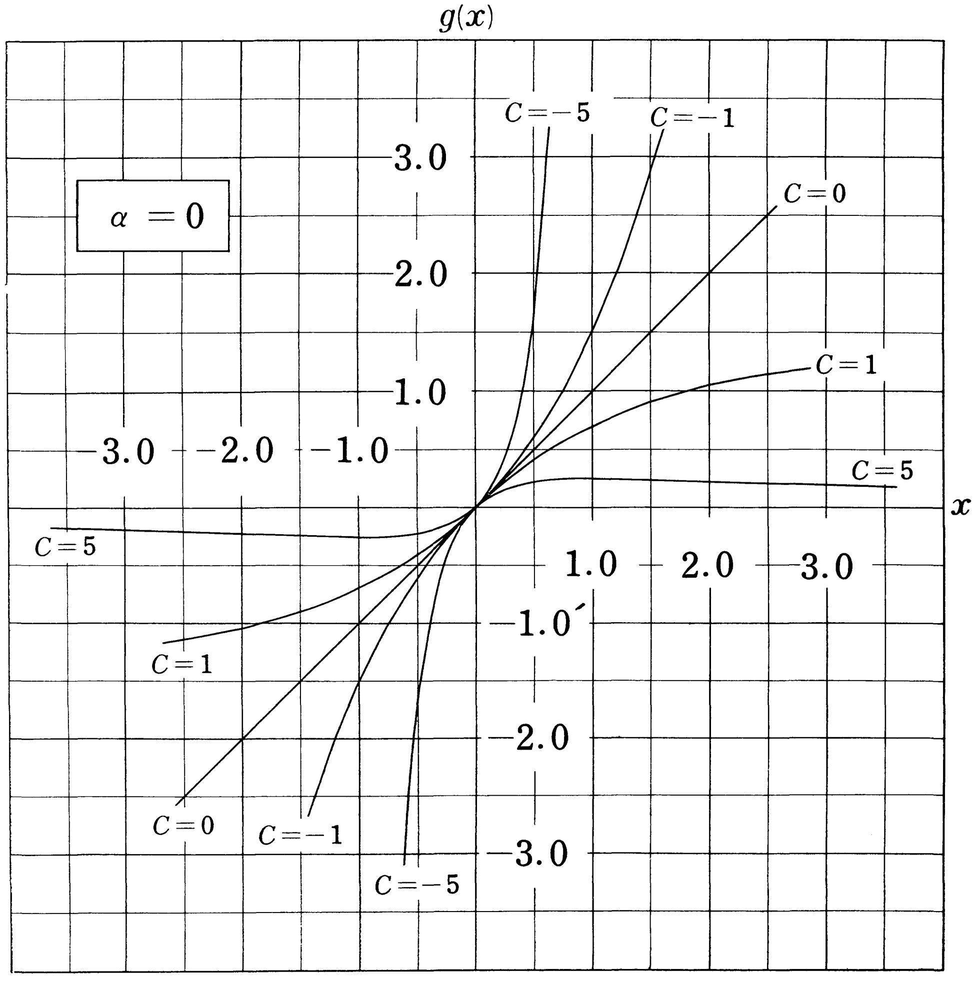
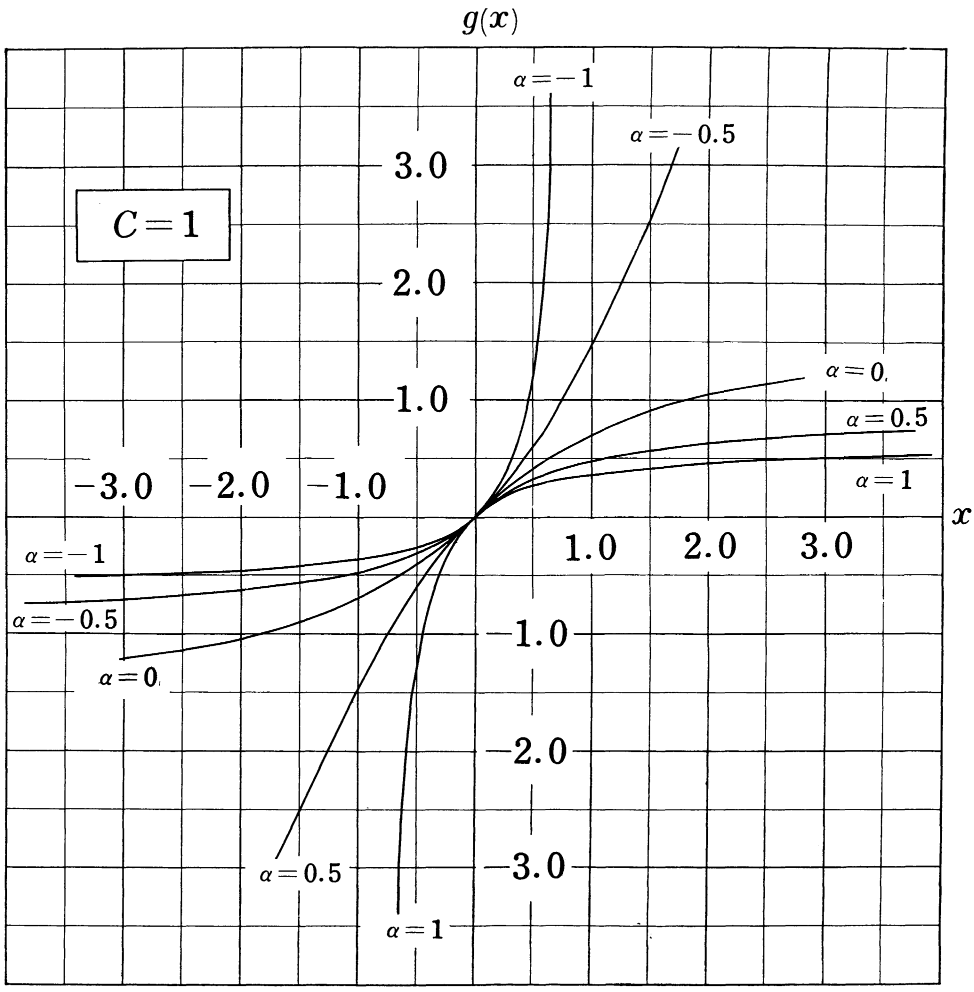


Fig. 1



$g(x)$
$C=1$
$\alpha=-1$
$\alpha=-0.5$
3.0
2.0
1.0
$\alpha=0$
$\alpha=0.5$
$\alpha=1$
-3.0
-2.0
-1.0
x
1.0
2.0
3.0
$\alpha=-1$
$\alpha=-0.5$
$\alpha=0$
-1.0
-2.0
-3.0
$\alpha=0.5$
$\alpha=1$

Fig. 2

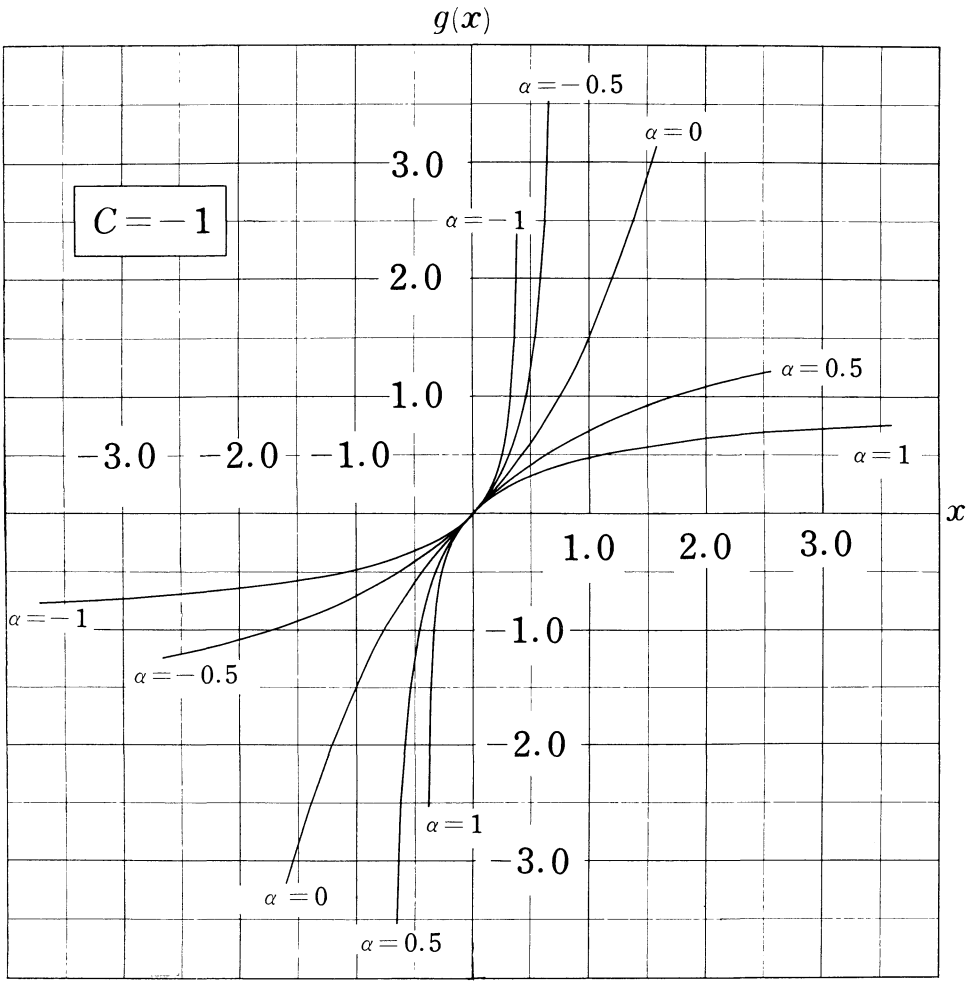
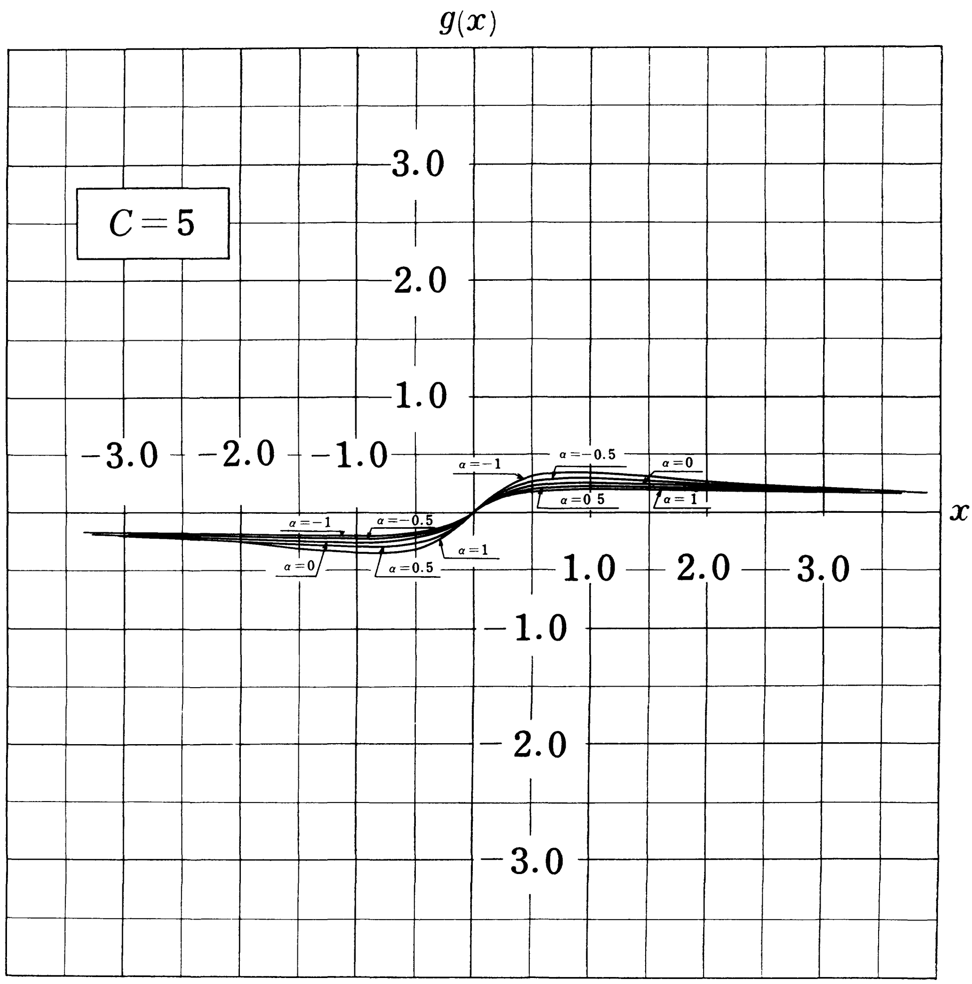

g(x)
x
C = −1
3.0
2.0
1.0
−3.0
−2.0
−1.0
1.0
2.0
3.0
−1.0
−2.0
−3.0
α = −0.5
α = 0
α = −1
α = 0.5
α = 1
α = −1
α = −0.5
α = 1
α = 0
α = 0.5

Fig. 3



$g(x)$
3.0
$C=5$
2.0
1.0
-3.0
-2.0
-1.0
$\alpha=-1$
$\alpha=-0.5$
$\alpha=0$
$\alpha=0.5$
$\alpha=1$
x
$\alpha=-1$
$\alpha=-0.5$
$\alpha=1$
$\alpha=0$
$\alpha=0.5$
1.0
2.0
3.0
-1.0
-2.0
-3.0

Fig. 4

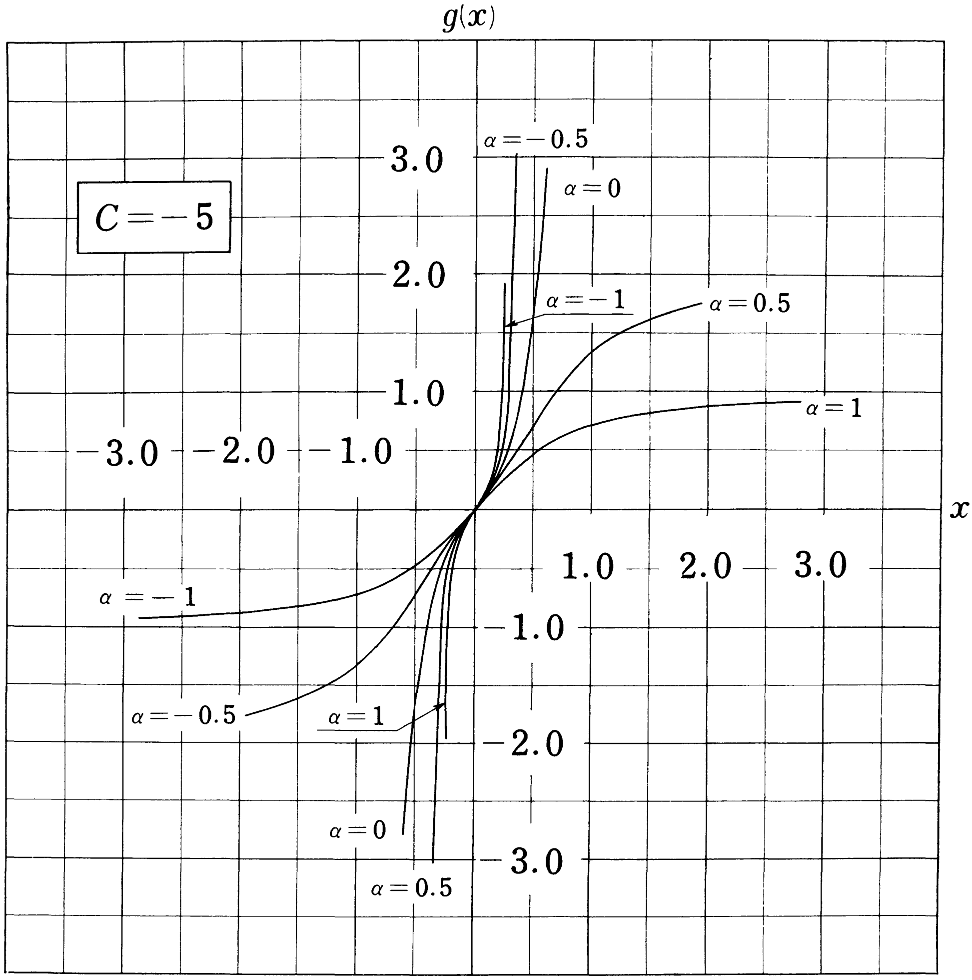


Fig. 5

References

[1] Urabe, M., *Potential forces which yield periodic motions of a fixed period*, J. Math. Mech., **10** (1961), 569–578.

[2] Urabe, M., *The potential force yielding a periodic motion whose period is an arbitrary continuous function of the amplitude of the velocity*, Arch. Rational Mech. Anal., **11** (1962), 27-33.

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