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Equilibrium Points of Stochastic Non-Cooperative n-Person Games

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A non-cooperative n-person game is originated by J. F. Nash [7]. It is a game in which each player acts independently without collaboration or communication with any of the others, thus it admits no coalitions [8] formed by the players of the game. He has introduced the notion of equilibrium points in an n-person game [6] which yields a generalization of the concept of the solution of a two-person zero-sum game, and has proved that any finite non-cooperative game has an equilibrium point. The purpose of this paper is to show the existence of an equilibrium point of a stochastic game, defined below, in which each component game is an infinite non-cooperative n-person game. The proof will be carried out by making use of a fixed point theorem due to K. Fan [2] and I. L. Glicksberg [4] which is a generalization of a theorem of Kakutani [5] to a locally convex space. This proof given here is closely related to that of A. M. Fink [3].

We shall concern ourselves with a stochastic non-cooperative n-person game. First we begin with its definition. Let $I=\{1, 2, \cdots, s\}$ be a finite set of states. There is assumed to be associated with each state i and Player h a compact space $\sum_h^i$ called a strategic space. Let us denote by $\mathfrak{M}_h^i$ the set of regular probability measures in $\sum_h^i$ which is referred to as the space of mixed strategies of Player h at the state i. We put on $\sum_h^i$ the vague topology so that it is a compact space [1]. Let us denote by $g_h^i(\vec{\sigma}^i)(=g_h^i(\sigma_1^i,\cdots, \sigma_n^i))$ the gain of Player h when each player k chooses a pure strategy $\sigma_k^i(\in \sum_k^i)$ at the state i. Here we assume that the function g_h^i is continuous in $\sum_1^i \times \cdots \times \sum_n^i$, so that there exists a positive number N independent of i, h such that $|g_h^i|\leq N$. The set $\Gamma^i=(\sum_1^i, \cdots, \sum_n^i, g_1^i, \cdots, g_n^i, \mathfrak{M}_1^i, \cdots, \mathfrak{M}_n^i)$ will be referred to as an i-th component game of the stochastic non-cooperative n-person game which will be defined below. At the state i, each player chooses a pure strategy $\sigma_h^i \in \sum_h^i$ independently of the others, where Player h is assumed to use a mixed strategy $\mu_h^i(\in \mathfrak{M}_h^i)$. Once the choice has been made, the game proceeds to a next state j with transition probability $p^{ij}(\vec{\sigma}^i)$ assumed to be continuous in $\prod_{h=1}^n \sum_h^i$, or stops with probability $p^{i0}(\vec{\sigma}^i)$ assumed to satisfy the condition

$$\inf_{i,\vec{\sigma}^i} p^{i0}(\vec{\sigma}^i)=p^0>0.$$

Let us denote by $\vec{\mu}^i$ an n-dimensional vector $(\mu_1^i, \cdots, \mu_n^i) \in \prod_{h=1}^{n} \mathfrak{M}_h^i$, by $(\vec{\mu}^i; \rho_h^i)$ an n-dimensional vector $(\mu_1^i, \cdots, \mu_{h-1}^i, \rho_h^i, \mu_{h+1}^i, \cdots, \mu_n^i)$, and by $\vec{\mu}_h$ an s-dimensional vector $(\mu_h^1, \cdots, \mu_h^s) \in \prod_{i=1}^{s} \mathfrak{M}_h^i$. A stochastic non-cooperative n-person game $\boldsymbol{\Gamma}$ is defined as a collection of all Γ^i, p^{ij}, and p^{i0} for $i, j=1, \cdots, s$, where the payments accumulate throughout the course of the play (cf. [9], [10]). There we note that each player uses the stationary strategies.

Now we consider the infinite game $\boldsymbol{\Gamma}^i$ which starts at the state i. Then the expected value $G_h^i(\vec{\mu}_1, \cdots, \vec{\mu}_n)$ of the gains of Player h is given by

$$
(1) \qquad G_h^i(\vec{\mu}_1, \cdots, \vec{\mu}_n) = g_h^i(\vec{\mu}^i) + \sum_{j=1}^{s} p^{ij}(\vec{\mu}^i) g_h^j(\vec{\mu}^j) + \sum_{j=1}^{s}\sum_{k=1}^{s} p^{ij}(\vec{\mu}^i) p^{jk}(\vec{\mu}^j) g_h^k(\vec{\mu}^k) + \cdots, \quad i = 1, \cdots, s; \; h = 1, \cdots, n.
$$

The right hand series of (1) is clearly absolutely convergent.

DEFINITION 1. We say that $(\vec{\mu}_1, \cdots, \vec{\mu}_n)$ is an equilibrium point of the infinite game $\boldsymbol{\Gamma}^i$ when

$$
(2) \qquad G_h^i(\vec{\mu}_1, \cdots, \vec{\mu}_{h-1}, \vec{\rho}_h, \vec{\mu}_{h+1}, \cdots, \vec{\mu}_n) \leqq G_h^i(\vec{\mu}_1, \cdots, \vec{\mu}_n)
$$

for any $\vec{\rho}_h \in \prod_{i=1}^{s} \mathfrak{M}_h^i$ and for every h.

It is our main purpose to prove that the infinite games $\boldsymbol{\Gamma}^i$ $(i=1, \cdots, s)$ have equilibrium points. Now it is obvious that $\{G_h^i(\vec{\mu}_1, \cdots, \vec{\mu}_n)\}$ is a unique solution of the simultaneous system of linear equations with unknowns v_h^i:

$$
(3) \qquad v_h^i = g_h^i(\vec{\mu}^i) + \sum_{j=1}^{s} p^{ij}(\vec{\mu}^i) v_h^j, \; i = 1, \cdots, s; \; h = 1, \cdots, n.
$$

For $\vec{v} = \{v_h^i\}$, $i=1, \cdots, s$; $h=1, \cdots, n$, we use the notations $\vec{v}_h = (v_h^1, \cdots, v_h^s)$ and $\vec{v}^i = (v_1^i, \cdots, v_n^i)$.

DEFINITION 2. We say that $(\vec{\mu}_1, \cdots, \vec{\mu}_n)$ is an equilibrium point of the stochastic game $\boldsymbol{\Gamma}$ when

$$
G_h^i(\vec{\mu}_1, \cdots, \vec{\mu}_{h-1}, \vec{\rho}_h, \vec{\mu}_{h+1}, \cdots, \vec{\mu}_n) \leqq G_h^i(\vec{\mu}_1, \cdots, \vec{\mu}_n)
$$

for any $\vec{\rho}_h \in \prod_{i=1}^{s} \mathfrak{M}_h^i$ and for every h and i.

We shall show the following

THEOREM. Any stochastic game Γ has an equilibrium point.

Proof. Let I be an interval $[-A, A]$ such that $N/p^\circ \leqq A$. Let us denote by $\vec{v}$, $\vec{w}$ ns-dimensional vectors $\in \underbrace{I\times\cdots\times I}_{ns}$, and by $\vec{\mu}$, $\vec{\nu}$, ns-dimensional vectors $\in \mathfrak{M}_1^1\times\cdots\times\mathfrak{M}_n^s$. Put $K=I\times\cdots\times I\times\mathfrak{M}_1^1\times\cdots\times\mathfrak{M}_n^s$. It is a compact convex set of a locally convex space. Consider a point to set mapping

$$\Phi\colon (\vec{v}, \vec{\mu})\ (\in K)\to(\vec{w}, \phi(\vec{v}, \vec{\mu})),$$

where $\vec{w}$ and $\phi(\vec{v}, \vec{\mu})$ are defined as follows:

$$w_h^i = \sup_{\rho^i{}_h\in\mathfrak{M}^i{}_h}[g_h^i(\vec{\mu}^i;\ \rho_h^i) + \sum_j p^{ij}(\vec{\mu}^i;\ \rho_h^i)v_h^j] \tag{4}$$

and $\vec{\nu}\in\phi(\vec{v}, \vec{\mu})$ if and only if

$$w_h^i = g_h^i(\vec{\mu}^i;\ \nu_h^i) + \sum_{j=1}^s p^{ij}(\vec{\mu}^i;\ \nu_h^i)v_h^j. \tag{5}$$

According to our choice of A, it is clear that $\vec{w}\in\overbrace{I\times\cdots\times I}^{ns}$ and that $\phi(\vec{v}, \vec{\mu})$ is a compact convex subset of $\mathfrak{M}_1^1\times\cdots\times\mathfrak{M}_n^s$. If we can show that the mapping Φ is upper semi-continuous, or the graph of the mapping is closed, then we can apply a theorem of Ky Fan [2] to conclude that there exists a $(\vec{v}, \vec{\mu})$ such that $(\vec{v}, \vec{\mu})\in\Phi(\vec{v}, \vec{\mu})$, that is, by (4),

$$v_h^i = \sup_{\rho^i{}_h\in\mathfrak{M}^i{}_h}[g_h^i(\vec{\mu}^i;\ \rho_h^i) + \sum_j p^{ij}(\vec{\mu}^i;\ \rho_h^i)v_h^j] \tag{6}$$

and by (5), we have

$$v_h^i = g_h^i(\mu_1^i, \cdots, \mu_n^i) + \sum_{j=1}^s p^{ij}(\mu_1^i, \cdots, \mu_n^i)v_h^j. \tag{7}$$

Now we proceed to the proof of the upper semi-continuity of our mapping Φ. In terms of nets, it will be sufficient to show that

$$\text{if}\quad \vec{v}_\delta\to\vec{v},\ \vec{w}_\delta\to\vec{w},\ \vec{\mu}_\delta\to\vec{\mu},\quad\text{and}\quad \vec{\nu}_\delta(\in\phi(\vec{v}_\delta, \vec{\mu}_\delta))\to\vec{\nu}, \tag{8}$$

then $(\vec{w}, \vec{\nu})\in\Phi(\vec{v}, \vec{\mu})$. In fact we have

$$w_{\delta h}^i \geqq g_h^i(\vec{\mu}_\delta^i;\ \rho_h^i) + \sum_{j=1}^s p^{ij}(\vec{\mu}_\delta^i;\ \rho_h^i)v^j{}_{\delta h},$$

and

$$w^i_{\delta h}=g^i_h(\vec{\mu}^i_\delta;\ \nu^i_{\delta h})+\sum_{j=1}^{s}p^{ij}(\vec{\mu}^i_\delta;\ \nu^i_{\delta h})v^j_{\delta h}.$$

Passing to the limit, we have

$$w^i_h\geqq g^i_h(\vec{\mu}^i;\ \rho^i_h)+\sum_{j=1}^{s}p^{ij}(\vec{\mu}^i;\ \rho^i_h)v^j_h,$$

and

$$w^i_h=g^i_h(\vec{\mu}^i;\ \nu^i_h)+\sum_{j=1}^{s}p^{ij}(\vec{\mu}^i;\ \nu^i_h)v^j_h,$$

which prove that $(\vec{w},\ \vec{\nu})\in\Phi(\vec{v},\ \vec{\mu})$.

Let us consider a $(\vec{v},\ \vec{\mu})\in\Phi(\vec{v},\ \vec{\mu})$, whose existence has been proved above. We shall show that $(\vec{\mu}_1,\ \cdots,\ \vec{\mu}_n)$ is the equilibrium point of the stochastic game Γ. By (6) we have

$$v^i_h\geqq g^i_h(\vec{\mu}^i;\ \rho^i_h)+\sum_{j=1}^{s}p^{ij}(\vec{\mu}^i;\ \rho^i_h)v^j_h.$$

Put

$$u^i_h=g^i_h(\vec{\mu}^i;\ \rho^i_h)+\sum_{j=1}^{s}p^{ij}(\vec{\mu}^i;\ \rho^i_h)v^j_h. \tag{9}$$

Then $u^i_h\leqq v^i_h$ for $i=1,\ \cdots,\ s$. We have

$$g^i_h(\vec{\mu}^i;\ \rho^i_h)+\sum_{j=1}^{s}p^{ij}(\vec{\mu}^i;\ \rho^i_h)u^j_h\leqq v^i_h. \tag{10}$$

By (9) and (10) we have

$$v^i_h\geqq g^i_h(\vec{\mu}^i;\ \rho^i_h)+\sum_{j=1}^{s}p^{ij}(\vec{\mu}^i;\ \rho^i_h)\{g^j_h(\vec{\mu}^j;\ \rho^j_h)+\sum_{k=1}^{s}p^{jk}(\vec{\mu}^j;\ \rho^j_h)v^k_h\} \tag{11}$$

$$\geqq g^i_h(\vec{\mu}^i;\ \rho^i_h)+\sum_{j=1}^{s}p^{ij}(\vec{\mu}^i;\ \rho^i_h)g^j_h(\vec{\mu}^j;\ \rho^j_h)+\sum_{j=1}^{s}\sum_{k=1}^{s}p^{ij}(\vec{\mu}^i;\ \rho^i_h)p^{jk}(\vec{\mu}^j;\ \rho^j_h)g^k_h(\vec{\mu}^k;\ \rho^k_h)+\cdots$$

$$=G^i_h(\vec{\mu}_1,\ \cdots,\ \vec{\mu}_{h-1},\ \vec{\rho}_h,\ \vec{\mu}_{h+1},\ \cdots,\ \vec{\mu}_n).$$

On the other hand, $\{v^i_h\}$ is a solution of (3), whence $v^i_h=G^i_h(\vec{\mu}_1,\ \cdots,\ \vec{\mu}_n)$ as already remarked. Then the inequalities yield

$$G^i_h(\vec{\mu}_1,\ \cdots,\ \vec{\mu}_{h-1},\ \vec{\rho}_h,\ \vec{\mu}_{h+1},\ \cdots,\ \vec{\mu}_n)\leqq G^i_h(\vec{\mu}_1,\ \cdots,\ \vec{\mu}_n)$$

for any $\vec{\rho}_h \in \prod_{i=1}^{s} \mathfrak{M}_h^i$. Thus our theorem is proved.

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