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Notes on Green Lines and Kuramochi Boundary of a Green Space.

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Introduction. It has been clarified that the ideal boundary of a Riemann surface introduced by Kuramochi [6] enjoys many properties possessed by the usual relative boundary of a bounded plane domain. A purpose of this note is to show that the Kuramochi boundary also has the property that almost every Green line tends to one boundary point.

On the other hand, an analogous definition of the Kuramochi boundary can be given for a Green space introduced by Brelot-Choquet [2]. In this note, we shall lay down a definition following the method of Constantinescu-Cornea [3] and we shall obtain the property mentioned above and some other related properties for the Kuramochi boundary of a Green space.

§ 1. Green lines and a compactification.

Let Ω be a Green space and $G(a, b)$ be the Green function for Ω. For the definitions and properties of these and following notions, we refer to Brelot-Choquet [2]. We consider the Green lines in Ω determined by $G_0(a)\equiv G(a, a_0)$ for a fixed point $a_0 \in \Omega$. The set L of all Green lines admits the Green measure γ. A Green line l for which $\inf_{a \subset l} G_0(a)=0$ is called a regular Green line. Any regular Green line tends to the ideal boundary of Ω as $G_0 \to 0$. The set of all regular Green lines will be denoted by L_r. It is known that $\gamma(L-L_r)=0$.

Given a real function f on Ω and $l \in L_r$, let $\overline{\lim_{l}} f$ (resp. $\underline{\lim_{l}} f$) denote the upper limit $\overline{\lim}_{\substack{a \in l \\ G_0(a)\to 0}} f(a)$ (resp. the lower limit $\underline{\lim}_{\substack{a \subset l \\ G_0(a)\to 0}} f(a)$). If $\underline{\lim_{l}} f=\overline{\lim_{l}} f$, then we say that f has limit along l. Let $C_{G_0}(\Omega)$ be the set of all bounded continuous functions on Ω having limit along almost every $l \in L_r$ (with respect to γ).

For a compactification Ω^* of Ω, let

$$C(\Omega^*)=\{f|_{\Omega}; f \text{ is continuous on } \Omega^*\}.$$

A family Q of real functions on Ω is said to separate points of $\Delta=\Omega^*-\Omega$, if for any b_1, $b_2 \in \Delta$ $(b_1 \neq b_2)$, there is $f \in Q$ such that $\underline{\lim}_{\substack{a\to b_1 \\ a \subset \Omega}} f(a) > \overline{\lim}_{\substack{a\to b_2 \\ a \in \Omega}} f(a)$.

THEOREM 1. *Let Ω^* be a compactification of Ω. If one of the following conditions is satisfied, then almost every Green line tends to one point of Δ:*

i) *There exists a countable family Q of functions on Ω such that any $f \in Q$ has limit along almost every $l \in L_r$ and Q separates points of Δ.*

ii) *Ω^* is metrizable and $C(\Omega^*) \subseteq C_{G_0}(\Omega)$.*

PROOF. First we assume condition i). For $f \in Q$ let $\Lambda_f = \{l \in L_r; \varliminf_l f \neq \varlimsup_l f\}$. Then, by assumption, $\gamma(\Lambda_f)=0$. Since Q is countable, we have $\gamma(\bigcup_{f \in Q} \Lambda_f)=0$. On the other hand, if $l \notin \bigcup_{f \in Q} \Lambda_f$, then $\varliminf_l f = \varlimsup_l f$ for all $f \in Q$. Let b_1, b_2 be limit points of l on Δ. It follows then that $\lim_{\substack{a \to b_1 \\ a \in l}} f(a)$ and $\lim_{\substack{a \to b_2 \\ a \in l}} f(a)$ exist and are equal to $\varliminf_l f = \varlimsup_l f$. Hence

$$\varliminf_{a \to b_1} f(a) \leq \lim_{\substack{a \to b_1 \\ a \in l}} f(a) = \lim_{\substack{a \to b_2 \\ a \in l}} f(a) \leq \varlimsup_{a \to b_2} f(a)$$

for all $f \in Q$. Since Q separates points of Δ, we cannot have $b_1 \neq b_2$. Therefore, there is only one limit point of l on Δ for $l \notin \bigcup_{f \in Q} \Lambda_f$.

If condition ii) is satisfied, then $C(\Omega^*)$ is separable in the uniform convergence topology, so that there exists a countable family Q which is dense in $C(\Omega^*)$. Then it is obvious that Q satisfies condition i).

REMARK: Conversely, if almost every Green line tends to one point of $\Delta = \Omega^* - \Omega$, then $C(\Omega^*) \subseteq C_{G_0}(\Omega)$ and condition i) without the countability of Q is valid.

§ 2. Application of Godefroid's result.

We follow Brelot [1] for the definition of BLD functions on a Green space Ω. The following result is due to Godefroid [5]:

LEMMA 1. *Any BLD function on Ω has limit along almost every Green line.*

Let $C_D(\Omega^*) = \{f \in C(\Omega^*); f$ is a BLD function on $\Omega\}$. Then Lemma 1 implies that $C_D(\Omega^*) \subseteq C_{G_0}(\Omega)$. Thus, the following theorem is an immediate consequence of this lemma and Theorem 1:

THEOREM 2. *Let Ω^* be a compactification of Ω. If one of the following conditions is satisfied, then almost every Green line tends to one point of $\Delta = \Omega^* - \Omega$:*

i) *There exists a countable family of BLD functions separating points* *of* Δ.

ii) Ω^* *is metrizable and* $C_D(\Omega^*)$ *separates points of* Δ.

COROLLARY. *Let* Ω *be a hyperbolic Riemann surface. Then almost every Green line tends to one point of the Kuramochi boundary*[1] *of* Ω.

PROOF. By [3], §16, we see that the Kuramochi boundary is determined by a family of BLD functions[2] on Ω. Since the Kuramochi compactification is metrizable, the corollary follows from the theorem.

REMARK: If Ω^* is not metrizable, then the fact that $C_D(\Omega^*)$ separates points of Δ does not necessarily imply that almost every Green line tends to one point of Δ. For example, if Ω_D^* is the Royden compactification ([3], §9), then no point of $\Delta_D=\Omega_D^*-\Omega$ is accessible, so that no point of Δ_D can be the end point of a Green line. By definition, $C_D(\Omega_D^*)$ separates points of Δ_D in this case.

§ 3. Kuramochi boundary of a Green space.

In this section, we shall give a definition of the Kuramochi boundary of a Green space following the method of [3]. We start with recalling some known results on BLD functions most of which are found in [1]. (There is also a summary on properties of BLD functions in the introduction of Doob [4].)

Let Ω be a Green space of dimension ≥ 3. For a set $\sigma \subseteq \Omega$, let $\check{\sigma}=\sigma-$ {points of infinity}. The set of all BLD functions on Ω will be denoted by $D \equiv D(\Omega)$. Obviously, D is a linear space. For any $f_1, f_2 \in D$ and for any measurable set σ in Ω, $<f_1, f_2>_\sigma = \int_{\check{\sigma}} (\text{grad } f_1, \text{grad } f_2)dv$ (dv is the volume element in $\check{\Omega}$) is defined. We write $\|f\|_\sigma=(<f, f>_\sigma)^{\frac{1}{2}}$ for $f \in D$. We often write $<f_1, f_2>$ and $\|f\|$ instead of $<f_1, f_2>_\Omega$ and $\|f\|_\Omega$ respectively.

We say that $f_1 \in D$ and $f_2 \in D$ are equivalent if $\|f_1-f_2\|=0$. For $f_1, f_2 \in D$, they are equivalent if and only if $f_1=f_2+\text{const.}$ q.p. ("q.p." is the abbreviation of "quasi-partout" meaning "except on a polar set". See [2] and [1].) The space $\hat{D} \equiv \hat{D}(\Omega)$ of the equivalence classes becomes a Hilbert space with respect to the inner product induced by $<f_1, f_2>$. We denote the equivalence class of f by $\hat{f}$. The linear space D is the direct sum of two subspaces HD and D_0, where HD is the space of harmonic BLD functions on Ω and D_0 is the space of BLD functions of "radiale nulle" (See [1]. They are called BLD functions

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1) For the definition of the Kuramochi boundary of a Riemann surface, see [6] and [3], §16.

2) It is not difficult to see that the family of BLD functions on a Riemann surface coincides with the family of Dirichlet's functions in the sense of [3].

of potential type in [4].) Then $\hat{D}=\widehat{HD}\oplus\hat{D}_0$, where the direct sum is topological, i.e., $\widehat{HD}$, $\hat{D}_0$ are mutually orthogonal closed subspaces of $\hat{D}$. Let $C_D^\infty\equiv C_D^\infty(\Omega)$ be the space of all functions on Ω each of which is infinitely differentiable on $\check{\Omega}$, has finite Dirichlet integral and has compact support. Then $C_D^\infty\subseteq D_0$ and $\overline{\hat{C}_D^\infty}=\hat{D}_0$.

LEMMA 2. *If f_n, $f\in D$, $\hat{f}_n\to\hat{f}$ and $f_n\to f$ on a non-polar set, then there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k}\to f$ q.p. on Ω.*

PROOF. This is immediate from [1], n° 16, α).

THEOREM 3. *Let δ be a non-polar closed set in Ω and let $D_\delta=\{f\in D;$ $f=0$ q.p. on $\delta\}$. Then $\hat{D}_\delta$ is a closed subspace of $\hat{D}$.*

PROOF. Obviously, D_δ is a linear space. Let $\{f_n\}\subseteq D_\delta$, $\hat{f}_n\to\hat{f}$ for an $f\in D$. Then, by the above lemma, there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k}\to f$ q.p. Then $f=0$ q.p. on δ, i.e., $f\in D_\delta$.

THEOREM 4. *Let $f\in D$ and let δ be a non-polar closed set in Ω. Let $(\hat{f})_\delta$ be the projection of $\hat{f}$ onto $\hat{D}_\delta$. Then there exists a uniquely determined $f_\delta\in D$ such that* a) $\hat{f}-\hat{f}_\delta=(\hat{f})_\delta$, b) $f=f_\delta$ *on δ and* c) f_δ *is harmonic on $\Omega-\delta$.*

PROOF. Let $f_1\in(\hat{f})_\delta$ and $f_0=f-f_1$. Since $f_1\in D_\delta$, $f_0=f$ q.p. on δ. Let ω be any component of $\Omega-\delta$. For any $\varphi\in C_D^\infty(\omega)$, we extend φ by 0 outside ω. Then $\varphi\in C_D^\infty(\Omega)$ and $\varphi\in D_\delta$. Since $(\hat{f})_\delta$ is the projection of $\hat{f}$ onto $\hat{D}_\delta$, $\hat{f}-(\hat{f})_\delta$ is orthogonal to $\hat{\varphi}$. Hence $\langle f-f_1,\varphi\rangle=0$ or $\langle f_0,\varphi\rangle=0$ or $\langle f_0,\varphi\rangle_\omega=0$. This implies that $f_0|_\omega$ is orthogonal to $D_0(\omega)$, so that $\hat{f}_0|_\omega\in\widehat{HD}(\omega)$. Therefore there is a harmonic function u_ω on ω such that $f_0=u_\omega$ q.p. on ω. Now let

$$f_\delta=\begin{cases} f & \text{on } \delta \\ u_\omega & \text{on each component } \omega \text{ of } \Omega-\delta. \end{cases}$$

Then, by the above argument, we see that $f_\delta=f_0$ q.p. on Ω. Hence $f_\delta\in D$ and $\hat{f}-\hat{f}_\delta=\hat{f}-\hat{f}_0=\hat{f}_1=(\hat{f})_\delta$. Therefore f_δ satisfies a), b) and c).

Next, suppose g satisfies the properties of f_δ given in a), b) and c). Then a) implies $\hat{g}=\hat{f}_\delta$. Therefore, by b), we have $g=f_\delta$ q.p. on Ω, everywhere on δ. Now it follows from c) that $g=f_\delta$ also on $\Omega-\delta$ and the uniqueness of f_δ follows.

DEFINITION. Let

$$N=\{f\in D;\, f=f_\delta \text{ for some non-polar compact set } \delta \text{ in } \Omega\}.$$

The compactification Ω_N^* of Ω determined by N, i.e., the compactification such that every function of N can be continuously extended over Ω_N^* and N separates points of $\Delta_N=\Omega_N^*-\Omega$, is[3] called the *Kuramochi compactification* of Ω and Δ_N is called the *Kuramochi boundary* of Ω.

The following properties of f_δ can be proved in the same way as Satz 15.1 in [3]:

(1) $\|f_\delta\|\leq\|f\|$, $<f_\delta, g>=0$ for all $g\in D_\delta$.
(2) $(\alpha f+\beta g)_\delta=\alpha f_\delta+\beta g_\delta$, where α, β are real numbers.
(3) If $f\equiv$const., then $f_\delta=f$.
(4) If $f\geq 0$, then $f_\delta\geq 0$.
(5) If δ, δ' are non-polar closed sets such that $\delta\subseteq\delta'$, then $f_\delta=(f_\delta)_{\delta'}=(f_{\delta'})_\delta$.
(6) If ω is a component of $\Omega-\delta$, then $f_\delta=f_{\partial\omega}$ on ω. ($\partial\omega$ is the boundary of ω in Ω.)

§ 4. Metrizability.

We may proceed to define the kernel of Kuramochi's type[4] on Ω and see that the Kuramochi compactification is metrizable. But, for our purpose here, we can directly show that there exists a countable family of BLD functions in N separating points of Δ_N.

Lemma 3. *Let f_n, $f\in D$ and δ be a non-polar closed set in Ω.*

(i) *If* $f_n\rightrightarrows f$ (uniformly convergent) *on $\partial\delta$, then $(f_n)_\delta\rightrightarrows f_\delta$ on $\Omega-\delta$.*

(ii) *If δ is compact, $\hat{f}_n\to\hat{f}$ and $f_n\to f$ q.p. on δ, then there exists a subsequence $\{f_{n_k}\}$ such that $(f_{n_k})_\delta\rightrightarrows f_\delta$ on $\Omega-\bar{\omega}$ for any relatively compact open set ω containing δ.*

Proof. (i) By (4) and (6) in the previous section, we see that $f\geq 0$ on $\partial\delta$ implies $f_\delta\geq 0$ on $\Omega-\delta$. If $f_n\rightrightarrows f$ on $\partial\delta$, then for any $\varepsilon>0$ there exists n_0 such that $n\geq n_0$ implies $|f_n-f|\leq\varepsilon$ on $\partial\delta$. Hence $|(f_n)_\delta-f_\delta|\leq\varepsilon_\delta=\varepsilon$ on $\Omega-\delta$. Therefore, $(f_n)_\delta\rightrightarrows f_\delta$ on $\Omega-\delta$.

(ii) By (2) and (1) in the previous section, we have

$$\|f_\delta-(f_n)_\delta\|=\|(f-f_n)_\delta\|\leq\|f-f_n\|.$$

Hence, $\hat{f}_n\to\hat{f}$ implies that $\widehat{(f_n)_\delta}\to\hat{f}_\delta$. Now, $(f_n)_\delta=f_n\to f=f_\delta$ q.p. on δ. Hence, by Lemma 2, there exists a subsequence $\{f_{n_k}\}$ such that $(f_{n_k})_\delta\to f_\delta$ q.p. Since $(f_{n_k})_\delta$, f_δ are harmonic on $\Omega-\delta$, the convergence is uniform on $\partial\omega$. Hence, by

3) The existence and the uniqueness (up to a homeomorphism) of such a compactification are assured by a general theory. Cf. [3], §9.

4) The kernel is denoted by $N(z, p)$ in [6], by $\tilde{g}$ in [3] for Riemann surfaces. To construct a potential theory on the Kuramochi compactification, it should be preferable to consider the kernel first.

(i), we have $((f_{n_k})_\delta)_{\bar{\omega}} \rightrightarrows (f_\delta)_{\bar{\omega}}$ on $\Omega-\bar{\omega}$. Then it follows from (5) that $(f_{n_k})_\delta \rightrightarrows f_\delta$ on $\Omega-\bar{\omega}$.

THEOREM 5. *There exists a countable subfamily of N which separates points of Δ_N.*

PROOF. Let $\{\omega_n\}_{n=1,2,\ldots}$ be an exhaustion of Ω, i.e., let each ω_n be a relatively compact domain, $\bar{\omega}_n \subset \omega_{n+1}$ and $\bigcup_n \omega_n = \Omega$. We assume that $\partial\omega_n$ does not contain points of infinity for each n. Let

$$C_n = \{f \in C_D^\infty;\ \text{the support of } f \text{ is contained in } \omega_{n+1}\},$$

$n=1, 2, \cdots$. Then C_n can be regarded as a subspace of $C(\bar{\omega}_{n+1})$. Since $C(\bar{\omega}_{n+1})$ is separable with respect to the uniform convergence topology, C_n is also separable. Hence there exists a countable family Q_n which is dense in C_n. Let $Q=\bigcup_{n=1}^{\infty} \{g_{\omega_n};\ g \in Q_n\}$. Obviously, $Q \subseteq N$ and Q is a countable family. We shall show that, for any $f \in N$ and for any $\varepsilon>0$, there exist n and $g \in Q_n$ such that $|g_{\bar{\omega}_n}-f|<\varepsilon$ on $\Omega-\bar{\omega}_{n+1}$. Then it follows that Q separates points of Δ_N.

Let $f \in N$ and $\varepsilon>0$ be given. There is a non-polar compact set δ such that $f=f_\delta$. Choose n such that $\omega_n \supset \delta$. Since $\partial\omega_n$ does not contain points of infinity, we can construct a function φ which is infinitely differentiable on $\check{\Omega}$, is equal to 1 on $\bar{\omega}_n$ and whose support is contained in the set $\omega_{n+1}-$ {points of infinity in $\omega_{n+1}-\omega_n$}. Then it is easy to see that $\varphi f \in D_0$ and $\varphi f=f$ on $\bar{\omega}_n$. Since $\varphi f \in D_0(\omega_{n+1})$ and $\hat{C}_D^\infty(\omega_{n+1})$ is dense in $\hat{D}_0(\omega_{n+1})$, there exists a sequence $\{f_m\} \subseteq C_n$ such that $\hat{f}_m \rightarrow \widehat{\varphi f}$. Since f_m, φf are zero outside ω_{n+1}, we can apply Lemma 2. Therefore, taking a subsequence, we may assume that $f_m \rightarrow \varphi f$ q.p. on Ω. By the above lemma, we can choose a subsequence $\{f_{m_k}\}$ such that $(f_{m_k})_{\bar{\omega}_n} \rightrightarrows (\varphi f)_{\bar{\omega}_n} = f_{\bar{\omega}_n} = f$ on $\Omega-\bar{\omega}_{n+1}$. Hence, there is $g_1 \in C_n$ such that $|(g_1)_{\bar{\omega}_n} - f| < \frac{\varepsilon}{2}$ on $\Omega-\bar{\omega}_{n+1}$. Since Q_n is dense in C_n, there exists $g \in Q_n$ such that $|g-g_1| < \frac{\varepsilon}{2}$. Then $|g_{\bar{\omega}_n}-(g_1)_{\bar{\omega}_n}| < \frac{\varepsilon}{2}$, so that $|g_{\bar{\omega}_n}-f|<\varepsilon$ on $\Omega-\bar{\omega}_{n+1}$.

COROLLARY. *Ω_N^* is metrizable.*

THEOREM 6. *For a Green space of dimension ≥ 3, almost every Green line tends to one point of the Kuramochi boundary.*

PROOF. This is an immediate consequence of the above theorem and Theorem 2, since N consists of BLD functions.

§ 5. Harmonic measure and Green measure.

For a compactification Ω^* of a Green space Ω, we can discuss the Dirichlet problem (cf. [3], §8, also [2], VI). Namely, given a real function φ on $\Delta=\Omega^*-\Omega$, we set $\bar{\mathfrak{S}}_\varphi=\{s;$ superharmonic and bounded below on Ω, $\varliminf_{a\to b} s(a)\geq\varphi(b)$ for each $b\in\Delta\}\cup\{\infty\}$, $\underline{\mathfrak{S}}_\varphi=\{-s; s\in\bar{\mathfrak{S}}_{-\varphi}\}$,

$$\bar{H}_\varphi=\inf\bar{\mathfrak{S}}_\varphi \qquad \text{and} \qquad \underline{H}_\varphi=\sup\underline{\mathfrak{S}}_\varphi.$$

If $\bar{H}_\varphi=\underline{H}_\varphi$ and is harmonic, then φ is called resolutive and if any $\varphi\in C(\Delta)$ (the space of continuous functions on Δ) is resolutive, then we say that Ω^* is a *resolutive* compactification (cf. [3], §8).

We know that, if Ω is a hyperbolic Riemann surface, then the Kuramochi compactification is resolutive ([3], §16). We shall show that this is true also for a Green space of dimension ≥ 3.

Lemma 4. *If $f\in D_0$, then there exists a Green potential p on Ω such that $|f|\leq p$.*

The proof of this lemma goes exactly in the same way as that of Hilfssatz 7.7 in [3], using Theorem 5.1 in [4].

Theorem 7. *The Kuramochi compactification of a Green space of dimension ≥ 3 is resolutive.*

Proof. For any $f\in N$, let $f=u+g$ with $u\in HD$ and $g\in D_0$. Then, by the above lemma, there exists a Green potential p such that $|g|\leq p$. Since $u-p\leq f\leq u+p$, we have $u-p\in\underline{\mathfrak{S}}_\varphi$ and $u+p\in\bar{\mathfrak{S}}_\varphi$, where $\varphi=f|_{\Delta_N}$. Hence $u-p\leq\underline{H}_\varphi\leq\bar{H}_\varphi\leq u+p$, so that $\underline{H}_\varphi$ and $\bar{H}_\varphi$ are harmonic and $u=\underline{H}_\varphi=\bar{H}_\varphi$. Therefore, $\varphi=f|_{\Delta_N}$ is resolutive for any $f\in N$.

Now consider the smallest subspace Q of $C(\Delta_N)$ with the property that it contains $\{f|_{\Delta_N}; f\in N\}$, it is closed under max. and min. operations and under uniform convergence. Since Q separates points of Δ_N and contains constants, $Q=C(\Delta_N)$ by the Stone-Weierstrass theorem. On the other hand, the space R of all resolutive functions on Δ_N is a linear space satisfying the above property. Hence $Q\subseteq R$, i.e., every function in $C(\Delta_N)$ is resolutive.

When a compactification Ω^* is resolutive, there is the harmonic measure $\mu=\mu_a$ $(a\in\Omega)$ on $\Delta=\Omega^*-\Omega$ defined by $\mu_a(e)=H_{\varphi_e}(a)$; φ_e is the characteristic function of a Borel set e on Δ.

Observing that Théorème 30 and its corollary in [2] are also valid for a resolutive metrizable compactification, we obtain the following theorem concerning the relation between the Green measure and the harmonic measure on the Kuramochi boundary:

Theorem 8. *Let Ω be a hyperbolic Riemann surface or a Green space of dimension ≥ 3, μ be the harmonic measure on the Kuramochi boundary Δ_N of Ω, let*

$$\Lambda_N=\{l \in L_r;\ l \text{ tends to one point } b(l) \text{ on } \Delta_N\}$$

and let

$$\Delta_L=\{b(l);\ l \in \Lambda_N\}.$$

Then

(i) *Δ_L is μ-measurable and $\mu(\Delta_N-\Delta_L)=0$,*

(ii) *for any μ-measurable set $\sigma \subset \Delta_L$, the set $\Lambda_\sigma=\{l \in \Lambda_N;\ b(l) \in \sigma\}$ is γ-measurable and $\mu_{a_0}(\sigma)=\gamma(\Lambda_\sigma)$.*

(Remark that Theorem 6 states $\gamma(L-\Lambda_N)=0$.)

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